Advanced Robotics

ENGG5402 Spring 2023

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Mid-Terms Revision

Rotation Matrix

1. Find the rotation matrix corresponding to the XYZ convention.

If you recall……

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi)$$

Where,

$$R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Rotation Matrix

1. Find the rotation matrix corresponding to the XYZ convention.

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi)$$

The product of $R_Z(\phi), R_Y(\theta)$ and $R_X(\psi)$ gives,

$$R_{RPY}(\psi,\theta,\phi) = \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}$$

Rotation Matrix

2. Find the Euler Angles corresponding to the XYZ convention.

If the rotation matrix is defined as:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The set of Euler angles XYZ are given by:

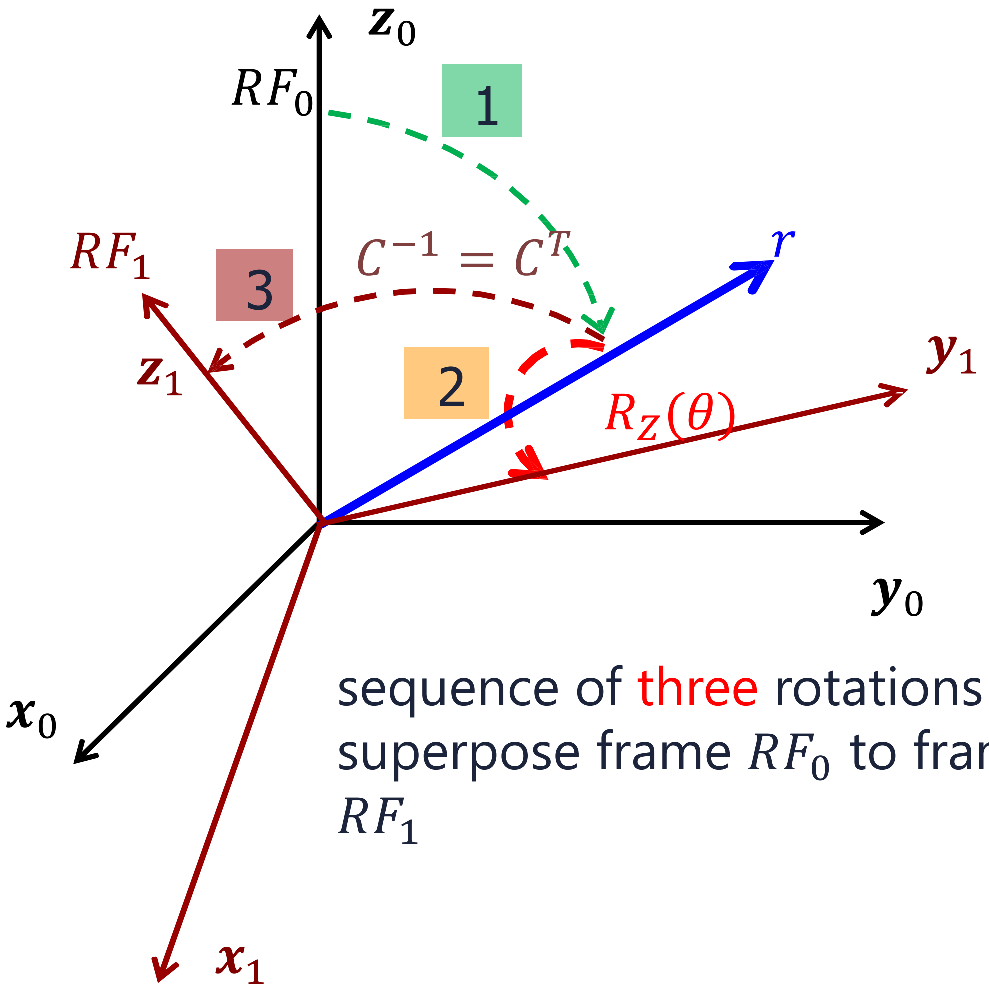
$$\theta = \operatorname{atan2}\left\{-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\right\}$$

$$\phi = \operatorname{atan2}\{r_{21}/c\theta, -r_{11}/c\theta\}$$

$$\psi = \operatorname{atan2}\{r_{32}/c\theta, r_{33}/c\theta\}$$

Axis/angle Representation

Note: from Week 2, slide no.17



sequence of three rotations that superpose frame RF_0 to frame RF_1

$$R(\theta, \boldsymbol{r}) = C R_z(\theta) C^T$$

$$C = [n \quad s \quad r]$$

$n, \mathrm{s}, \mathrm{r}$ denotes the columns of a rotation matrix

Prove the equations in green frame

$$CC^T = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

$$\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$$

Axis/angle Representation

Rotation Matrix Orthonormal Condition (det = +1):

- Orthogonality:

$$R^T = R^{-1}$$

The column vectors of R are mutually orthogonal as they represent unit vectors of an orthonormal frame

$$x'^T y' = 0, \qquad y'^T z' = 0, \qquad z'^T x' = 0$$

- Unit Normal

The product of the rotation matrix and its transpose has the following properties

$$x'^T x' = 1\,, \qquad y'^T y' = 1, \qquad z'^T z' = 1$$

The rotation matrix satisfies the orthonormal condition

$$R^T R = \begin{bmatrix} x'^T x' & y'^T x' & z'^T x' \\ x'^T y' & y'^T y' & z'^T y' \\ x'^T z' & y'^T z' & z'^T z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Axis/angle Representation

$$Prove:\ CC^T = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

$$CC^T = [\boldsymbol{n} \quad \boldsymbol{s} \quad \boldsymbol{r}] \cdot \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix}$$

$$= \begin{bmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{bmatrix} \cdot \begin{bmatrix} x_x & x_y & x_z \\ y_y & y_y & y_z \\ z_x & z_y & z_z \end{bmatrix}$$

$$= \begin{bmatrix} x_x^2 + y_y^2 + z_z^2 & x_x x_y + y_x y_y + z_x z_y & x_x x_z + y_x y_z + z_x z_z \\ x_y x_x + y_y y_x + z_y z_x & x_y^2 + y_y^2 + z_y^2 & x_y x_z + y_y y_z + z_y z_z \\ x_z x_x + y_z y_x + z_z z_x & x_z x_y + y_z y_y + z_z z_y & x_z^2 + y_z^2 + z_z^2 \end{bmatrix}$$

Axis/angle Representation

$$Prove: CC^T = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

Apply rotation matrix conditions:

The unit normal condition

- $x_x^2 + y_y^2 + z_z^2 = 1$
- $x_y^2 + y_y^2 + z_y^2 = 1$
- $x_z^2 + y_z^2 + z_z^2 = 1$

The Orthogonality Condition

- $x_y x_x + y_y y_x + z_y z_x = 0$
- $x_z x_x + y_z y_x + z_z z_x = 0$
- $x_z x_y + y_z y_y + z_z z_y = 0$

$$CC^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Axis/angle Representation

Prove: $sn^T - ns^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$

$$sn^T - ns^T = \begin{bmatrix} y_x \\ y_y \\ y_z \end{bmatrix} \cdot [x_x x_y x_z] - \begin{bmatrix} x_x \\ x_y \\ x_z \end{bmatrix} \cdot [y_x y_y y_z]$$

$$= \begin{bmatrix} 0 & y_x x_y - x_x y_y & y_x x_z - x_x y_z \\ y_y x_x - x_y y_x & 0 & y_y x_z - x_y y_z \\ y_z x_x - x_z y_x & y_z x_y - x_z y_y & 0 \end{bmatrix}$$

Apply the Orthogonality condition:

$$c^{-1} = c^T$$

Axis/angle Representation

Prove: $sn^T - ns^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$

$$c^T = c^{-1}$$

$$c^T = \begin{bmatrix} x_x & x_y & x_z \\ y_x & y_y & y_z \\ z_x & z_y & z_z \end{bmatrix}$$

$$c^{-1} = \frac{1}{\det(c)} \begin{bmatrix} \begin{vmatrix} y_y & z_y \\ y_z & z_z \end{vmatrix} & \begin{vmatrix} z_x & y_x \\ z_z & y_z \end{vmatrix} & \begin{vmatrix} y_x & z_x \\ y_y & z_y \end{vmatrix} \\ \begin{vmatrix} z_y & x_y \\ z_z & x_z \end{vmatrix} & \begin{vmatrix} x_x & z_x \\ x_z & z_z \end{vmatrix} & \begin{vmatrix} z_x & x_x \\ z_y & x_y \end{vmatrix} \\ \begin{vmatrix} x_y & y_y \\ x_z & y_z \end{vmatrix} & \begin{vmatrix} y_x & x_x \\ y_z & x_z \end{vmatrix} & \begin{vmatrix} x_x & y_x \\ x_y & y_y \end{vmatrix} \end{bmatrix}$$

det(c) is +1 when the frame is right handed

$$z_z = \frac{1}{\det(c)} \begin{vmatrix} x_x & y_x \\ x_y & y_y \end{vmatrix} = x_x y_y - x_y y_x$$

Axis/angle Representation

The same applies for z_y and z_x.

$$z_y = y_x x_z - y_z x_x$$

$$z_x = x_y y_z - x_z y_y$$

$$sn^T - ns^T = \begin{bmatrix} 0 & y_x x_y - x_x y_y & y_x x_z - x_x y_z \\ y_y x_x - x_y y_x & 0 & y_y x_z - x_y y_z \\ y_z x_x - x_z y_x & y_z x_y - x_z y_y & 0 \end{bmatrix}$$

Substitute z_x, z_y, z_z ,

$$sn^T - ns^T = \begin{bmatrix} 0 & -z_z & z_y \\ z_z & 0 & -z_x \\ -z_y & z_x & 0 \end{bmatrix}$$

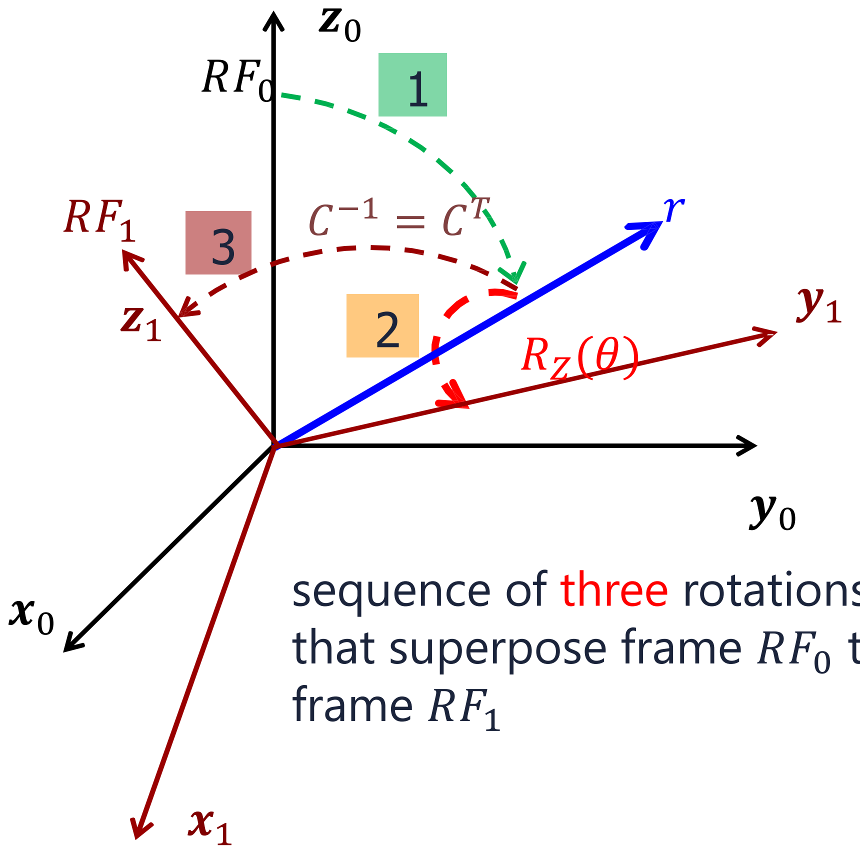
Since the r axis is aligned to the z axis at this point, the z-axis and r-axis are the same.

Axis/angle Representation

Hence,

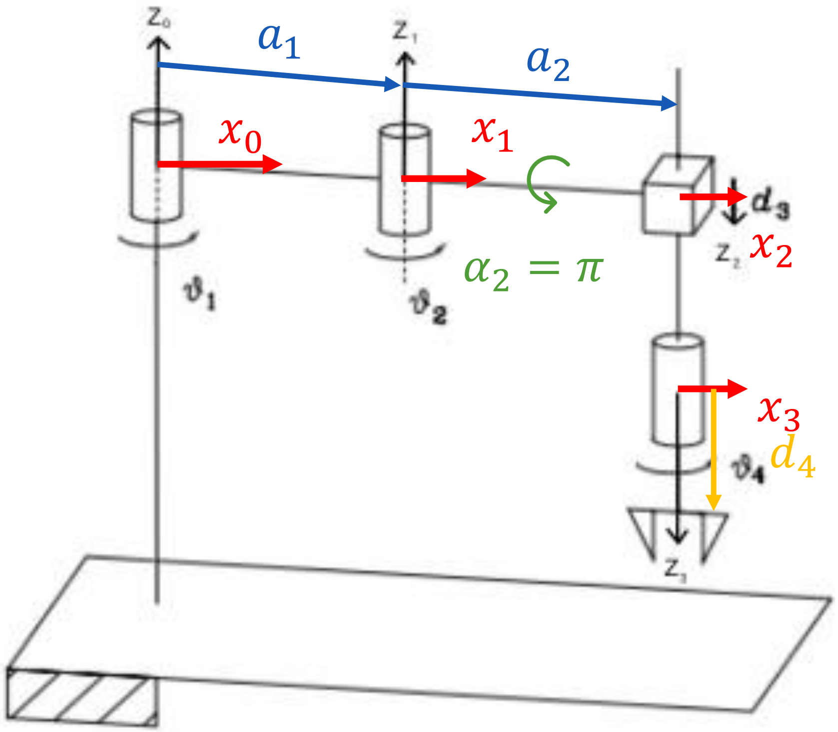
$$sn^T - ns^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$$

Where $S(\boldsymbol{r})$ is the skew symmetric matrix of axis r.



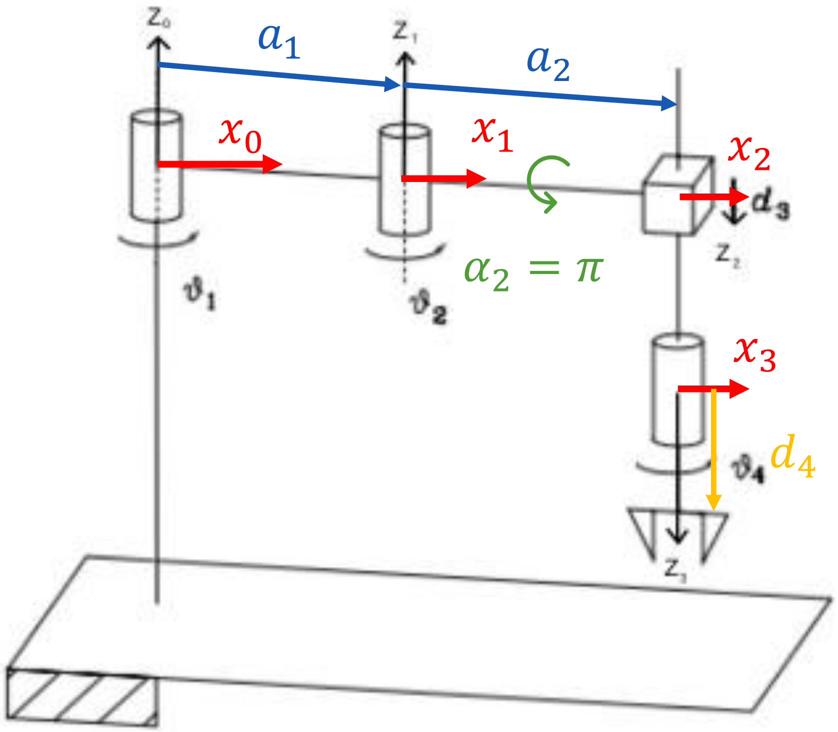
sequence of three rotations that superpose frame RF_0 to frame RF_1

Homework 1 – Question 1



joint i	α	a	d	θ
1	0	a_1	d_1	q_1
2	0	a_2	0	q_2
3	0	0	d_3	0
4	π	0	d_4	q_4

Homework 1 – Question 1



joint i	α	a	d	θ
1	0	a_1	0	q_1
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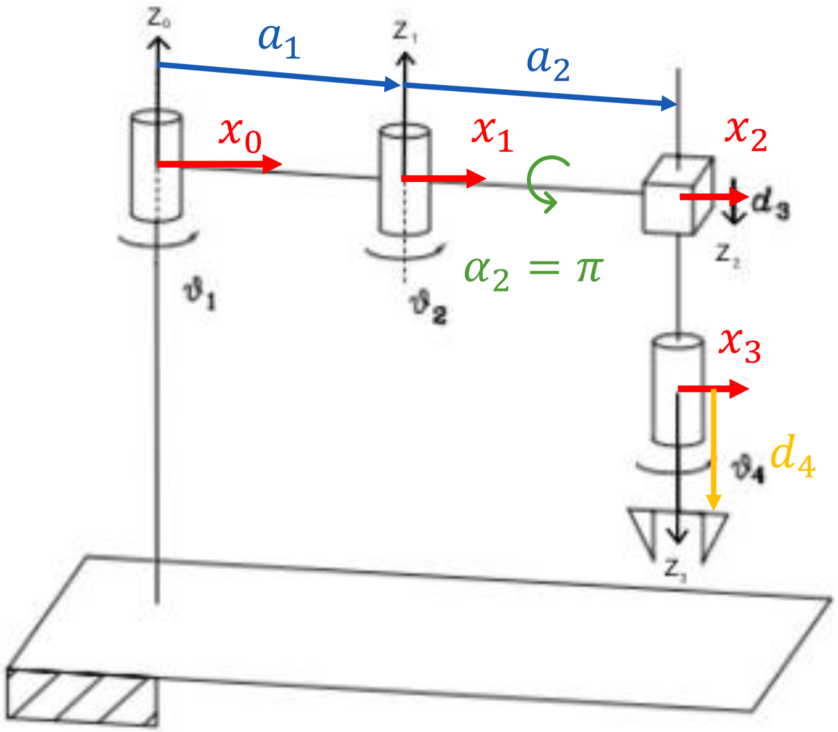
Homework 1 – Question 1



joint i	α	a	d	θ
1	0	a_1	0	q_1
2	π	a_2	0	q_2
3	0	0	d_3	0
4	0	0	d_4	q_4

Homework 1 – Question 1

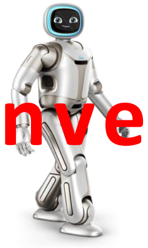
$$
{}^{0}A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2 c_2 \\ s_2 & -c_2 & 0 & a_2 s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}T_4 = {}^{0}A_1\,{}^{1}A_2\,{}^{3}A_4\,{}^{3}A_4 = \begin{bmatrix} c_{12-4} & s_{12-4} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12-4} & -c_{12-4} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Inverse of a homogeneous transformation

Prove the equations in the red frame starting from the green frame

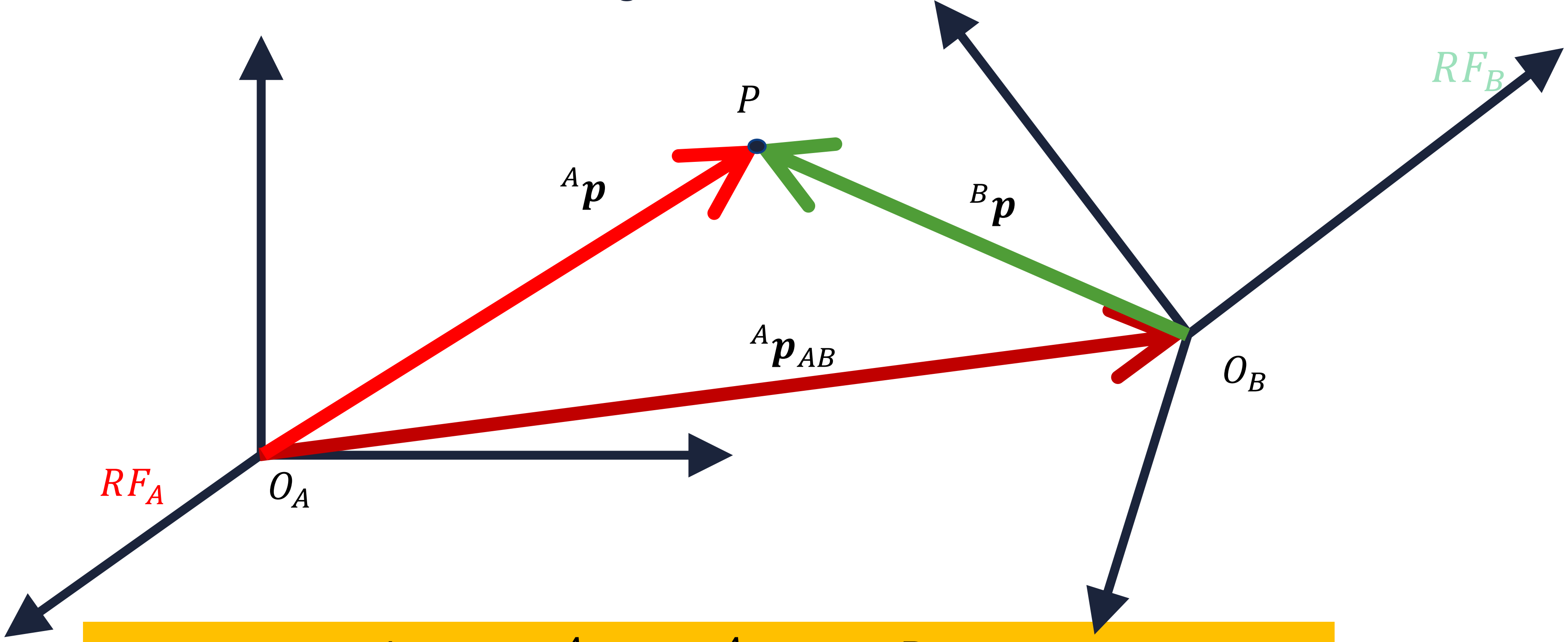
$$ {}^{B}\boldsymbol{p} = {}^{B}\boldsymbol{p}_{BA} + {}^{B}R_{A}\,{}^{A}\boldsymbol{p} = -{}^{A}R_{B}^{T}\,{}^{A}\boldsymbol{p}_{AB} + {}^{A}R_{B}^{T}\,{}^{A}\boldsymbol{p} $$

$$ \left[\begin{array}{ccc|c} & {}^{B}R_{A} & & {}^{B}\boldsymbol{p}_{BA} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^{A}R_{B}^{T} & & -{}^{A}R_{B}^{T}\,{}^{A}\boldsymbol{p}_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{B}T_{A} \qquad\qquad ({}^{A}T_{B})^{-1} $$

Basic Definitions

Homogeneous transformations



$$ {}^{A}\mathbf{p} = \begin{bmatrix} {}^{A}\mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_B & {}^{A}\mathbf{p}_{AB} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\mathbf{p} \\ 1 \end{bmatrix} = {}^{A}T_B \, {}^{A}\mathbf{p} $$

← linear relationship

vector in homogeneous coordinates

4×4 matrix of homogeneous transformation

Prove $RS(\omega)R^T = S(R\omega)$

$$\text{Let } R = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$RS(\omega)R^T = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} S(\omega) \begin{bmatrix} x & y & z \end{bmatrix}$$

$$= \begin{bmatrix} x^T S(\omega) x & x^T S(\omega) y & x^T S(\omega) z \\ y^T S(\omega) x & y^T S(\omega) y & y^T S(\omega) z \\ z^T S(\omega) x & z^T S(\omega) y & z^T S(\omega) z \end{bmatrix}$$

$$= \begin{bmatrix} x^T(\omega \times x) & x^T(\omega \times y) & x^T(\omega \times z) \\ y^T(\omega \times x) & y^T(\omega \times y) & y^T(\omega \times z) \\ z^T(\omega \times x) & z^T(\omega \times y) & z^T(\omega \times z) \end{bmatrix}$$

Prove $RS(\omega)R^T = S(R\omega)$

$$RS(\omega)R^T = \begin{bmatrix} x^T(\omega \times x) & x^T(\omega \times y) & x^T(\omega \times z) \\ y^T(\omega \times x) & y^T(\omega \times y) & y^T(\omega \times z) \\ z^T(\omega \times x) & z^T(\omega \times y) & z^T(\omega \times z) \end{bmatrix}$$

There was an error during the review class, it is now corrected

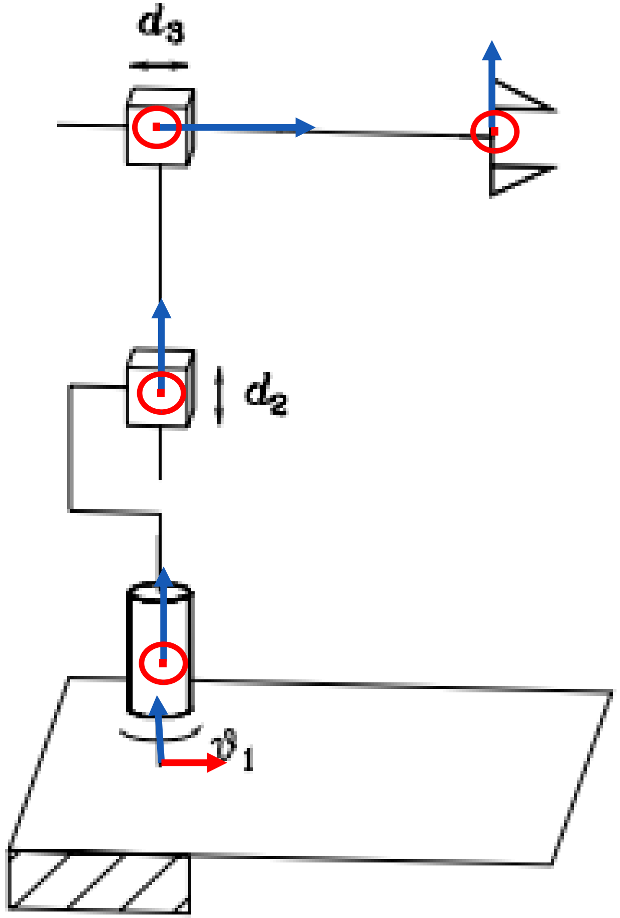
$$= \begin{bmatrix} \omega^T(x \times x) & \omega^T(y \times x) & \omega^T(z \times x) \\ \omega^T(x \times y) & \omega^T(y \times y) & \omega^T(z \times y) \\ \omega^T(x \times z) & \omega^T(y \times z) & \omega^T(z \times z) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\omega^T z & \omega^T y \\ \omega^T z & 0 & -\omega^T x \\ -\omega^T y & \omega^T x & 0 \end{bmatrix}$$

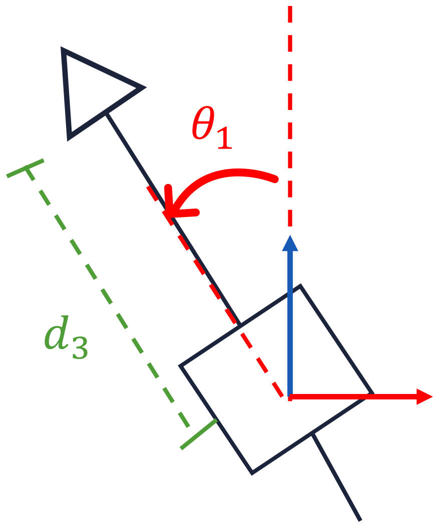
$$= S(R\omega)$$

Compute the Jacobians

Compute the Geometric Jacobian



Front View



Top View

Recall the Geometric Jacobian method:

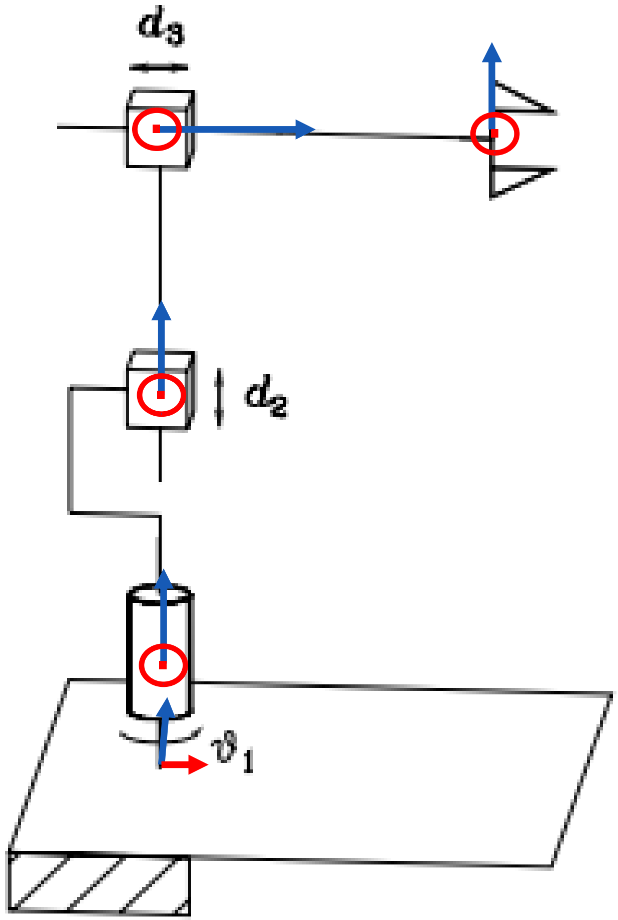
$$J(q) = \begin{bmatrix} z_0 \times (p - p_0) & z_1 & z_2 \\ z_0 & 0 & 0 \end{bmatrix}$$

Where:

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad p = \begin{bmatrix} -d_3 s_1 \\ d_3 c_1 \\ d_2 \end{bmatrix}$$

Compute the Jacobian

Compute the Geometric Jacobian



The z-axis of each joint can be derived as,

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad z_2 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

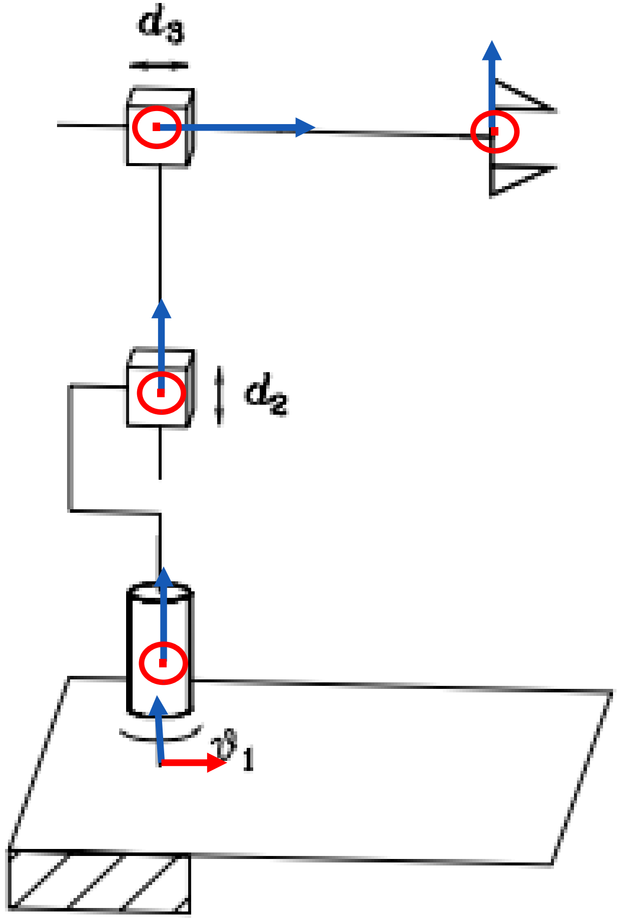
Hence, the geometric Jacobian is given as,

$$J = \begin{bmatrix} -d_3 c_1 & 0 & -s_1 \\ -d_3 s_1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Compute the Jacobian

There was an error during the review class, it is now corrected

Compute the Analytical Jacobian



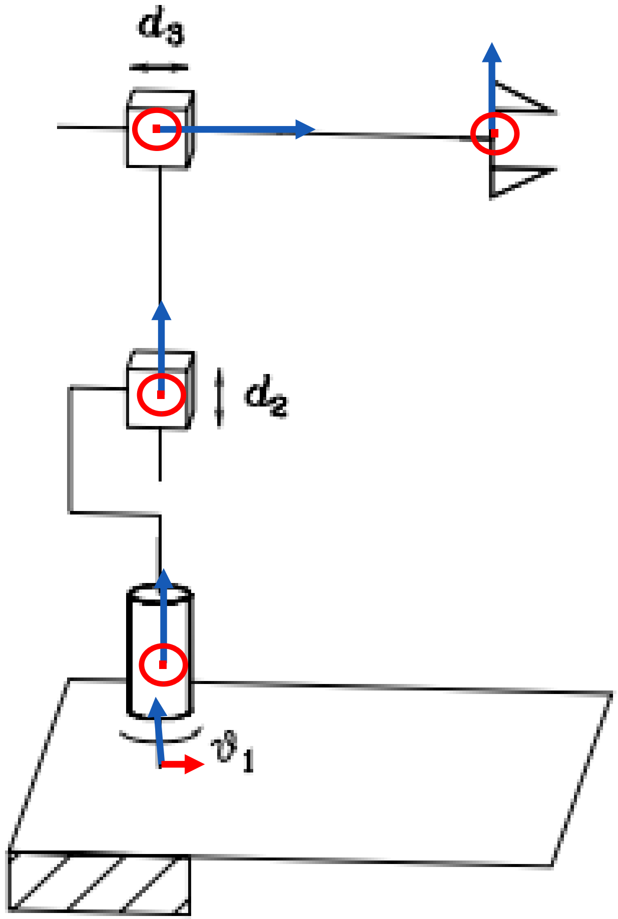
$$p_x = -d_3 s_1 \qquad \dot{p}_x = -d_3 c_1 \dot{\theta}_1 - \dot{d}_3 s_1$$
$$p_y = d_3 c_1 \qquad \dot{p}_y = d_3 s_1 \dot{\theta}_1 + \dot{d}_3 c_1$$
$$p_z = d_2 \qquad \dot{p}_z = \dot{d}_2$$

$$\phi = \theta_1 \qquad \dot{\phi} = \dot{\theta}_1$$

The analytical Jacobian is then,

$$J = \begin{bmatrix} -d_3 c_1 & 0 & -s_1 \\ -d_3 s_1 & 0 & c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Computing the Singularity



First consider the linear velocity component of the Jacobian,

$$J = \begin{bmatrix} -d_3c_1 & 0 & -s_1 \\ -d_3s_1 & 0 & c_1 \\ 0 & 1 & 0 \end{bmatrix}$$

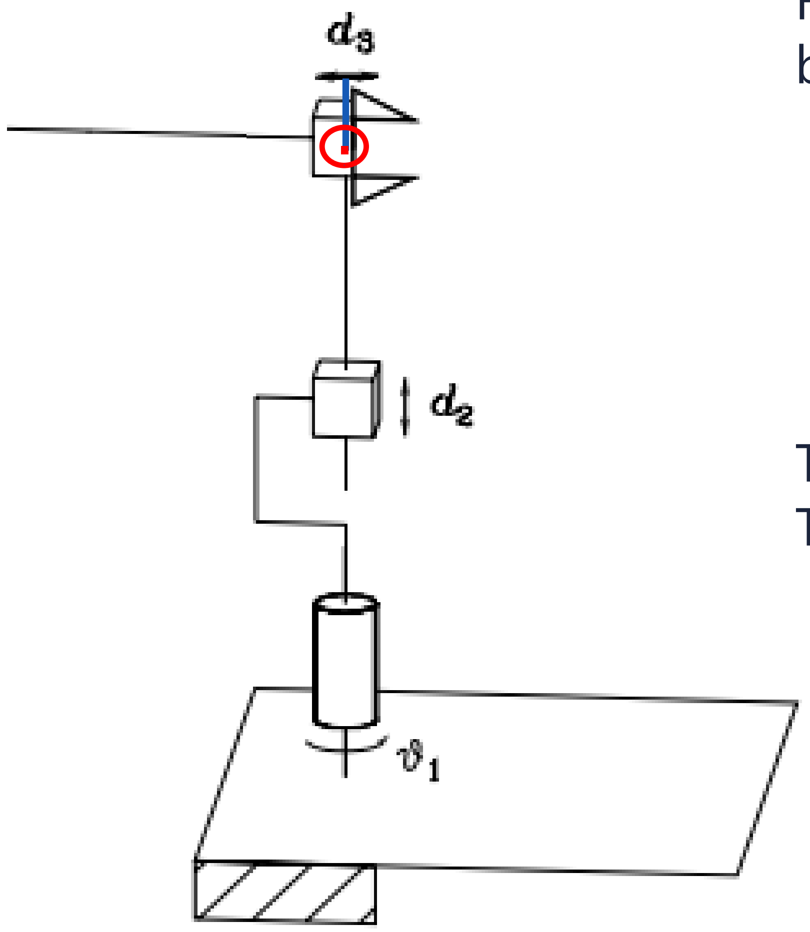
Singularity occurs when the robot loses rank, and its determinant equals to zero.

$$\det(J_P) = -d_3c_1 \begin{vmatrix} 0 & c_1 \\ 1 & 0 \end{vmatrix} - 0 - s_1 \begin{vmatrix} -d_3s_1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= d_3({c_1}^2 + {s_1}^2) = d_3$$

Hence, the singularity occurs when $d_3 = 0$. Thus, singularity occurs when the end effector is located along Joint 1 axis

Singularity Analysis



Here is a visual representation. When $d_3 = 0$, the linear Jacobian becomes

$$J = \begin{bmatrix} 0 & 0 & -s_1 \\ 0 & 0 & c_1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -s_1 \\ 0 & c_1 \\ 1 & 0 \end{bmatrix}$$

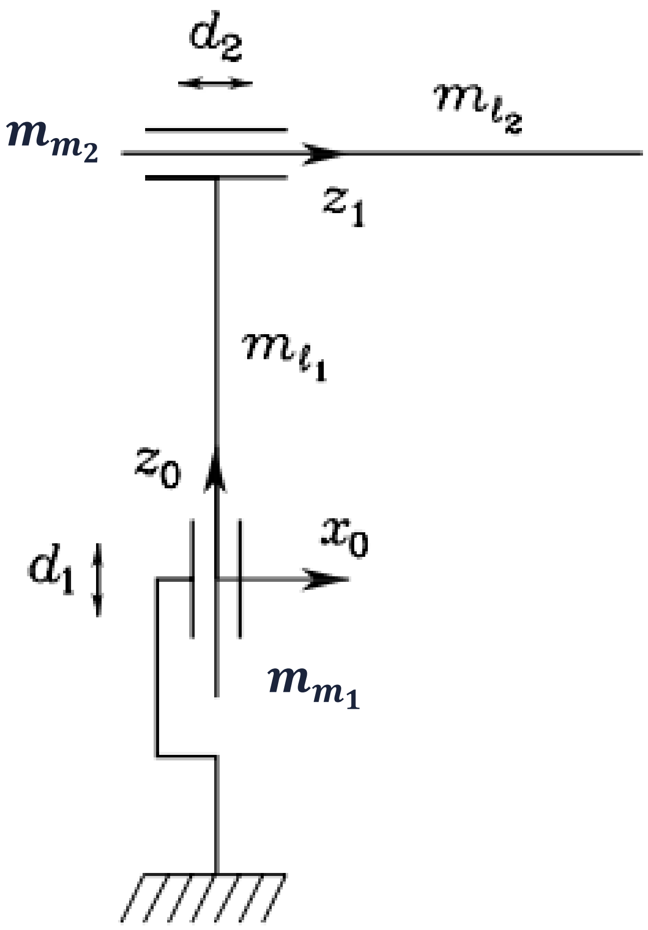
The matrix has clearly lost rank, and is no longer a square matrix. The null space, range and rank are as follows,

$$N(J) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad R(J) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\rho(J) = 3 - \dim\big(N(J)\big) = 2$$

$$\rho(J) < 3$$

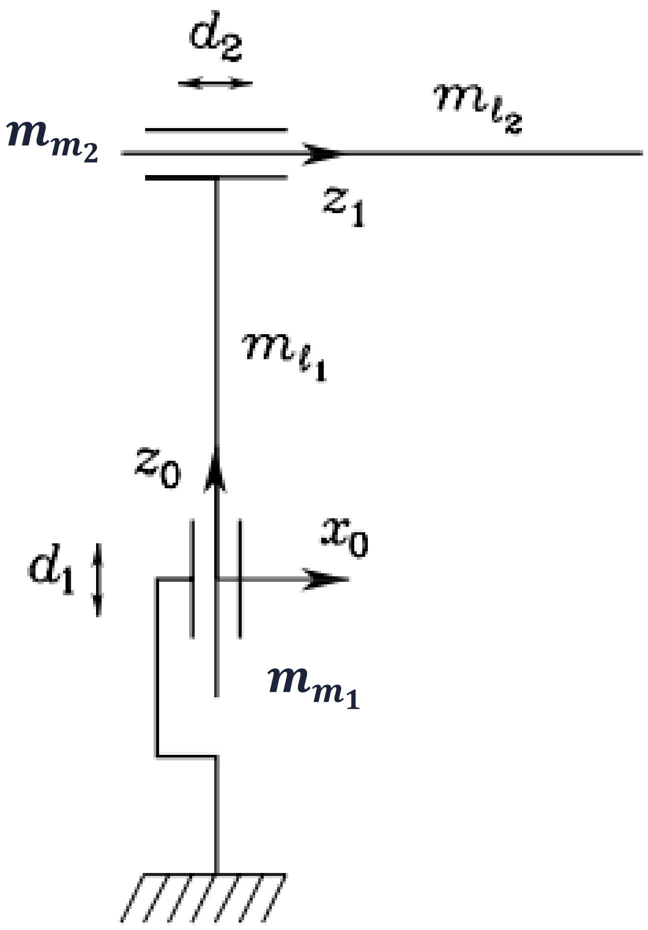
Find the forces applied to joint 1



Let m_{l_1},m_{l_2} be the masses of the two links, and m_{m_1}, m_{m_1} the masses of the rotors of the two joint motors. Also let I_{m_1},I_{m_2} be the moments of inertia with respect to the axes of the two rotors. The motors are located on the joint axes with centers of mass located at the origins of the respective frames.

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

Find the forces applied to joint 1



First, compute the kinetic energy.

$$T_1 = \frac{1}{2} m_{l1} \dot{d}_1^{\ 2} + \frac{1}{2} n_{r1}^2 I_{m1} \dot{d}_1^{\ 2}$$

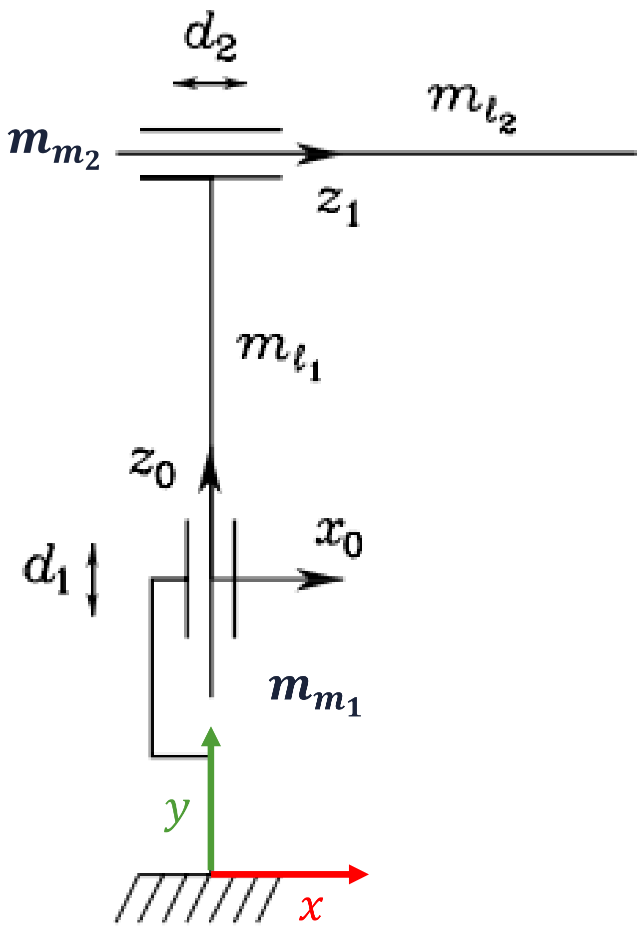
$$T_2 = \frac{1}{2} m_{l2} \dot{d}_2^{\ 2} + \frac{1}{2} n_{r2}^2 I_{m2} \dot{d}_2^{\ 2} + \frac{1}{2} (m_{l2} + m_{m2}) \dot{d}_1^{\ 2}$$

Apply the following formula

$$T = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{1}{2} [\dot{d}_1 \quad \dot{d}_2] M(q) \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix}$$

Find the forces applied to joint 1



The robot does not have a revolute joint, hence there is no I_{xy} *or* I_{yx} component. The inertia matrix is then given as,

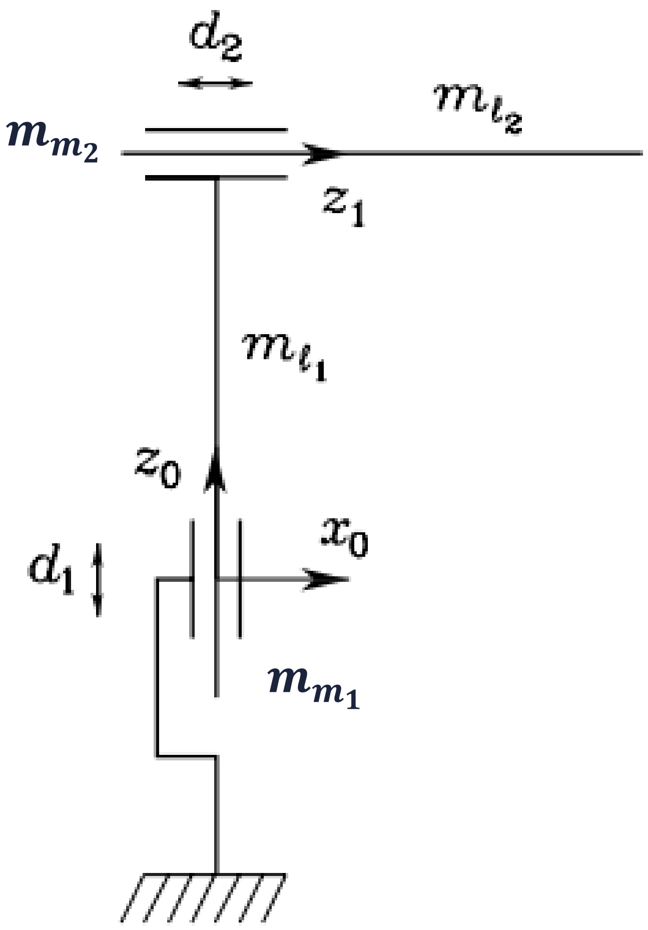
$$M = \begin{bmatrix} m_{l1} + m_{m2} + n_{r1}^2 I_{m1} + m_{l2} & 0 \\ 0 & m_{l2} + n_{r2}^2 I_{m2} \end{bmatrix}$$

As the inertia matrix is constant (there is no d_1 *or* d_2 terms), there is no Coriolis or centrifugal forces. *If there were, the computation of Christoffel symbols are as follows…*

$$c_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

$$c(q, \dot{q}) = \begin{pmatrix} c_1(q, \dot{q}) \\ c_2(q, \dot{q}) \end{pmatrix}$$

Find the forces applied to joint 1



The gravitational terms can be denoted as,

Since $g_0 = [0 \quad 0 \quad -g]^T$,

$$g_1 = (m_{l1} + m_{m2} + m_{l2})g \qquad g_2 = 0$$

In the absence of friction and tip contact forces, equation of motion for Joint 1 is ,

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$$(m_{l1} + m_{m2} + k_{r1}^2 I_{m1} + m_{l2})\ddot{d}_1 + (m_{l1} + m_{m2} + m_{l2})g = u_1$$

You can try to calculate u_2!