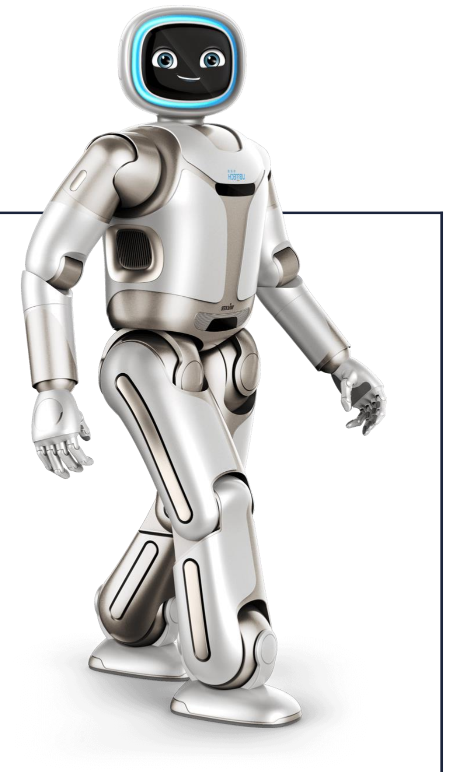
Advanced Robotics

ENGG5402 Spring 2023

Fei Chen

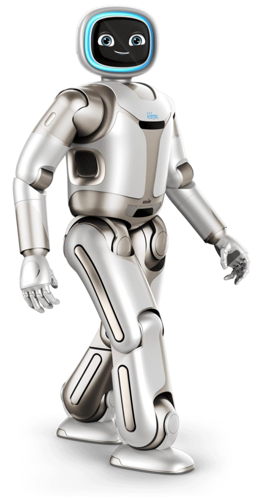


Topics:

- Position Regulation (with an introduction to stability)

Readings:

- Siciliano: Sec. 8, 8.5



Equilibrium

Equilibrium states of a robot

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

$$\Rightarrow \quad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1,x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} u$$

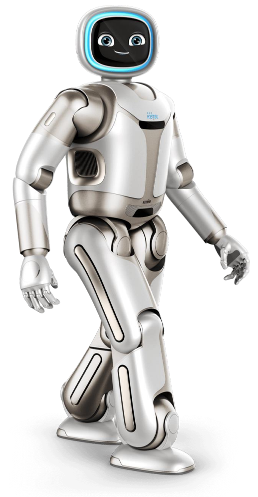
$$= f(x) + G(x_1)u$$

x_e unforced equilibrium $(u = 0)$ $\Rightarrow$ $f(x_e) = 0$ $\Rightarrow$ $\begin{cases} x_{e2} = 0 \\ g(x_{e1}) = 0 \end{cases}$

x_e forced equilibrium $(u = u(x))$ $\Rightarrow$ $f(x_e) + G(x_{e1})u(x_e) = 0$ $\Rightarrow$ $\begin{cases} x_{e2} = 0 \\ u(x_e) = g(x_{e1}) \end{cases}$

all equilibrium states of mechanical systems have zero velocity!

joint torques must balance gravity at the equilibrium!



Stability of Dynamical Systems

$$\dot{x} = f(x)$$

e.g., a closed-loop system (under feedback control)

x_e equilibrium: $f(x_e) = 0$
(sometimes we consider as equilibrium state $x_e = 0$, e.g., when using errors as variables)

stability of x_e

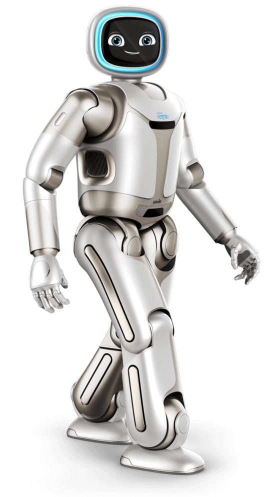
$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0: \|x(t_0) - x_e\| < \delta_\varepsilon \Rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq t_0$$

asymptotic stability of x_e — stability +

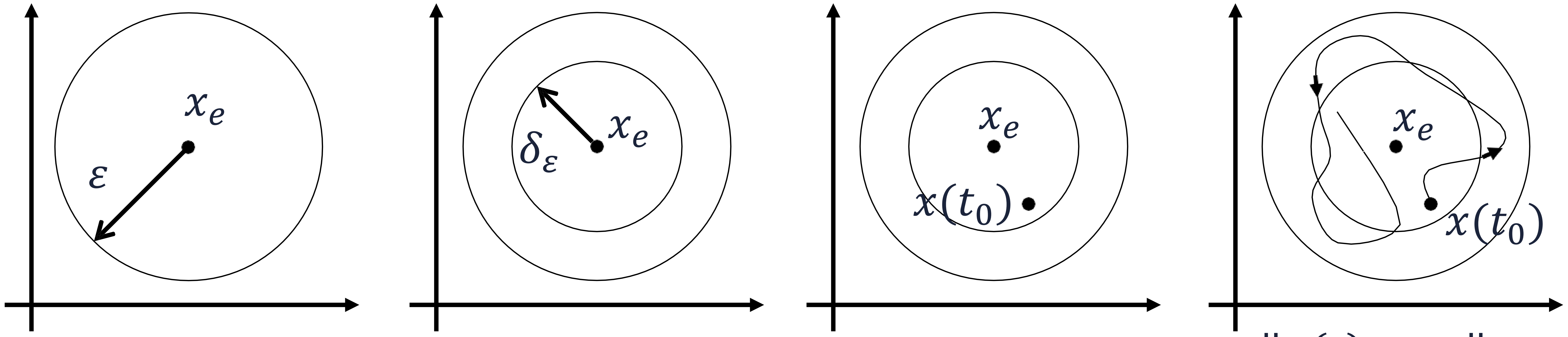
$$\exists \delta > 0: \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \to 0, \text{ for } t \to \infty$$

asymptotic stability may become global ($\forall \delta > 0$, finite)

note: these are definitions of stability “in the sense of Lyapunov”



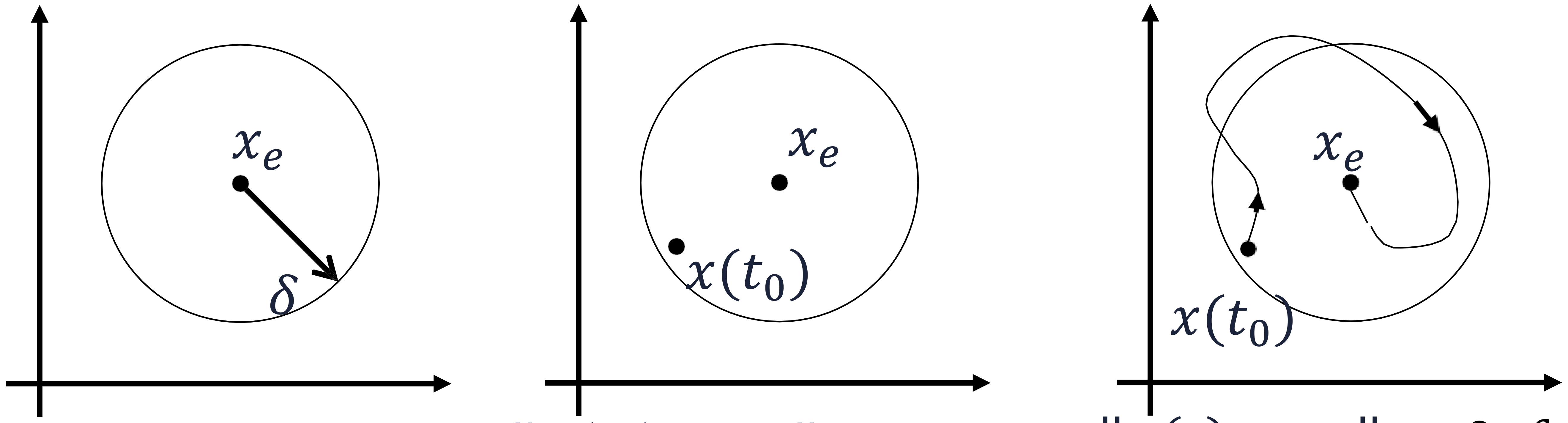
Stability vs. Asymptotic Stability



$\forall \varepsilon > 0$ $\quad \exists \delta_\varepsilon > 0$ $\quad \|x(t_0) - x_e\| < \delta_\varepsilon \Rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq t_0$

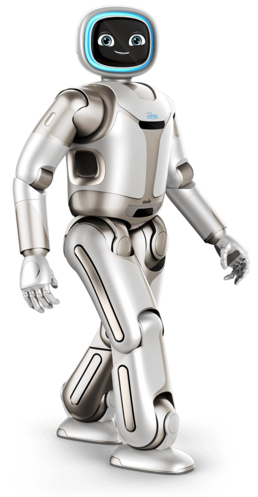
equilibrium state x_e is stable

\+



$\exists \delta > 0$ $\quad \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \to 0$, for $t \to \infty$

equilibrium state x_e is asymptotically stable



Stability of Dynamical Systems

exponential rate λ

exponential stability of x_e

$$\exists \delta, c, \lambda > 0: \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \leq c e^{-\lambda(t-t_0)} \|x(t_0) - x_e\|$$

- allows to estimate the time needed to "approximately" converge: for $c = 1$, in $t - t_0 = 3 \times$ the time constant $\tau = 1/\lambda$, the initial error is reduced to 5%
- typically, this is a local property only (within some maximum finite radius δ)

$\Rightarrow$ such "domain of attraction" is hard to be estimated accurately

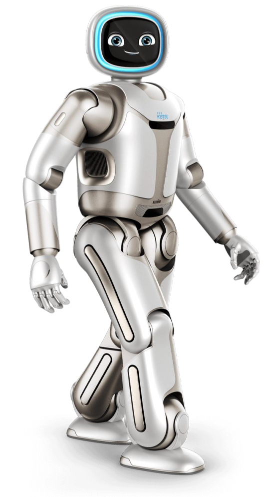
"practical" stability of a set S

$$\exists T(x(t_0), S) \in \mathbb{R}: x(t) \in S, \forall t \geq t_0 + T(x(t_0), S)$$

a finite time

also known as u.u.b. stability

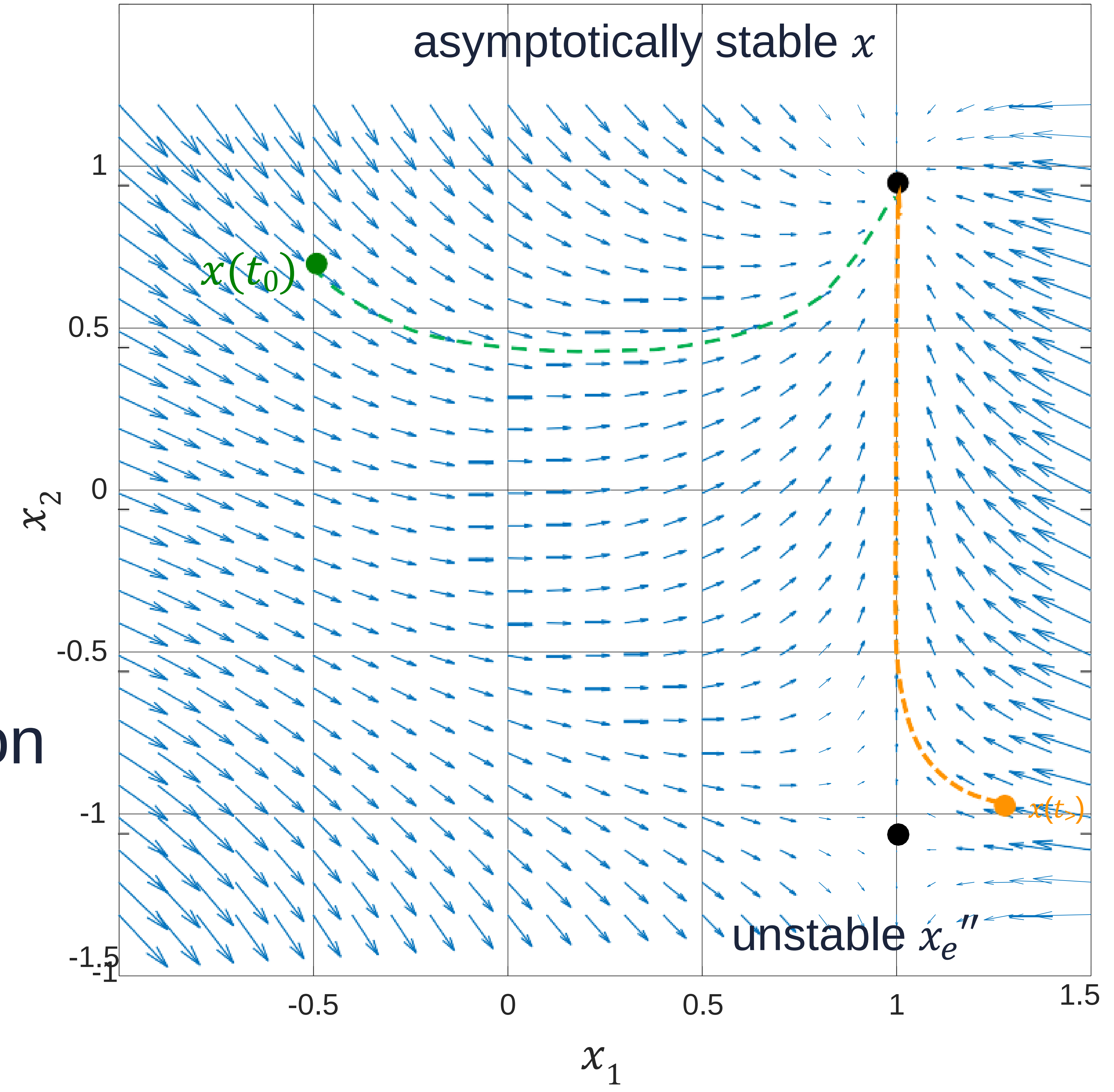
$\Rightarrow$ trajectories $x(t)$ are "ultimately uniformly bounded" (use in robust control)



Analysis and Criteria

a nonlinear system $\dot{x} = f(x)$ in $\mathbb{R}^2$ two equilibria $f(x_e) = 0$

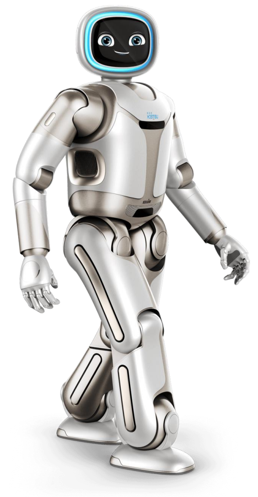
$$\begin{cases} \dot{x}_1 = 1 - x_1^3 \\ \dot{x}_2 = x_1 - x_2^2 \end{cases} \Rightarrow \quad x_e' = (1,1), \quad x_e'' = (1,-1)$$



to assess (asymptotic) stability [or not] of equilibria, do we need to compute all system trajectories, starting from all possible initial states $x(t_0)$?

⬇

rather, we may be able to just look at the time evolution of a scalar function V, evaluated analytically along the state trajectories of the system (even in $\mathbb{R}^n$!)



Lyapunov Theory

Lyapunov candidate

$V(x): \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$V(x_e) = 0, V(x) > 0, \forall x \neq x_e$

positive definite function

typically quadratic (e.g., $\frac{1}{2}(x - x_e)^T P(x - x_e)$ with level surfaces = ellipsoids) may also be a local candidate only $\forall x \neq x_e: \|x - x_e\| < \delta$

sufficient condition of stability

$\exists V$ candidate: $\dot{V}(x) \leq 0$,along the trajectories of $\dot{x} = f(x)$

negative semi-definite function

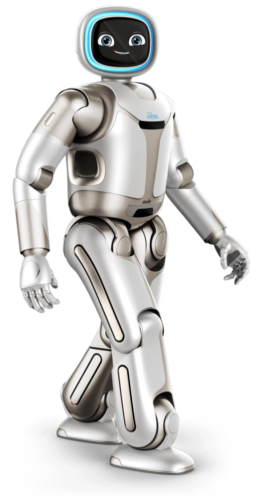
sufficient condition of asymptotic stability

$\exists V$ candidate: $\dot{V}(x) < 0,$ along the trajectories of $\dot{x} = f(x)$

negative definite function

sufficient condition of instability

$\exists V$ candidate:$\dot{V}(x) > 0$,along the trajectories of $\dot{x} = f(x)$



Lyapunov Theory

sufficient condition of u.u.b. stability of a set S

$\exists V$ candidate: i) S is a level set of V for a given c_0

$$S = S(c_0) = \{x \in \mathbb{R}^n : V(x) \le c_0\}$$

ii) $\dot{V}(x) < 0$ along trajectories of $\dot{x} = f(x), \ x \notin S$

LaSalle Theorem

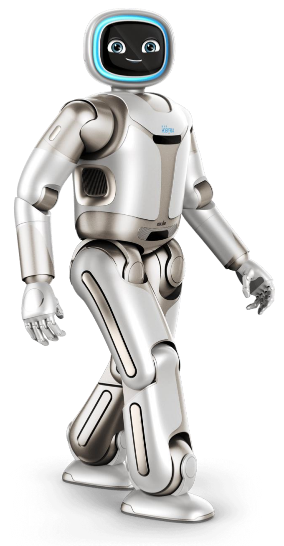
if $\exists V$ candidate: $\dot{V}(x) \le 0$ along the trajectories of $\dot{x} = f(x)$

then system trajectories asymptotically converge to the largest invariant set $\mathcal{M} \subseteq S = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$

$\mathcal{M}$ is invariant if $x(t_0) \in \mathcal{M} \Longrightarrow x(t) \in \mathcal{M}, \ \forall t \ge t_0$

Corollary

$\mathcal{M} \equiv \{x_e\}$ ⇒ asymptotic stability



Lyapunov Analysis

a mass m at the end of an unforced (passive) pendulum of length l

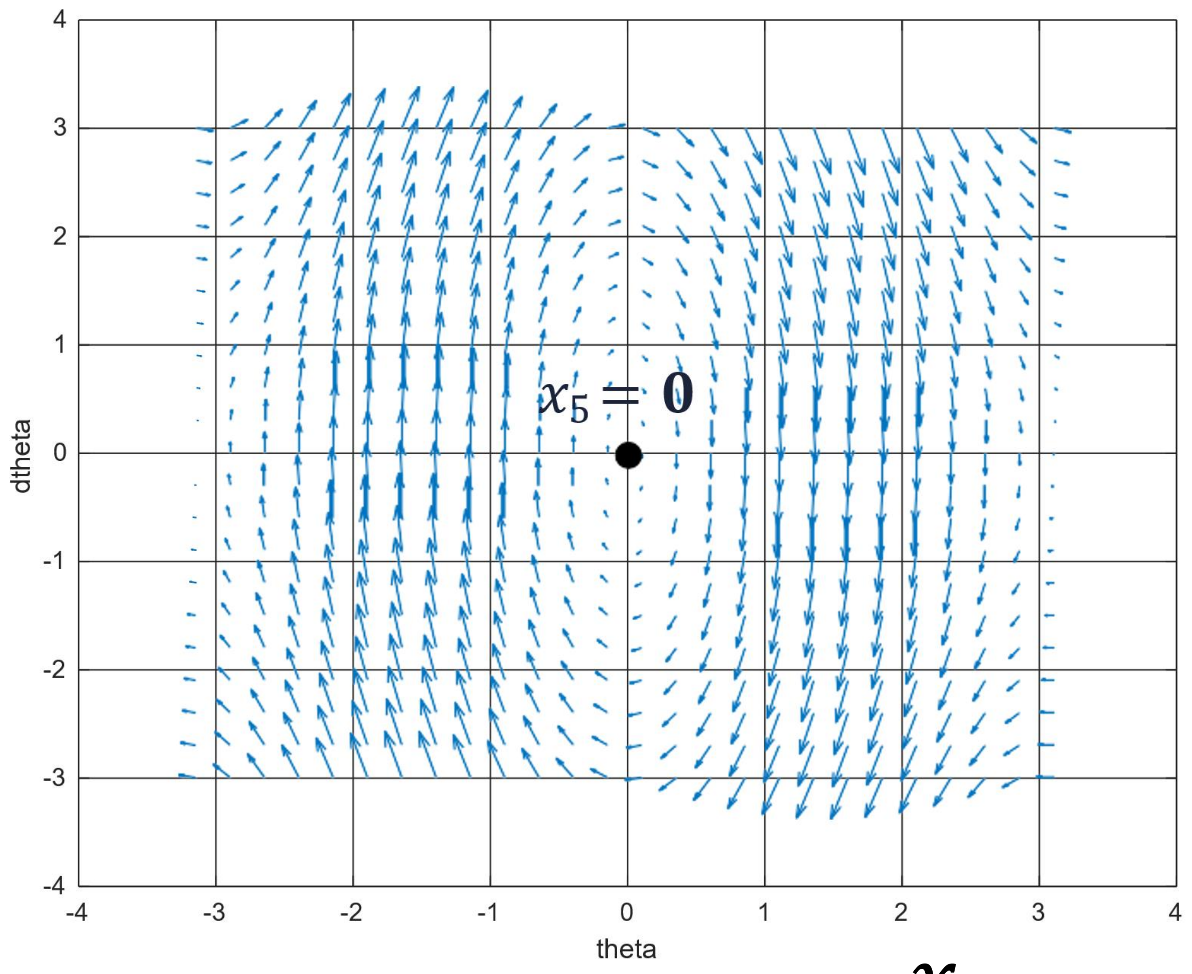
$$ml^2\ddot{\theta} + b\dot{\theta} + mlg_0 \sin\theta = 0$$

lower equilibrium at $\theta = 0$

$$\Rightarrow \quad x = (x_1, x_2) = (\theta, \dot{\theta}) \in \mathbb{R}^2 \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\frac{g_0}{l}\right)\sin x_1 - \left(\frac{b}{ml^2}\right) x_2 \end{cases}$$

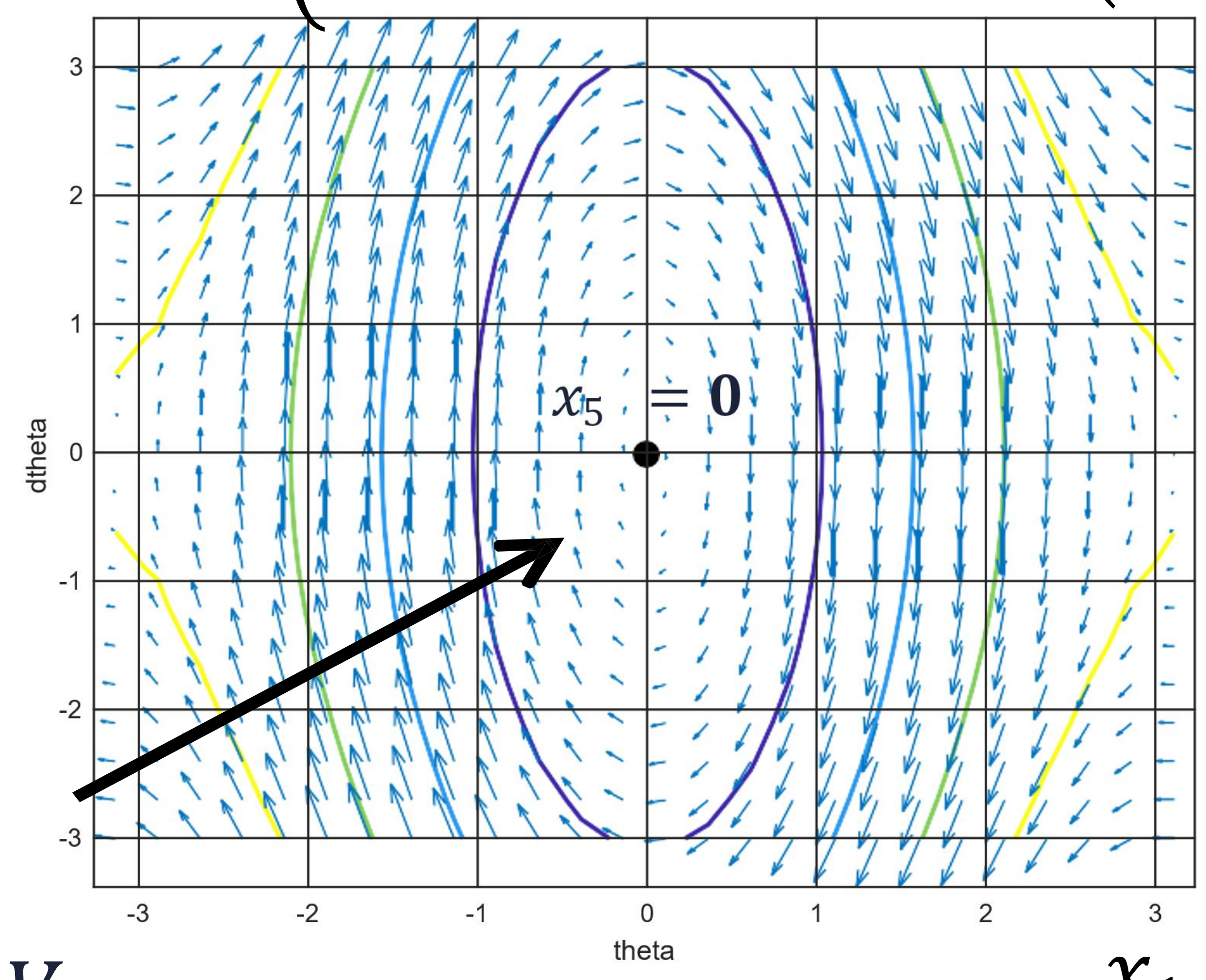
x_2

phase plane



x_1

x_2



level sets of V

x_1

$$V = E = \frac{1}{2}ml^2\dot{\theta}^2 + mlg_0(1 - \cos\theta) \geq 0$$

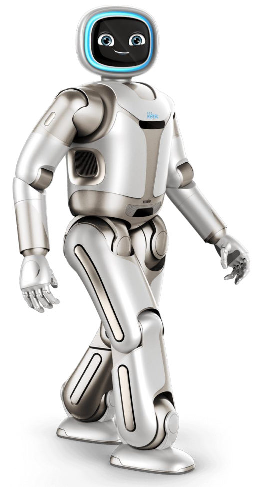
$$\dot{V} = \dot{\theta}\left(ml^2\ddot{\theta} + m\,l g_0 \sin\theta\right) = -b\dot{\theta}^2 \leq 0$$

$$V = 0 \Leftrightarrow x_e = (\theta_e, \dot{\theta}_e) = (0,0)$$

$\Rightarrow$ stability of equilibrium $x_e = 0$ (… at least!)

$\Rightarrow$ use LaSalle $\quad \dot{V} = 0 \Leftrightarrow \dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\left(\frac{g_0}{l}\right)\sin\theta \neq 0 \quad$ unless $\theta = \theta_e = 0(\text{ or } \pi!)$

$\Rightarrow$ local asymptotic stability



Stability of Dynamical Systems

- previous results are also valid for periodic time-varying systems

$$\dot{x} = f(x,t) = f(x,t+T_p) \Rightarrow V(x,t) = V(x,t+T_p)$$

- for general time-varying systems (e.g., in robot trajectory tracking control)

$$\dot{x} = f(x,t)$$

Barbalat Lemma

i) a function $V(x,t)$ is lower bounded

ii) $\dot{V}(x,t) \leq 0$

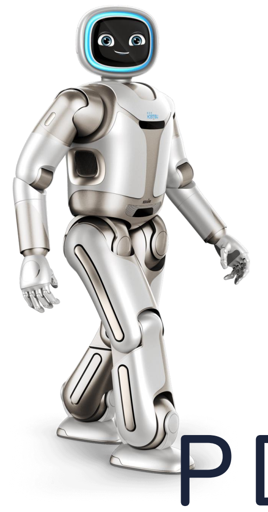
then $\Rightarrow \exists \lim_{t\to\infty} V(x,t)$ (but this does not imply that $\lim \lim_{t\to\infty} \dot{V}(x,t) = 0$)

if in addition iii) $\ddot{V}(x,t)$is bounded

then $\Rightarrow \lim \lim_{t\to\infty} \dot{V}(x,t) = 0$

Corollary

if a Lyapunov candidate $V(x,t)$ satisfies Barbalat Lemma along the trajectories of $\dot{x} = f(x,t)$,then the conclusions of LaSalle Theorem hold



Regulation PD Control

PD control (proportional + derivative action on the error)

robot $M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$

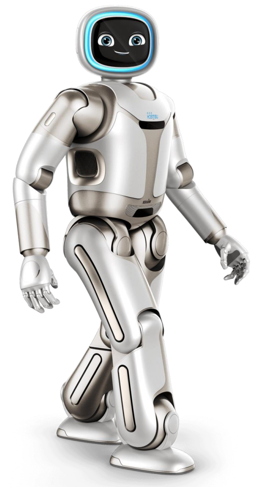
goal: asymptotic stabilization (= regulation) of the closed-loop equilibrium state

$$q = q_d, \dot{q} = 0$$

possibly obtained from kinematic inversion: $q_d = f^{-1}(r_d)$

control law $u = K_P(q_d - q) - K_D\dot{q}$

$K_P > 0, K_D > 0$ (positive definite), symmetric



Regulation PD Control

Asymptotic stability with PD control

Theorem 1

In the absence of gravity ($g(q) = 0$), the robot state $(q_d, 0)$ under the given PD joint control law is globally asymptotically stable

Proof

let $e = q_d - q$ (q_d constant)

Lyapunov candidate

$$V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e^T K_P e \geq 0$$

$$V = 0 \Leftrightarrow e = \dot{e} = 0$$

$$\dot{V} = \dot{q}^T M\ddot{q} + \frac{1}{2}\dot{q}^T\dot{M}\dot{q} - e^T K_P\dot{q} = \dot{q}^T\left(u \underbrace{- S\dot{q} + \frac{1}{2}\dot{M}\dot{q}}\right) - e^T K_P\dot{q}$$

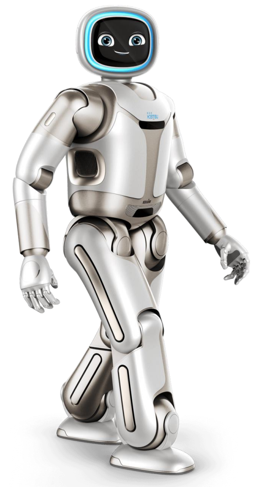
$= 0$, due to energy conservation (*check appendix slides, also check equation 8.49 in Bruno's book*)

$$= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D\dot{q} - \cancel{e^T K_P\dot{q}} = -\dot{q}^T K_D\dot{q} \leq 0$$

($K_D > 0$, symmetric)

up to here, we proved stability only but $\dot{V} = 0 \Leftrightarrow \dot{q} = 0$

continues ...



Regulation PD Control

Asymptotic stability with PD control

$\dot{V} = 0 \Leftrightarrow \dot{q} = 0$ —LaSalle→ system trajectories converge to the largest invariant set of states $\mathcal{M}$ where $\dot{q} \equiv 0$ that is $\dot{q} = \ddot{q} = 0$)

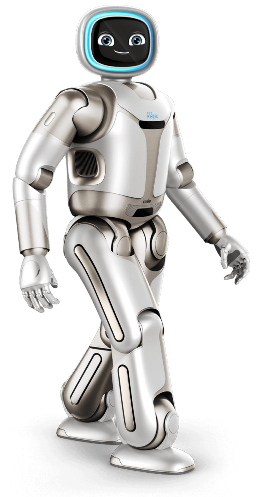
$\dot{q} = 0$ → $\underbrace{M(q)\ddot{q} = K_p e}_{\text{closed-loop dynamics}}$ → $\ddot{q} = \underbrace{M^{-1}(q)}_{\text{invertible}} KPe$

$$\dot{q} = 0, \ddot{q} = 0 \Leftrightarrow e = 0$$

→ the only invariant state in $\dot{V} = 0$ is given by $q = q_d, \dot{q} = 0$

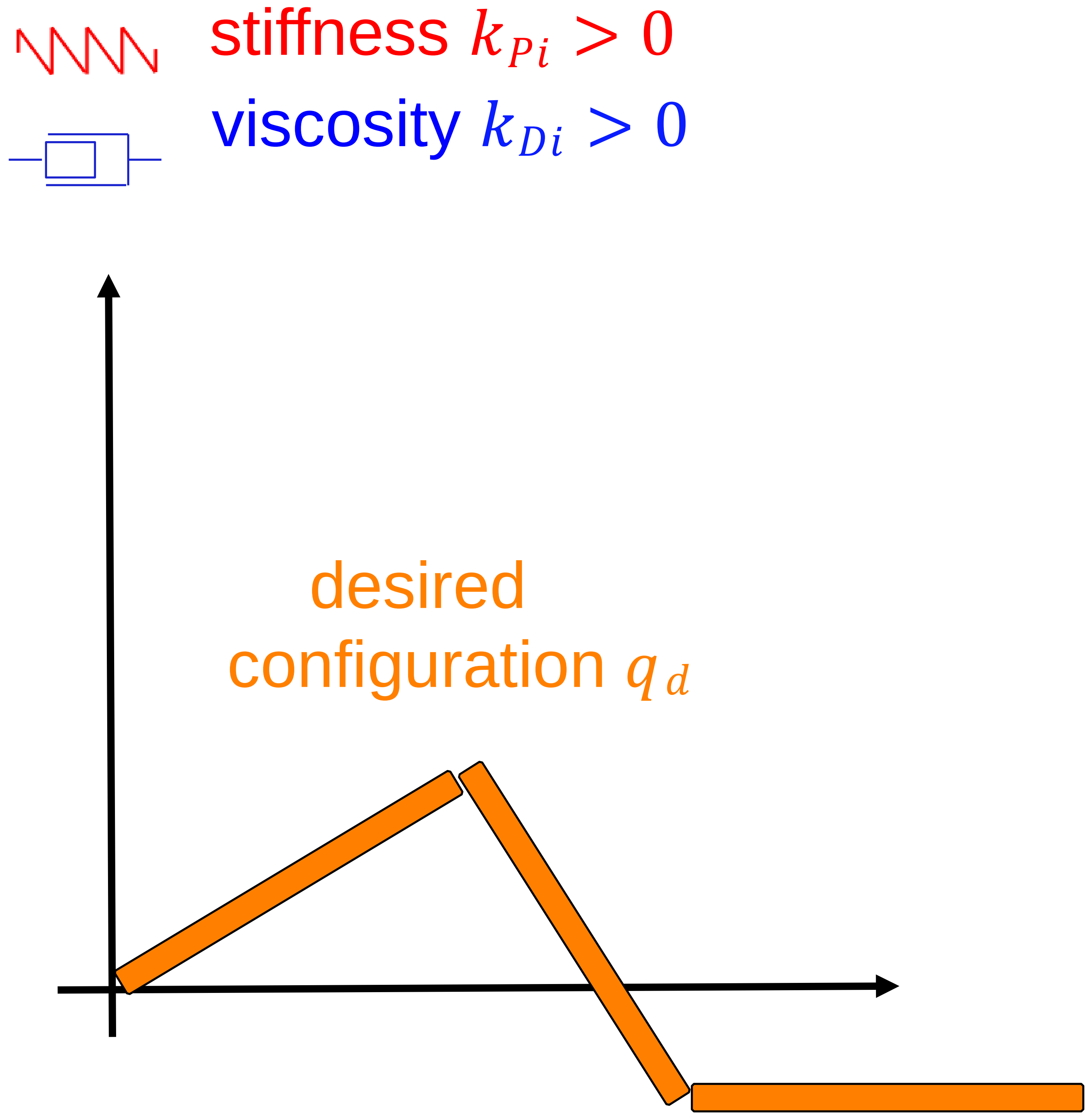
note: typically, $K_P = diag\{k_{Di}\}, K_D = diag\{k_{Di}\}$,

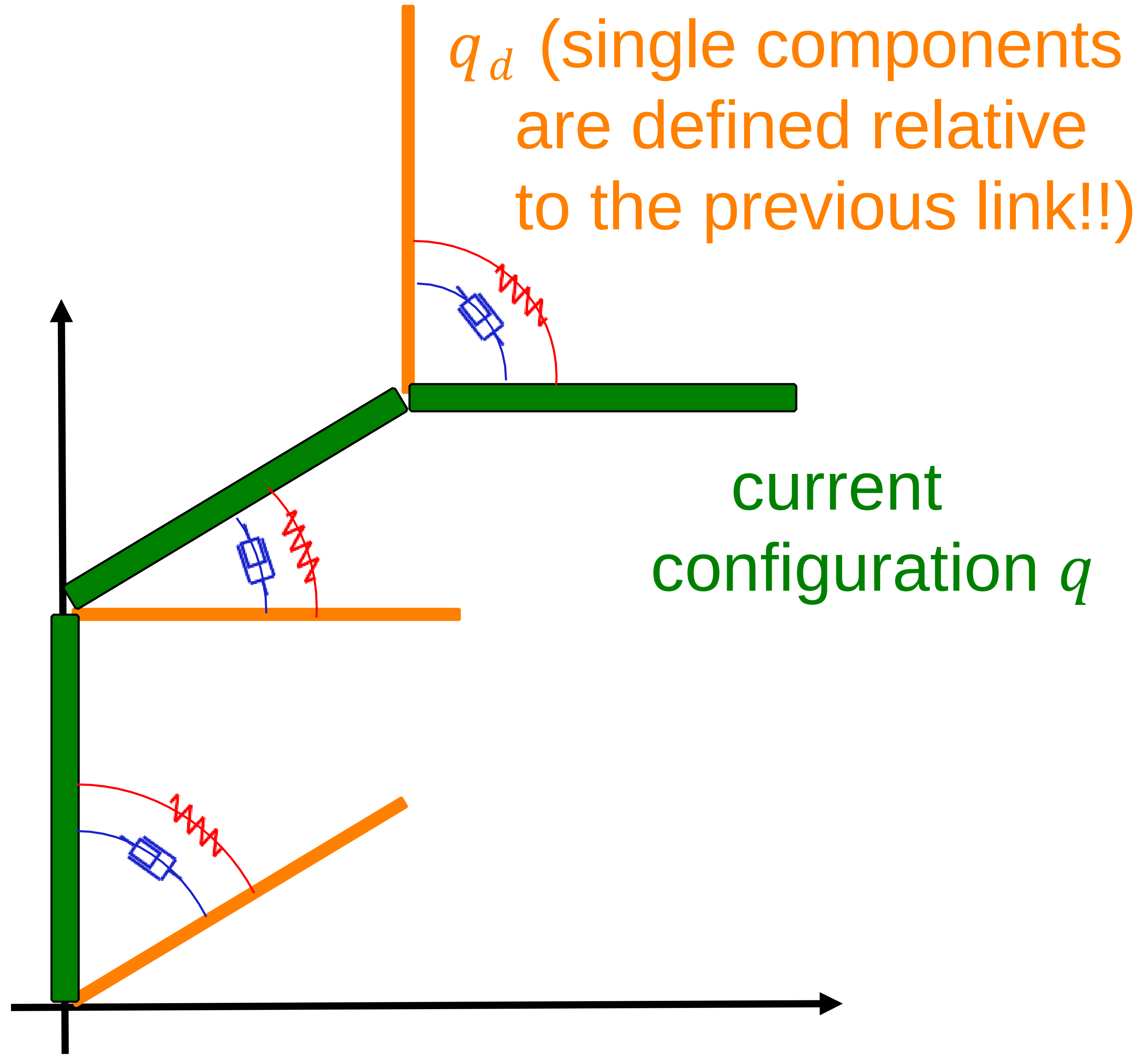
→ decentralized linear control (local to each joint)

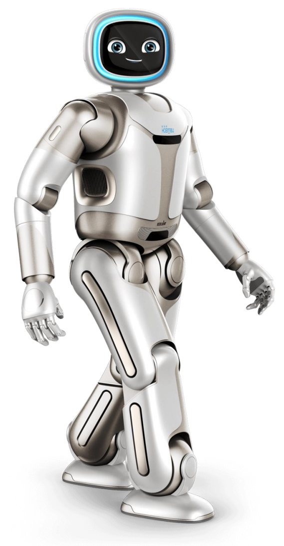


Mechanical Interpretation

- for diagonal positive definite gain matrices K_P and K_D (thus, with positive diagonal elements), such values correspond to stiffness of "virtual" springs and viscosity of "virtual" dampers placed at the joints



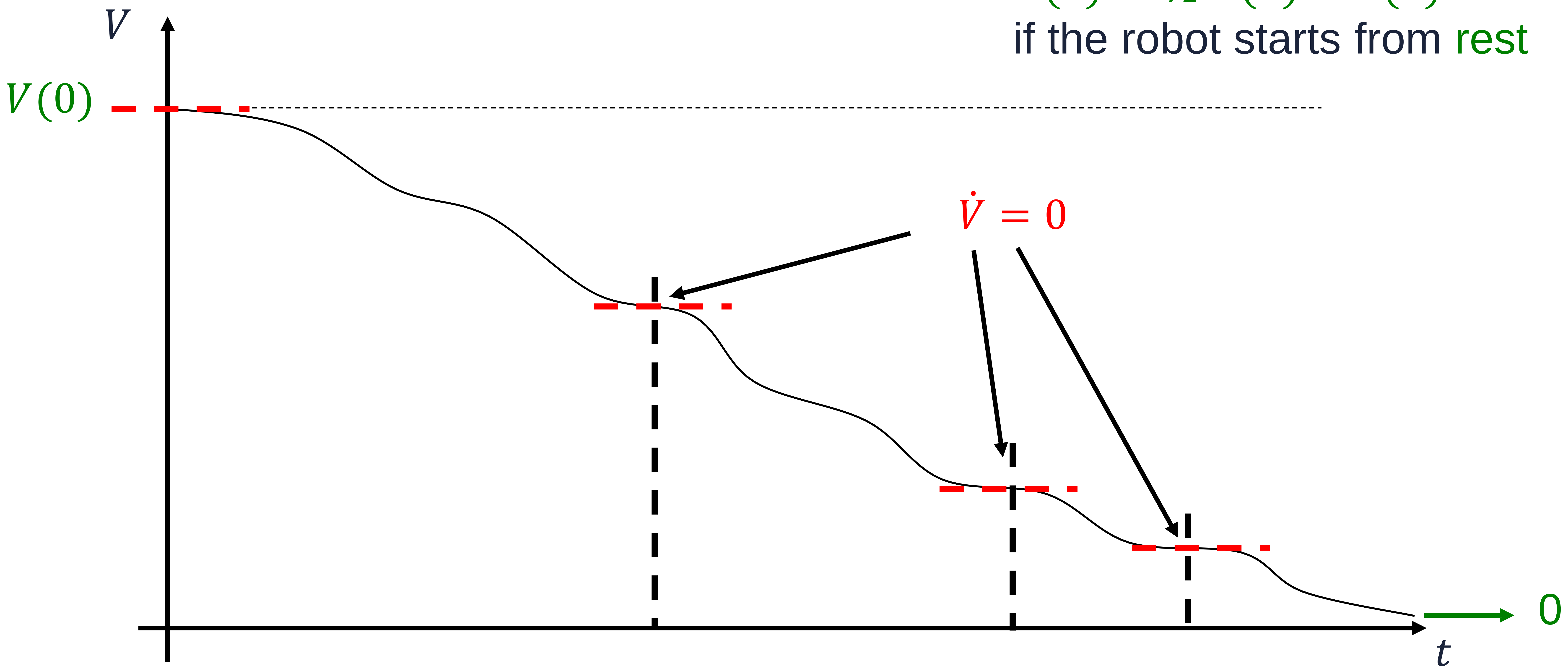




Plot of the Lyapunov Function V

- time evolution of the Lyapunov candidate

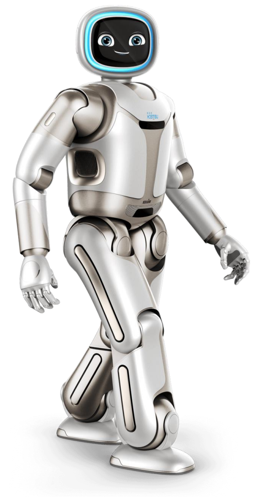
$V(0) = ½e^T(0)KPe(0)$
if the robot starts from rest



(isolated) instants of global “motion inversion” ($\dot{q} = 0, but\ \ddot{q} \neq 0!$ (

Comments on PD Control

- choice of control gains affects robot evolution during transients and practical settling times
  - hard to define values that are "optimal" in the whole workspace
  - "full" K_P and K_D gain matrices allow to assign desired eigenvalues to the linear approximation of the robot dynamics around the final desired state $(q_d, 0)$
- when (joint) viscous friction is present, the derivative term in the control law is *not strictly necessary*
  - $-FV\dot{q}$ in the robot model acts similarly to $-KD\dot{q}$ in the control law, but the latter can be modulated at will
- in the absence of tachometers, the actual realization of the derivative term in the feedback law requires some processing of joint position data measured by digital encoders (or analog resolvers/potentiometers)



Inclusion of Gravity

- in the presence of gravity, the same previous arguments (and proof) show that the control law

$$u = K_P(q_d - q) - K_D\dot{q} + g(q) \qquad K_P > 0, K_D > 0$$

will make the equilibrium state $(q_d, 0)$ globally asymptotically stable (nonlinear cancellation of gravity)

- if gravity is not cancelled or only approximately cancelled

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q) \qquad \hat{g}(q) \neq g(q)$$

it is $q \to q* \neq q_d, \dot{q} \to 0$, with steady-state position error

  - q^* is not unique in general, except when K_P is chosen large enough
  - explanation in terms of linear systems: there is no integral action before the point of access of the constant “disturbance” acting on the system

Inclusion of Gravity

PD control + constant gravity compensation

since $g(q)$ contains only trigonometric and/or linear terms in q, the following structural property holds

finite

$$\exists \alpha > 0 : \left\| \frac{\partial^2 U}{\partial q^2} \right\| = \left\| \frac{\partial g}{\partial q} \right\| \le \alpha, \forall q$$

consequence ➡ $\|g(q) - g(q_d)\| \le \alpha \|q - q_d\|$

note: Induced norm of a matrix

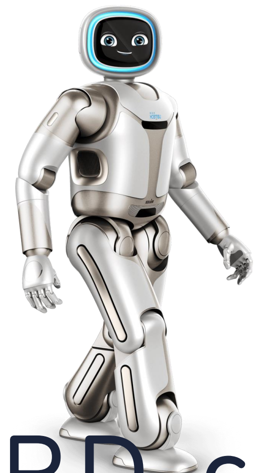
$$\|A\| = \sqrt{\lambda(A^T A)_{\max}} \triangleq A_M \ge A_m \triangleq \sqrt{\lambda_{min}(A^T A)}$$

LINEAR CONTROL law

$$u = K_P(q_d - q) - K_D \dot{q} + g(q_d)$$

$K_P, K_D > 0$ symmetric

linear feedback + constant feedforward



Inclusion of Gravity

PD control + constant gravity compensation (stability analysis)

Theorem 2

If $K_{P,m} > \alpha$, the state $(q_d, 0)$ of the robot under joint-space PD control + constant gravity compensation at $q_{|}$ is globally asymptotically stable

Proof

1. $(q_d, 0)$ is the unique closed-loop equilibrium state

in fact, for $\dot{q} = 0$ and $\ddot{q} = 0$, it is $K_P e = g(q) - g(q_d)$
which can hold only for $q = q_d$, because when $q \neq q_d$

$$\|K_P e\| \geq K_{P,m}\|e\| > \alpha\|e\| \geq \|g(q) - g(q_d)\|$$

Inclusion of Gravity

PD control + constant gravity compensation (stability analysis)

with $e = q_d - q, g(q) = \left(\frac{\partial U}{\partial q}\right)^T$ consider as Lyapunov candidate

$$V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e^T K_P e + U(q) - U(q_d) + e^T g(q_d)$$

2. V is convex in $\dot{q}$ and e, and zero only for $e = \dot{q} = 0$

$$\left(\frac{\partial V}{\partial \dot{q}}\right)^T = M(q)\dot{q} = 0 \quad \text{only for } \dot{q} = 0$$

$$\frac{\partial^2 V}{\partial \dot{q}^2} = M(q) > 0$$

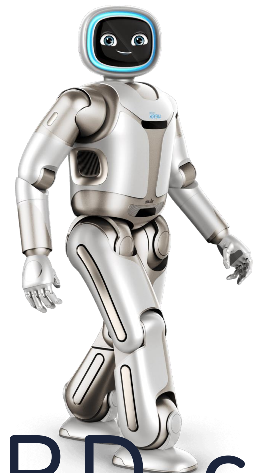
$$\left(\frac{\partial V_{|\dot{q}=0}}{\partial e}\right)^T = K_P e - \left(\frac{\partial U}{\partial q}\right)^T + g(q_d) = K_P e + g(q_d) - g(q) = 0$$

$\partial e / \partial q = -I$

only for $q = q_d$

$$\frac{\partial^2 V_{|\dot{q}=0}}{\partial e^2} = K_P + \frac{\partial^2 U}{\partial q^2} > 0, \text{ since } \|K_P\| = K_{P,M} \geq K_{P,m} > \alpha$$

$(q_d, 0)$ is a global minimum of $V \geq 0$



Inclusion of Gravity

PD control + constant gravity compensation (stability analysis)

differentiating
$$V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e^T K_P e + U(q) - U(q_d) + e^T g(q_d)$$

$$\dot{V} = \dot{q}^T\left(M(q)\ddot{q} + \frac{1}{2}\dot{M}(q)\dot{q}\right) - e^T K_P\dot{q} + \frac{\partial U(q)}{\partial q}\dot{q} - \dot{q}^T g(q_d)$$

$$= \dot{q}^T\left(u - \underbrace{S(q,\dot{q})\dot{q} + \frac{1}{2}\dot{M}(q)\dot{q}}_{=0} - g(q)\right) - e^T K_P\dot{q} + \dot{q}^T\left(g(q) - g(q_d)\right)$$

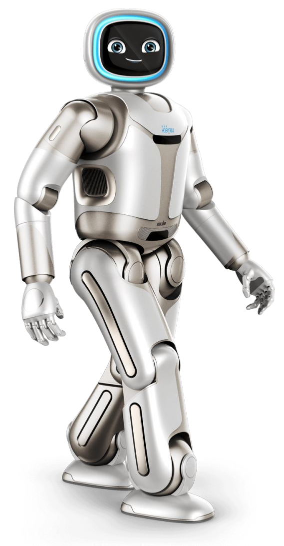
$$= \dot{q}^T K_P e - \dot{q}^T K_D\dot{q} + \dot{q}^T(g(q_d) - g(q)) - e^T K_P\dot{q} + \dot{q}^T\left(g(q) - g(q_d)\right)$$

$$= -\dot{q}^T K_D\dot{q} \le 0$$

for $\dot{V} = 0 (\Leftrightarrow \dot{q} = 0)$, we have in the closed-loop system

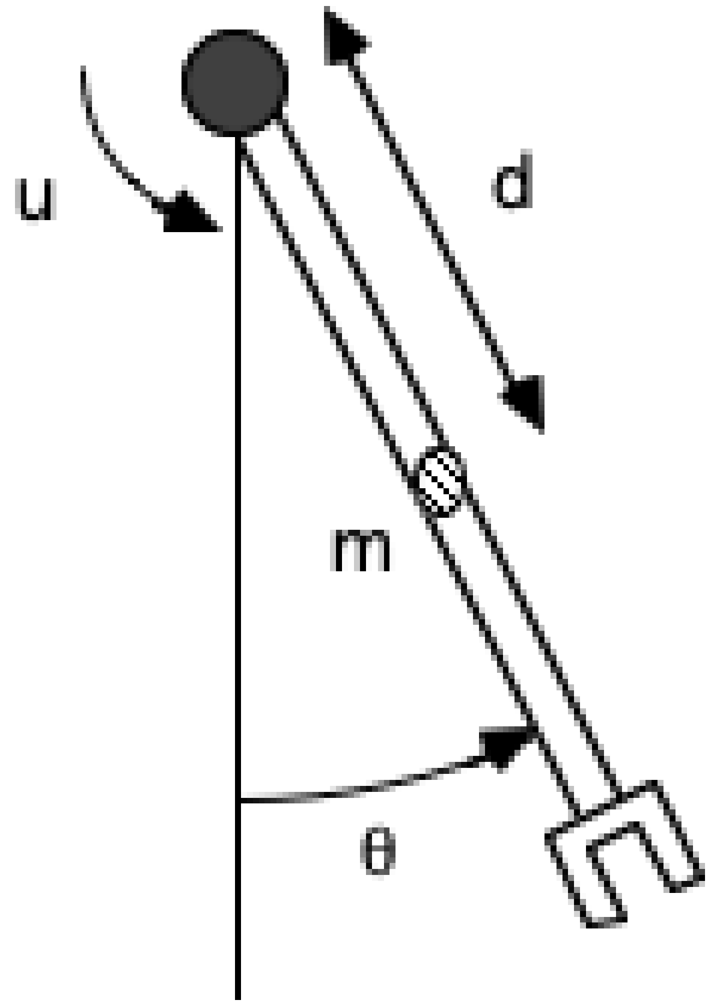
$$M(q)\ddot{q} + g(q) = K_P e + g(q_d) \;\Rightarrow\; \ddot{q} = M^{-1}(q)(K_P e + g(q_d) - g(q)) = 0 \Leftrightarrow e = 0$$

by LaSalle Theorem, the thesis follows



Example

Example of a single-link robot (stability analysis)



$$I\ddot{\theta} + mg_0 d\sin\theta = u$$

task: regulate the link position to the upward equilibrium

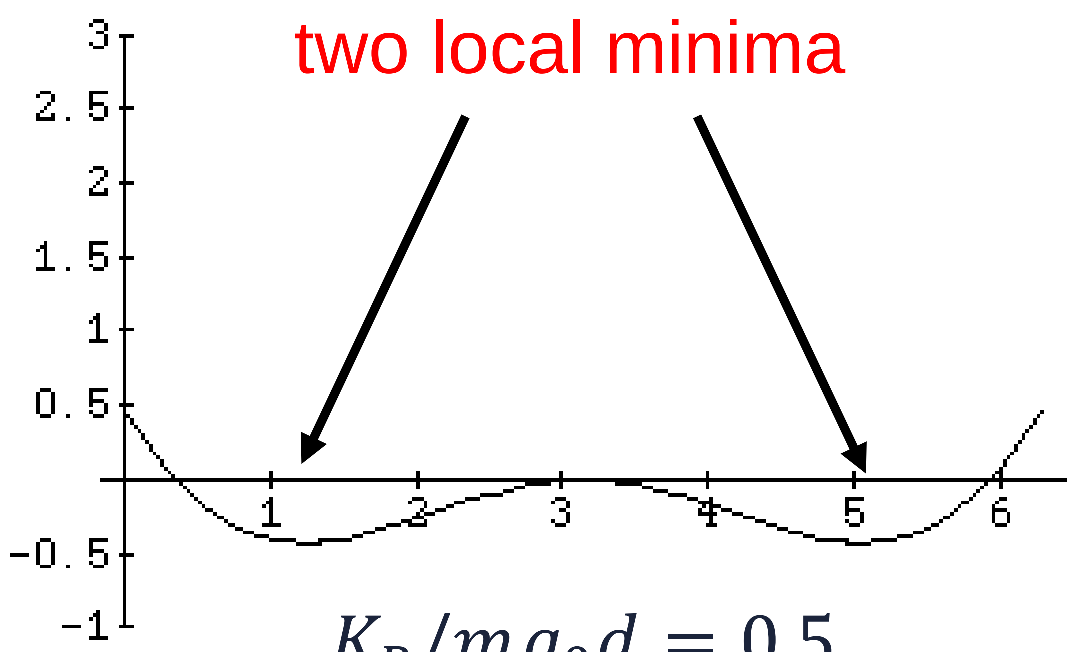
$$\theta_d = \pi \rightarrow g(\theta_d) = 0$$

PD control + constant gravity compensation (here, zero!)

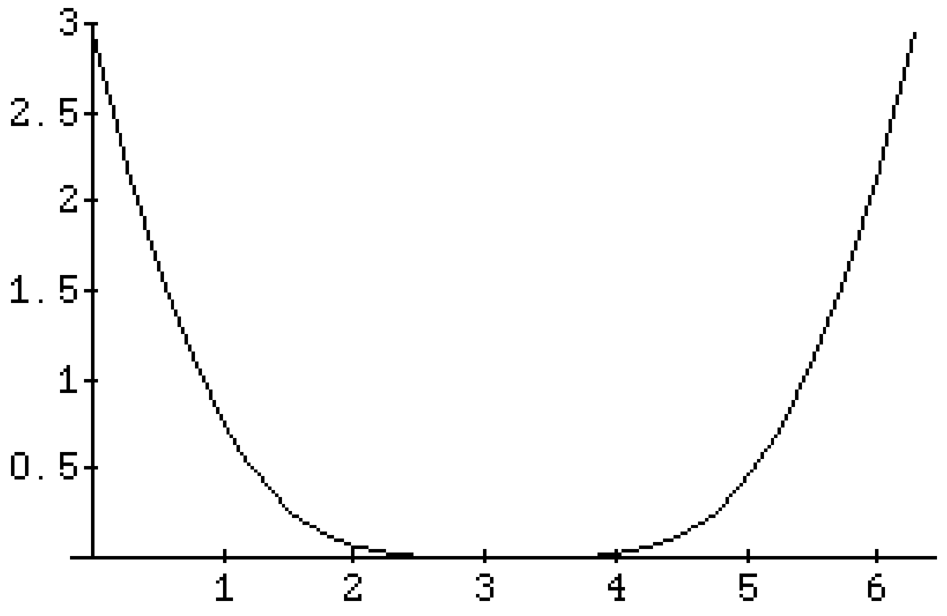
$$u = k_P\,(\pi - \theta) - k_D\dot{\theta}$$

by Theorem 2, it is sufficient (here, also necessary*) to choose

$$k_P > \alpha = mg_0 d,\ k_D > 0$$

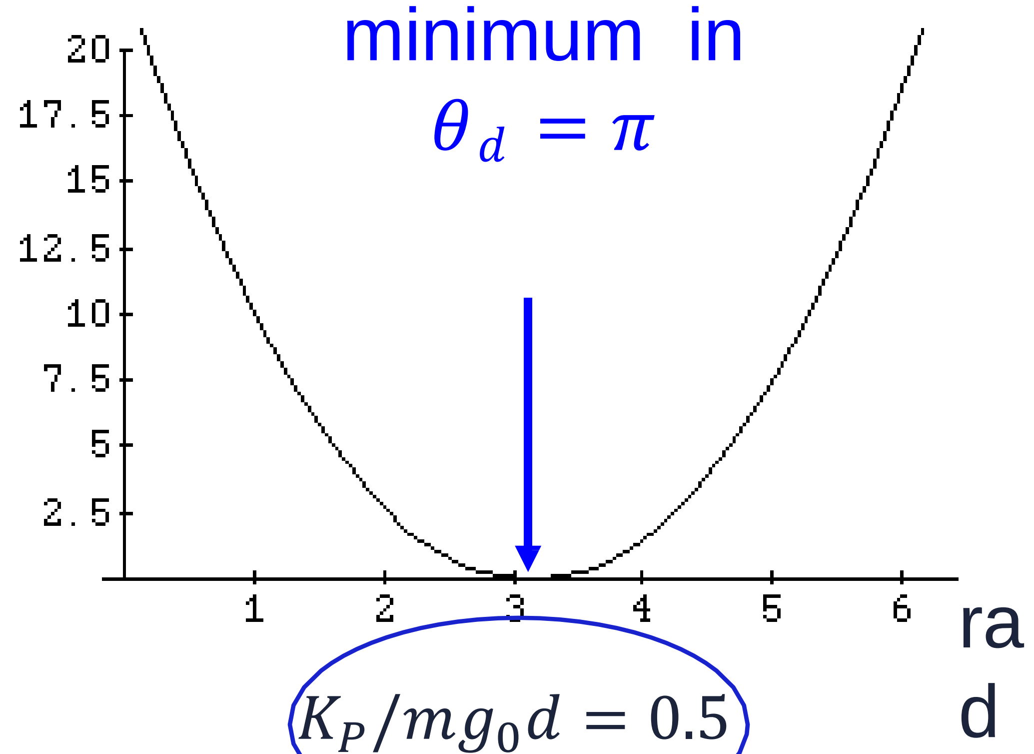


$K_P/mg_0d = 0.5$



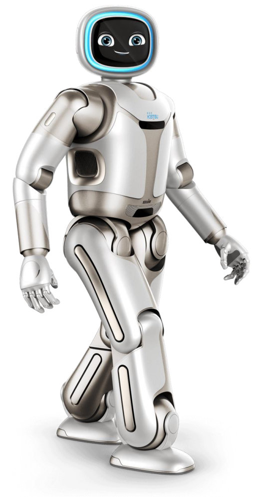
$K_P/mg_0d = 1$

plots of $V(\theta)$ (for $\dot{\theta} = 0$)



$K_P/mg_0d = 0.5$

* by a local analysis of the linear approximation at π



Example

PD control + constant gravity compensation
(simulations with data: $\boldsymbol{I} = \mathbf{0.9333}, \boldsymbol{mg_0d} = \mathbf{19.62}(=\boldsymbol{\alpha})$)

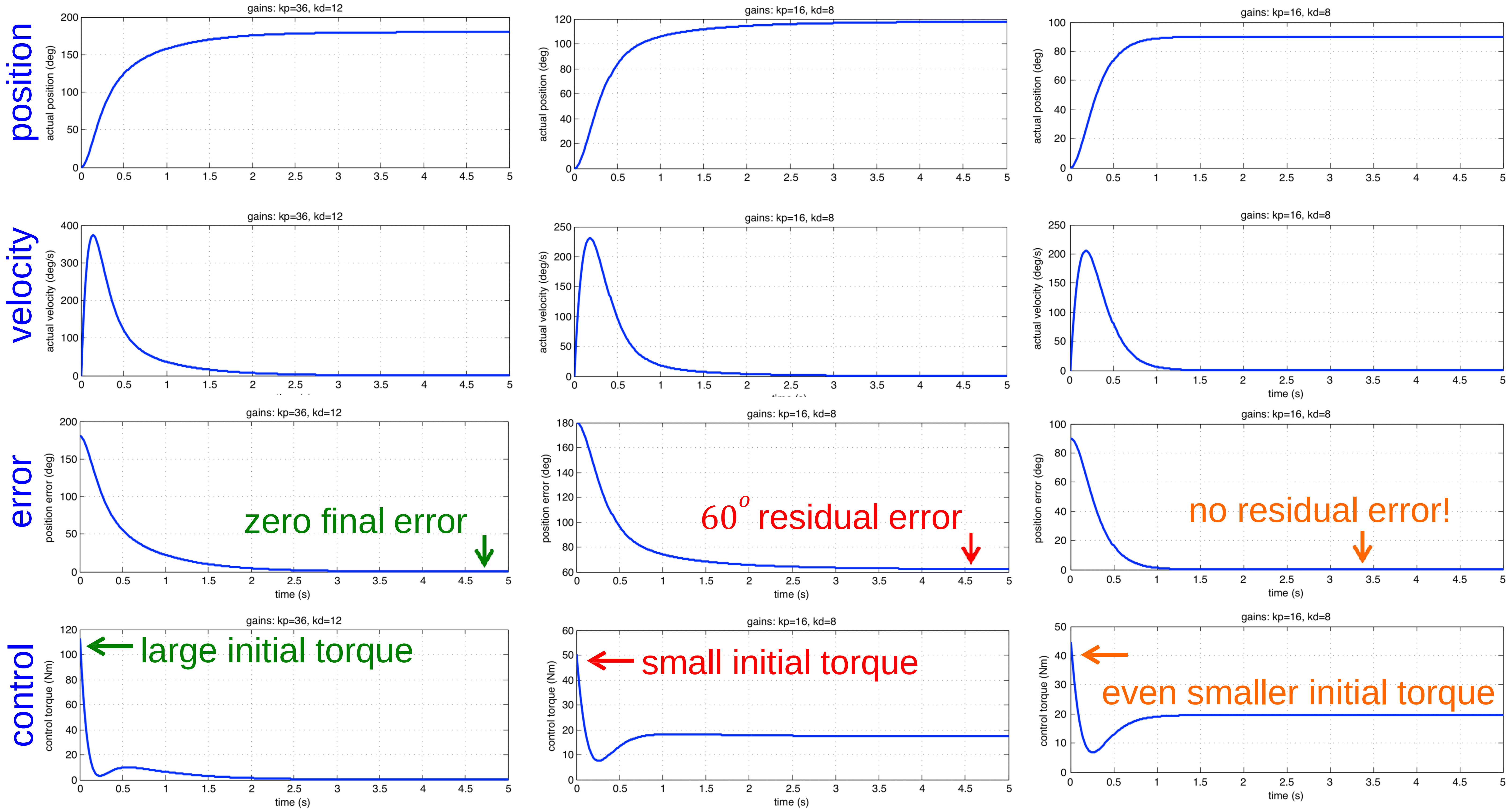
$\theta_d = 180^\circ \rightarrow g(\theta_d) = 0$

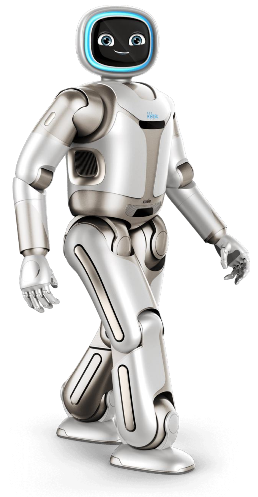
$\theta_d = 90^\circ \rightarrow g(\theta_d) = mg_0d$

sufficient P gain: $k_P = 36, k_D = 12$

low P gain: $k_P = 16, k_D = 8$

low P gain: $k_P = 16, k_D = 8$





Inclusion of Gravity

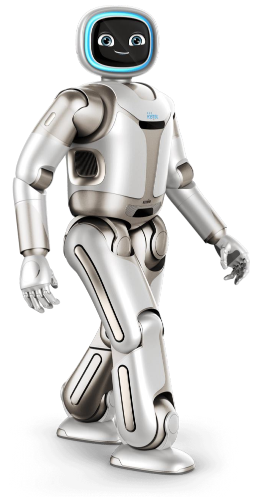
Approximate gravity compensation

the approximate control law

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q_d)$$

- leads, under similar hypotheses, to a closed-loop equilibrium q^*
  - its uniqueness is not guaranteed (unless $K_{É}$ is large enough)
  - for $K_P \to \infty$, one has $q^* \to q_d$

Conclusion: In the presence of gravity, the previous regulation control laws require an accurate knowledge of the gravity term in the dynamic model in order to guarantee the zeroing of the position error (since we can only use "finite" control gains ⇒ in practice, not too large)



PID Control

- in linear systems, the addition of an integral control action is used to eliminate a constant error in the step response at steady state
- in robots, a PID may be used to recover such a position error due to an incomplete (or absent) gravity compensation/cancellation

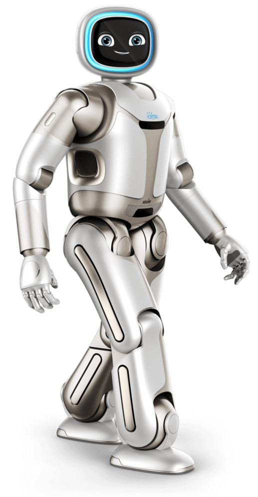
➡ the control law

$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t (q_d - q(\tau))\, d\tau - K_D\dot{q}(t)$$

- is independent from any robot dynamic model term
- **if** the desired closed-loop equilibrium is asymptotically stable under PID control, the integral term is "loaded" at steady state to the value

$$K_I \int_0^\infty (q_d - q(\tau))\, d\tau = g(q_d)$$

- however, one can show only local asymptotic stability of this law, i.e., for $q(0) \in \Delta(qd)$, under complex conditions on K_P, K_I, K_D and $e(0)$



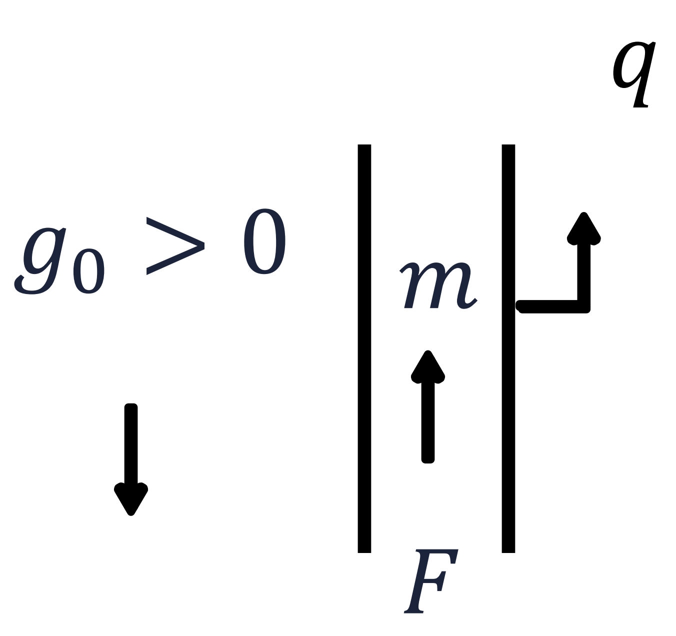
PID Control

Linear example with PID control

$$m\ddot{q} + mg_0 = F$$

(no friction)

$e(t) = q_d - q(t)$
$\dot{e}(t) = -\dot{q}(t)$



$$F = k_P(q_d - q) - k_D\dot{q}$$

(PD ⇒ steady-state error $e = q_d - \bar{q}$, with $\bar{q} = q_d - \frac{mg_0}{k_P}$

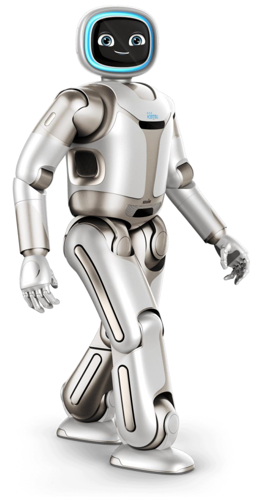
$$F = k_P(q_d - q) - k_D\dot{q} + mg_0$$

(PD + gravity cancellation ⇒ regulation $\forall k_P > 0, k_D > 0$)

$$F = k_P(q_d - q) - k_D\dot{q} + k_I\int_0^t (q_d - q(\tau))\, d\tau$$

with global exponential stability!

(PID ⇒ regulation $\forall k_I > 0,\ \ k_D > 0,\ \ k_P > \frac{mk_I}{k_D} > 0$)



PID Control

Saturated PID control

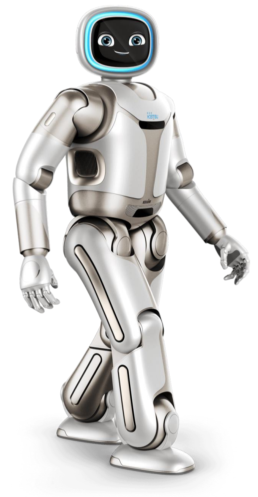
- more in general, one can prove global asymptotic stability of $(q_d, 0)$, under lower bound limitations for K_P, K_I, K_D (depending on suitable “bounds” on the terms in the dynamic model), for a nonlinear PID law

$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t \Phi\left(q_d - q(\tau)\right) d\tau - K_D \dot{q}$$

where $\Phi(q_d - q)$ is a saturation-type function, such as

$$\Phi(x) = \begin{cases} \sin x, & |x| \leq \pi/2 \\ 1, & x > \pi/2 \\ -1, & x < -\pi/2 \end{cases} \qquad \Phi(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

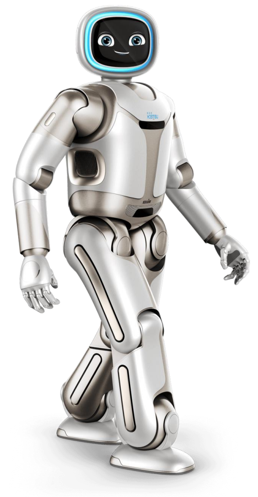
See paper by R. Kelly, IEEE TAC, 1998:
Global positioning of robot manipulators via PD control plus a class of nonlinear integral actions, IEEE Transactions on Automatic Control, 43 (7) (1998), pp. 934-938



Limits Discussion

Limits of robot regulation controllers

- response times needed for reaching the desired steady state are not easily predictable in advance
  - depend heavily on robot dynamics, on PD/PID gains, on the required total displacement, and on the interested area of robot workspace
  - integral term (when present) needs some time to “unload” itself from the error history accumulated during transients
    - large initial errors are stored in the integral term
    - anti-windup schemes stop the integration when commands saturate
    - … an intuitive explanation for the success of “saturated” PID law
- control efforts in the few first instants of motion typically exceed by far those required at steady state
  - especially for high positional gains
  - may lead to saturation (hard nonlinearity) of robot actuators



Limits Discussion

Regulation in industrial robots

- in industrial robots, the planner generates a reference trajectory $q_r(t)$ even when the task requires only positioning/regulation of the robot
  - "smooth" enough, with a user-defined transfer time T
  - reference trajectory interpolates initial and final desired position

$$q_r(0) = q(0) \qquad q_r(t \geq T) = q_d$$

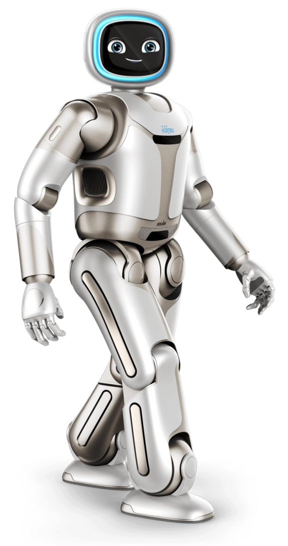
- $q_r(t)$ is used within a control law of the form

$$u = K_P(q_r(t) - q) + K_D(\dot{q}_r(t) - \dot{q}) + g(q)$$

e.g., PD with gravity cancellation

often neglected

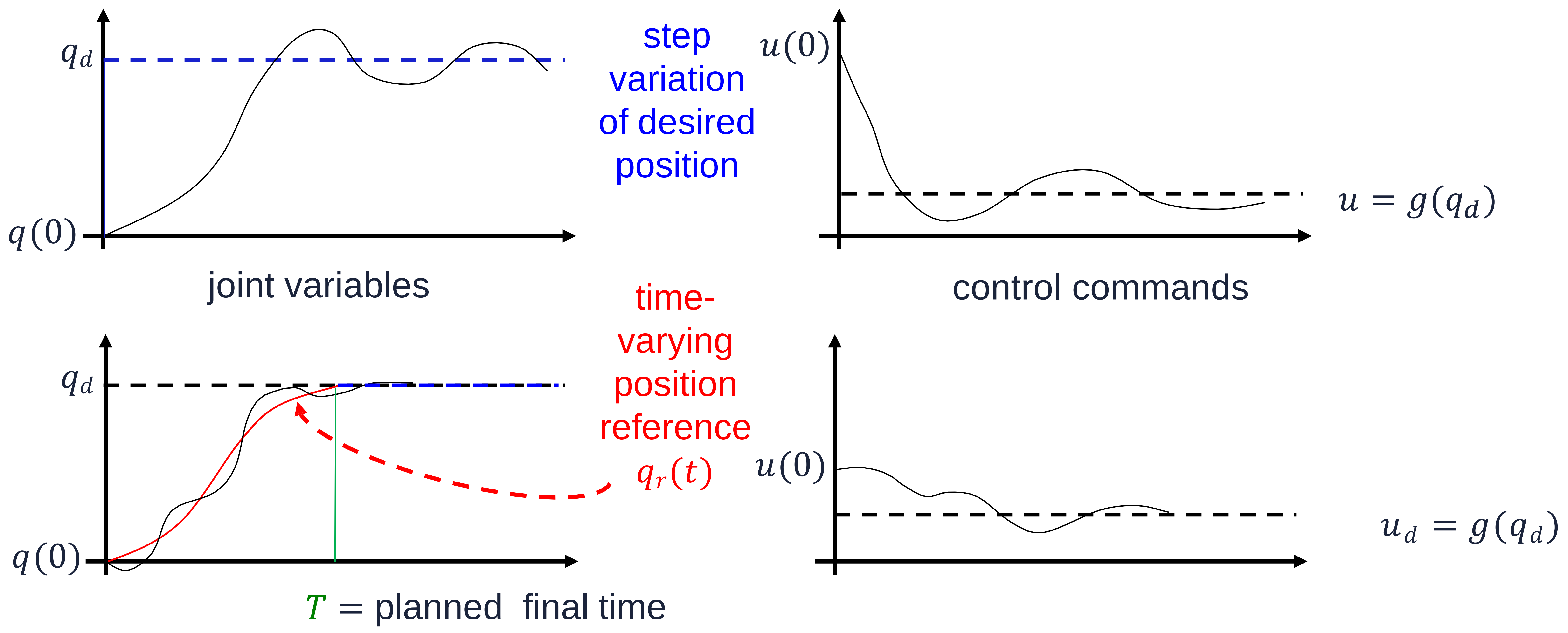
  - in this way, the position error is initially zero
  - robot motion stays only "in the vicinity" of the reference trajectory until $t = T$, typically with small position errors (gains can be larger!)
  - final regulation is only a "local" problem ($e(T) = q_d - q(T)$ is small)

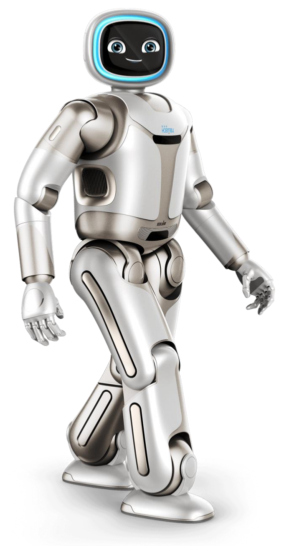


Limits Discussion

Qualitative comparison

- no saturation of commands: in principle, much larger gains can be used
- better prediction of settling times: local exponential convergence (designed on the linear approximation of the robot dynamics around $(q_d, 0)$)
- "fine tuning" of control gains is easier, but still a tedious and delicate task





Robot Dynamic Model

Appendix: Robot dynamic model
(in vector formats)

1. $$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

$k - th$ component of vector c

$$C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

$k - th$ column of matrix $M(q)$

symmetric matrix!

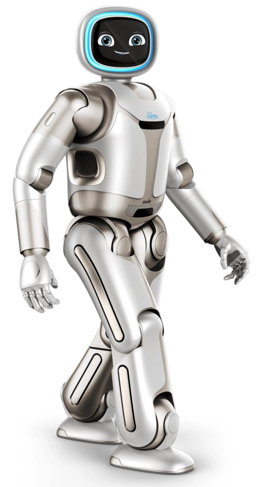
2. $$M(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u$$

NOTE: the model is in the form $\Phi(q,\dot{q},\ddot{q}) = u$ as expected

NOT a symmetric matrix in general

$$s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i$$

factorization of c by S is **not unique!**



Structural Property

Appendix: A structural property

Matrix $\dot{M}-2S$ is skew-symmetric
(when using Christoffel symbols to define matrix S)

Proof

$$\dot{m}_{kj}=\sum_i \frac{\partial m_{kj}}{\partial q_i}\dot{q}_i \qquad 2s_{kj}=\sum_i 2\,c_{kij}\dot{q}_i=\sum_i\left(\frac{\partial m_{kj}}{\partial q_i}+\frac{\partial m_{ki}}{\partial q_j}-\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i$$

$$\dot{m}_{kj}-2s_{kj}=\sum_i\left(\frac{\partial m_{ij}}{\partial q_k}-\frac{\partial m_{ki}}{\partial q_j}\right)\dot{q}_i=n_{kj}$$

$$n_{jk}=\dot{m}_{jk}-2s_{jk}=\sum_i\left(\frac{\partial m_{ik}}{\partial q_j}-\frac{\partial m_{ji}}{\partial q_k}\right)\dot{q}_i=-n_{kj}$$

using the symmetry of M

$$x^T(\dot{M}-2S)x=0,\forall x$$

Structural Property

Appendix: Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\dot{E} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{\partial U}{\partial q}\dot{q}$$

$$= \dot{q}^T\left(u - S(q,\dot{q})\dot{q} - g(q)\right) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T g(q)$$

$$= \dot{q}^T u + \frac{1}{2}\dot{q}^T\left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q}$$

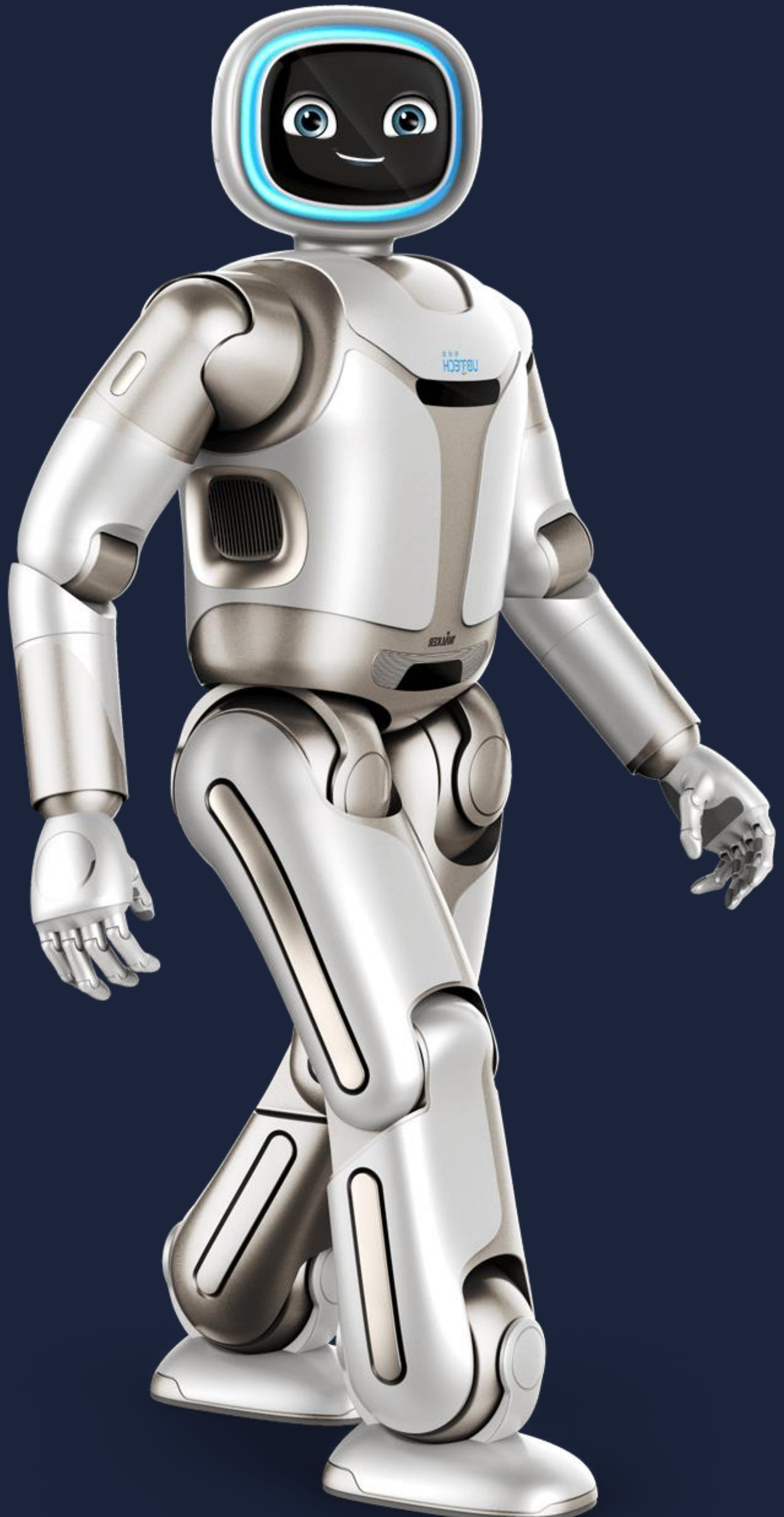
here, any factorization of vector c by a matrix S can be used

- if $u \equiv 0$, total energy is constant (no dissipation or increase)

$$\dot{E} = 0 \;\Rightarrow\; \boxed{\dot{q}^T\left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q} = 0, \forall q, \dot{q}} \;\Rightarrow\; \dot{E} = \dot{q}^T u$$

weaker property than skew-symmetry, as the external vector in the quadratic form is the same velocity $\dot{q}$ that appears also inside the two internal matrices $\dot{M}$ also S

in general, the variation of the total energy is equal to the work of non-conservative forces



Q&A