# Advanced Robotics

ENGG5402 Spring 2023

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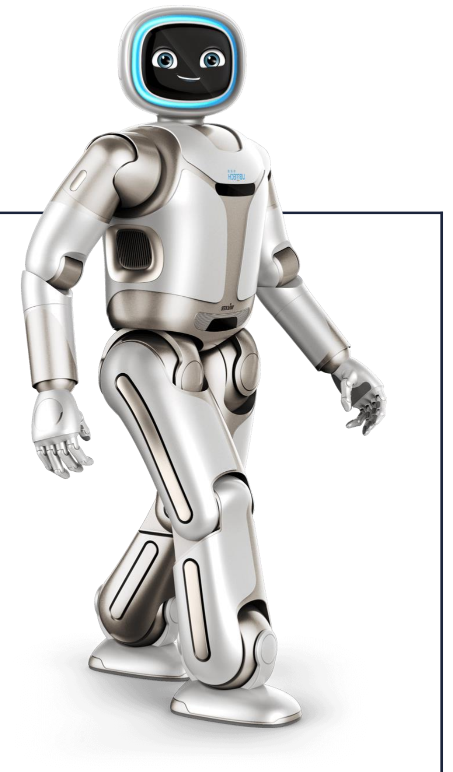
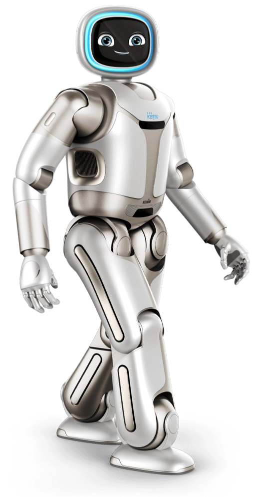


Topics:
- Position and Orientation of Rigid Bodies

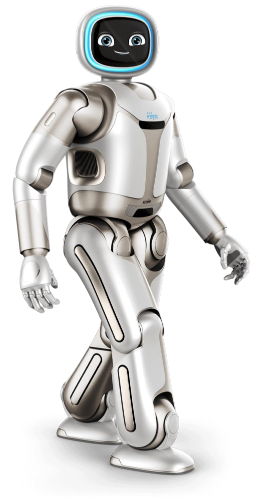
Readings:
- Siciliano: Sec. 2.1-2.6, 2.10



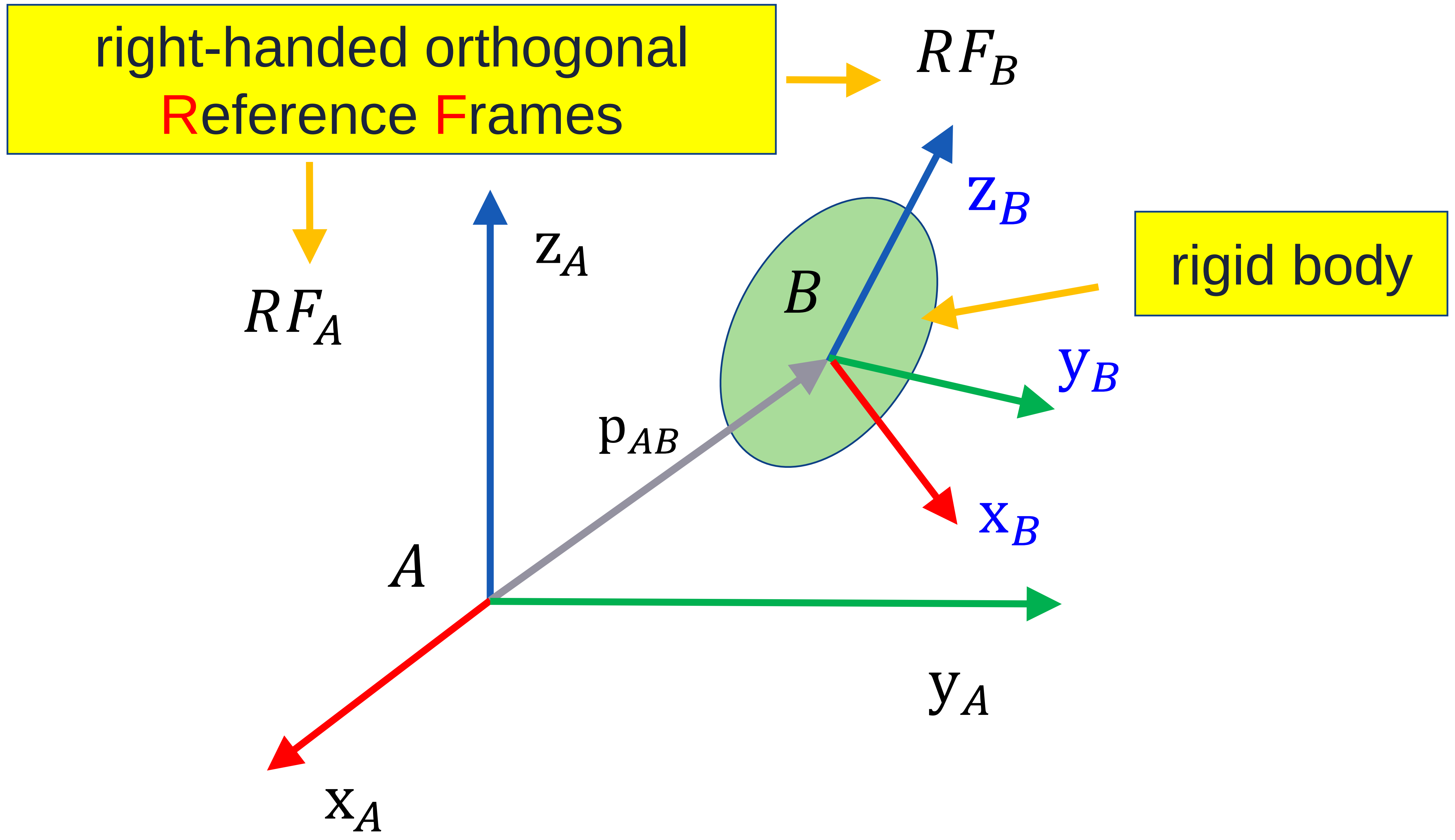
# Outline

Position and Orientation of Rigid bodies

- Basic Definitions



# Position and Orientation




- Position: ${}^{A}\boldsymbol{p}_{AB}$ (vector$\in \mathbb{R}^3$ )

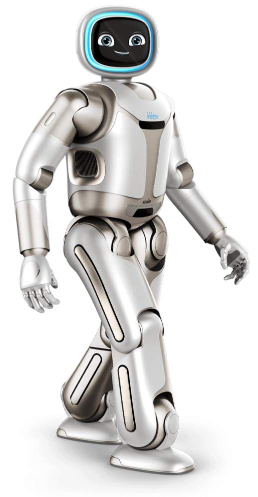
  Cartesian coordinates of vector expressed in $RF_A$

- Orientation:

  Orthonormal (Orthogonal + Normal) $3 \times 3$ matrix $(R^T = R^{-1} \Rightarrow R^T R = I)$ with Determinant (a.k.a., det) = +1

$$ {}^{A}R_B = [\, {}^{A}\mathbf{x}_B \quad {}^{A}\boldsymbol{y}_B \quad {}^{A}\boldsymbol{z}_B ] $$

- $\{\boldsymbol{x}_A, \boldsymbol{y}_A, \boldsymbol{z}_A\}\{\boldsymbol{x}_B, \boldsymbol{y}_B, \boldsymbol{z}_B\}$are axis vectors (of unitary norm) of frame $RF_A$ *and* $RF_B$
- Components in ${}^{A}R_B$ are the direction cosines of the axes of $RF_B$ with respect to $RF_A$



# Rotation Matrix

$${}^{A}R_{B} = \begin{bmatrix} x_A^T x_B & x_A^T y_B & x_A^T z_B \\ y_A^T x_B & y_A^T y_B & y_A^T z_B \\ z_A^T x_B & z_A^T y_B & z_A^T z_B \end{bmatrix}$$

orthonormal,

with det = +1

direction cosine of $z_B$ w.r.t. $x_A$

$$x_A^T z_B = \|x_A\| \|z_B\| \cos\beta$$

chain rule property

$${}^{k}R_{i}\,{}^{i}R_{j} = {}^{k}R_{j}$$

orientation of $RF_i$, w.r.t. $RF_k$

orientation of $RF_j$, w.r.t. $RF_i$

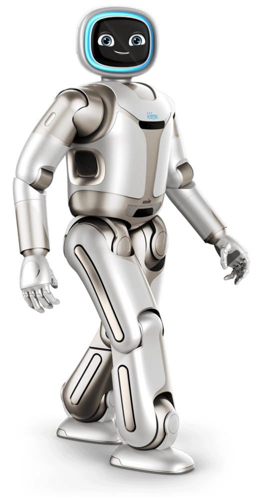
orientation of $RF_j$, w.r.t. $RF_k$

algebraic structure of a group $SO(3)$:

neutral element = $I$,

inverse element = $R^T$

NOTE: in general, the product of rotation matrices does **not** commute!



# Rotation Matrix

orthonormal,

with det = +1

$$^{A}R_{B} = \begin{bmatrix} x_A^T x_B & x_A^T y_B & x_A^T z_B \\ y_A^T x_B & y_A^T y_B & y_A^T z_B \\ z_A^T x_B & z_A^T y_B & z_A^T z_B \end{bmatrix}$$

direction cosine of $z_B$ w.r.t. $x_A$

$$x_A^T z_B = \|x_A\| \|z_B\| \cos \beta$$

chain rule property

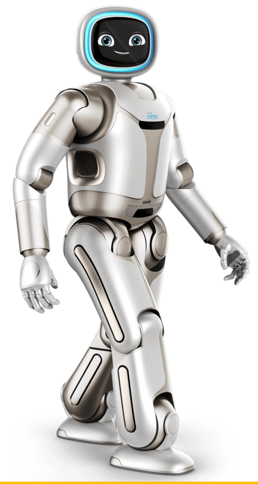
$$^{k}R_{i} \cdot {}^{i}R_{j} = {}^{k}R_{j}$$

orientation of $RF_i$, w.r.t. $RF_k$

orientation of $RF_j$, w.r.t. $RF_i$

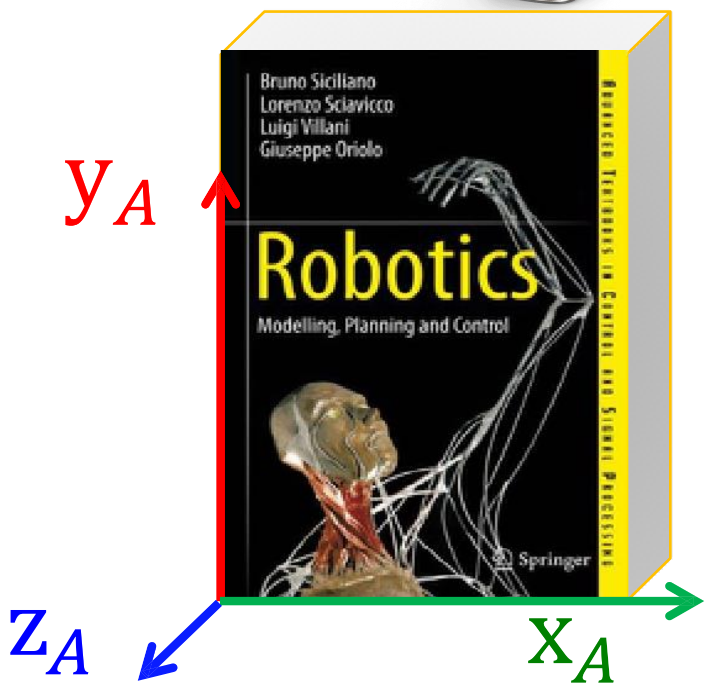
orientation of $RF_j$, w.r.t. $RF_k$

NOTE: in general, the product of rotation matrices does **not** commute!



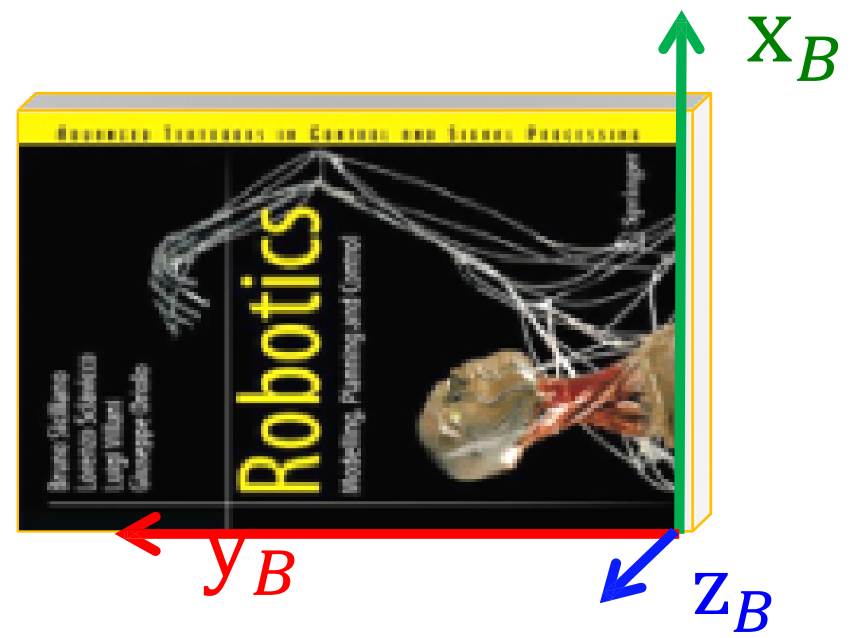
# Orientation of a rigid body

A simple example




90° around

$z$-axis



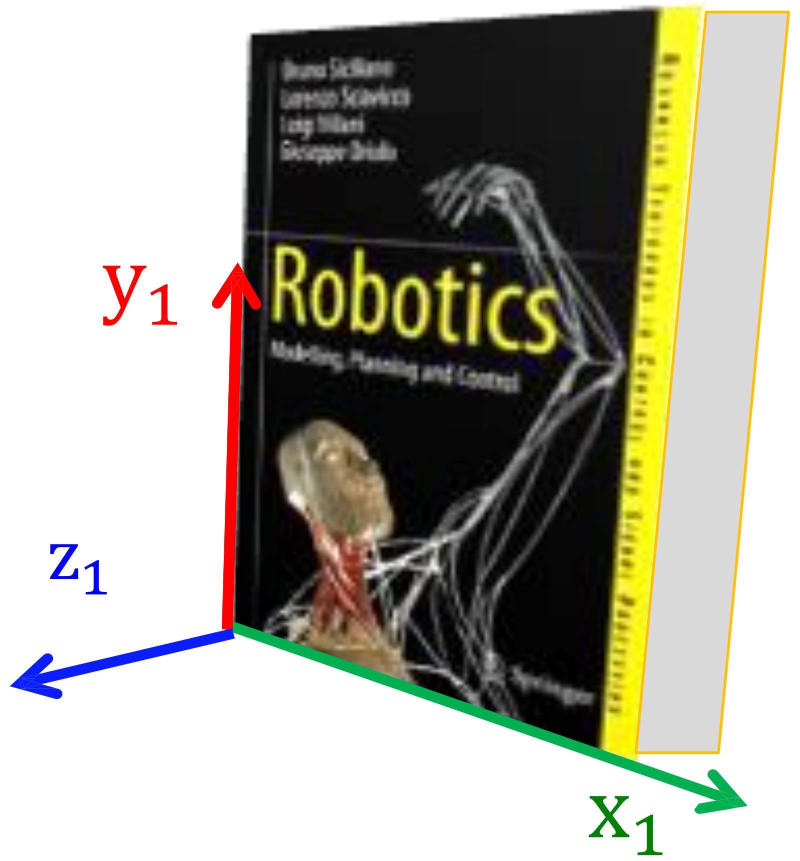

-90°
around
$x$-axis

$$^{B}R_A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = {}^{A}R_B^T$$

$$^{A}R_B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

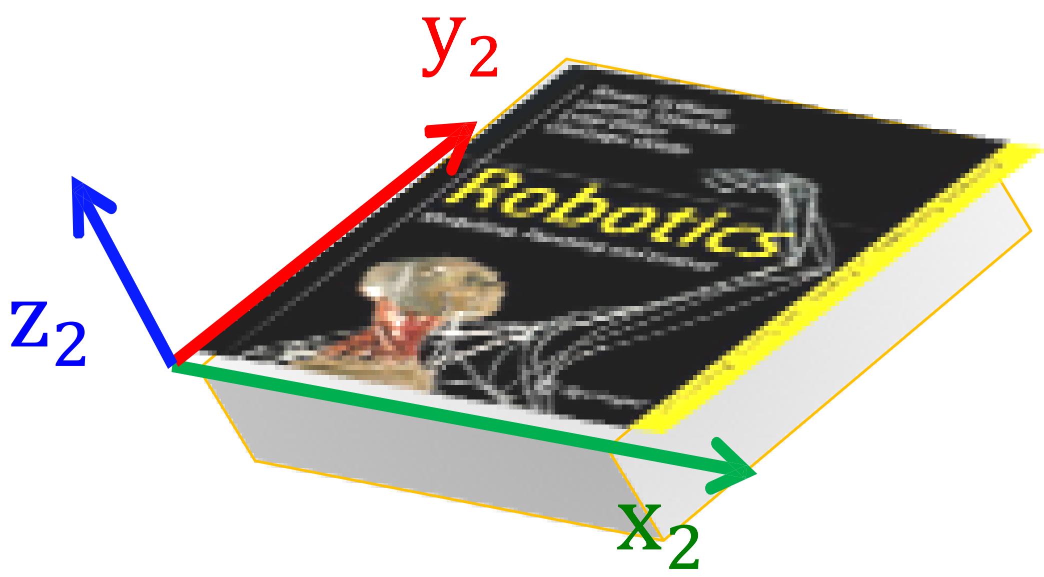

$$^{B}R_C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = {}^{B}R_A\,{}^{A}R_C = {}^{A}R_B^T\,{}^{A}R_C$$

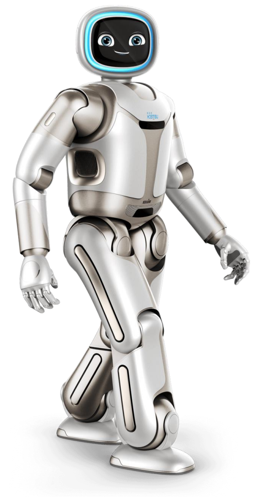
$$^{A}R_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$



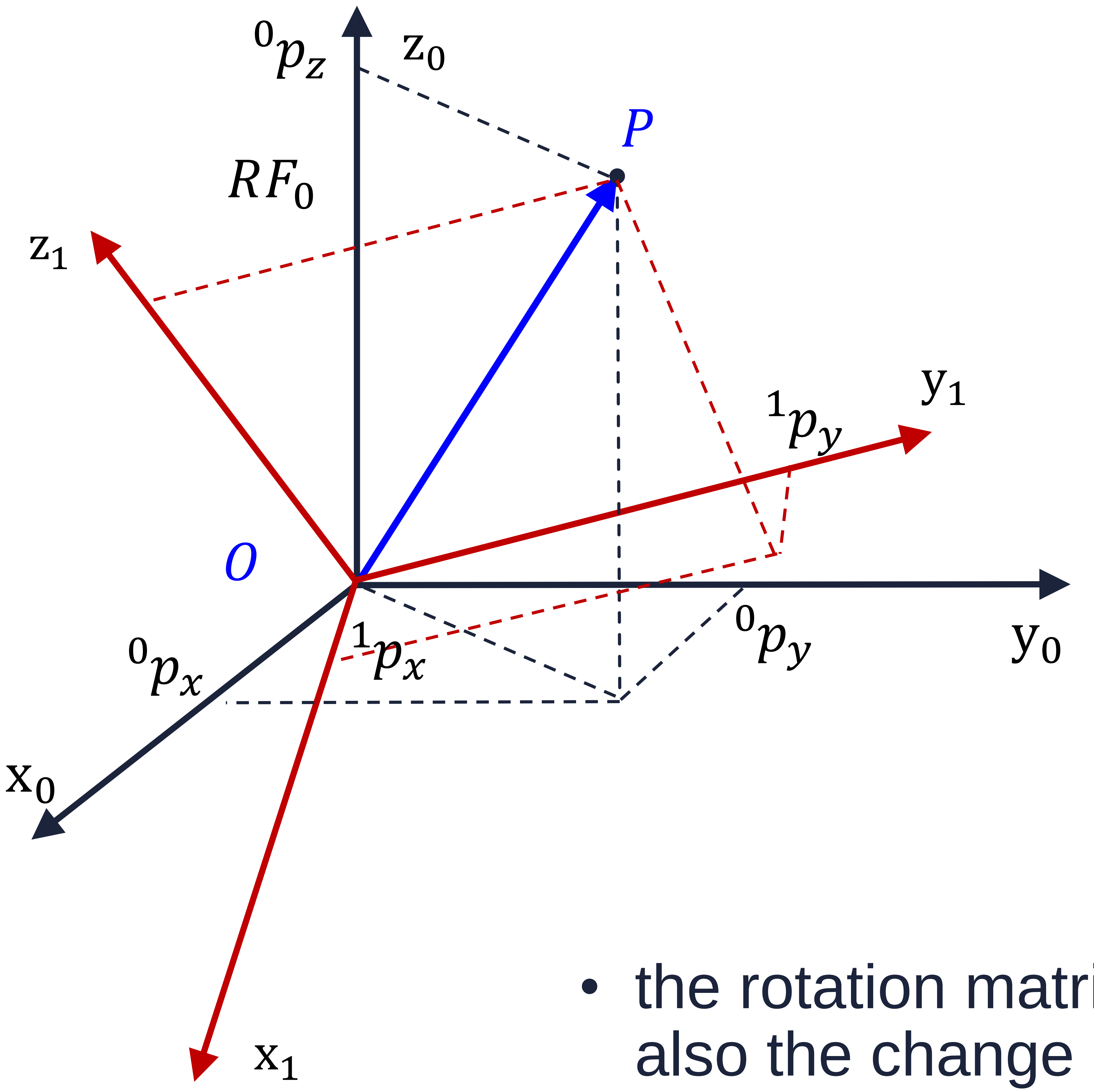

$^{A}R_1 = ?$

$^{A}R_2 = ?$





# Change of Coordinates




$$
\begin{aligned}
{}^0\boldsymbol{p} = \begin{bmatrix} {}^0p_x \\ {}^0p_y \\ {}^0p_z \end{bmatrix} &= {}^0p_x \, \overset{(1\ 0\ 0)^T}{\overset{\downarrow}{{}^0\mathbf{x}_0}} + {}^0p_y \, \overset{(0\ 1\ 0)^T}{\overset{\downarrow}{{}^0\mathbf{y}_0}} + {}^0p_z \, \overset{(0\ 0\ 1)^T}{\overset{\downarrow}{{}^0\mathbf{z}_0}} \\
&= {}^1p_x \, {}^0\mathbf{x}_1 + {}^1p_y \, {}^0\mathbf{y}_1 + {}^1p_z \, {}^0\mathbf{z}_1 \\
&= \begin{bmatrix} {}^0\boldsymbol{x}_1 & {}^0\boldsymbol{y}_1 & {}^0\boldsymbol{z}_1 \end{bmatrix} \begin{bmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \end{bmatrix}
\end{aligned}
$$

$$\boxed{{}^0\boldsymbol{p} = {}^0R_1 \, {}^1\boldsymbol{p}}$$

- the rotation matrix ${}^0R_1$ (*i.e*, the orientation of $RF_1$ *w.r.t.* $RF_0$) represents also the change of coordinates of a vector from $RF_1$ to $RF_0$

# Change of Coordinates

A simple example

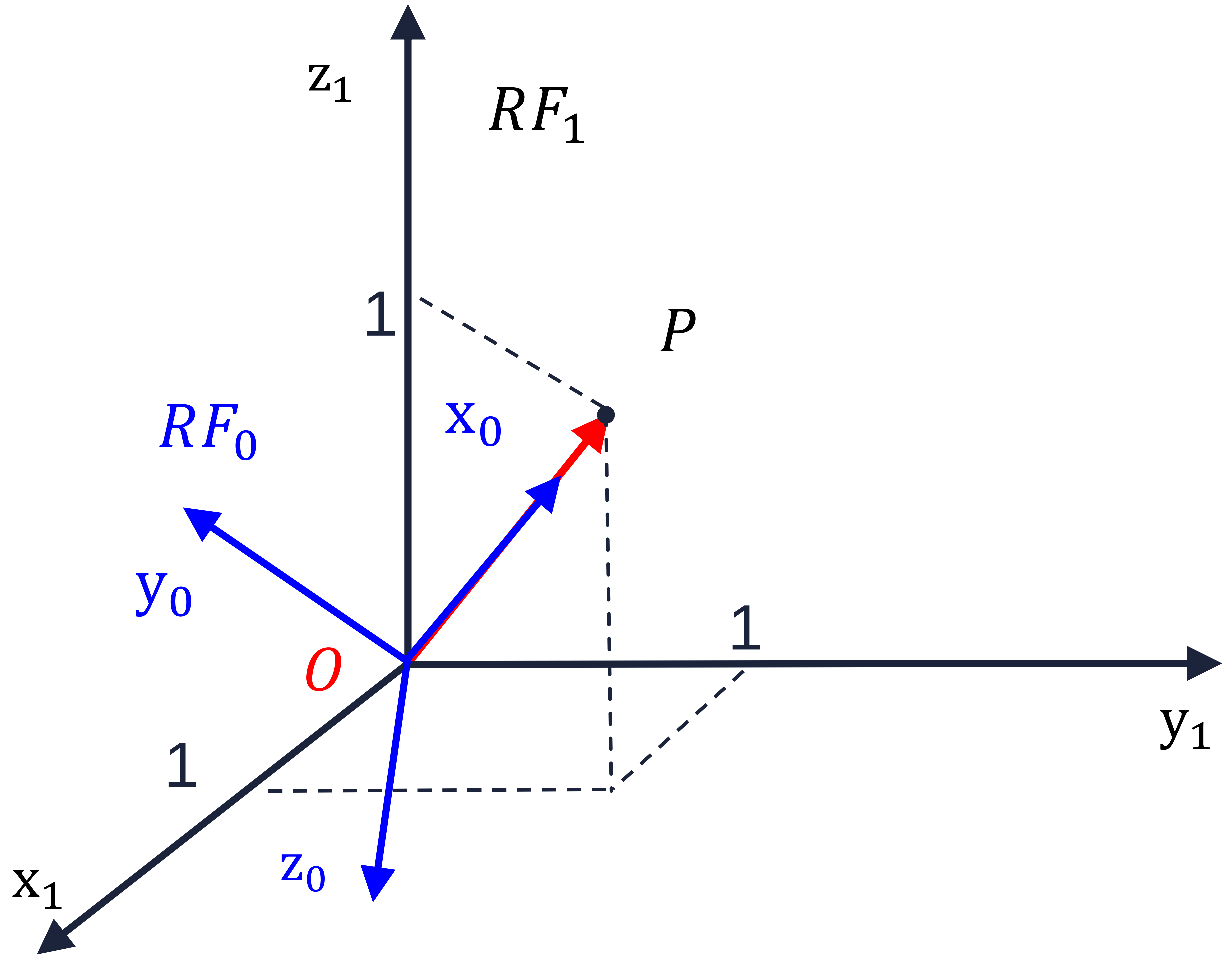
$$ {}^{1}\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} $$

$$ {}^{0}R_{1} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} $$

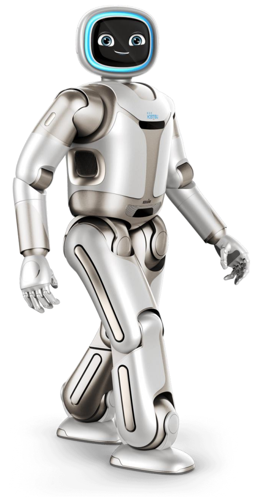
$$ {}^{0}\boldsymbol{p} = {}^{0}R_{1}\,{}^{1}\boldsymbol{p} = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix} $$

$$ \|\boldsymbol{p}\| = \|{}^{0}\boldsymbol{p}\| = \|{}^{1}\boldsymbol{p}\| = \sqrt{3} $$

… and where is $RF_0$ ?

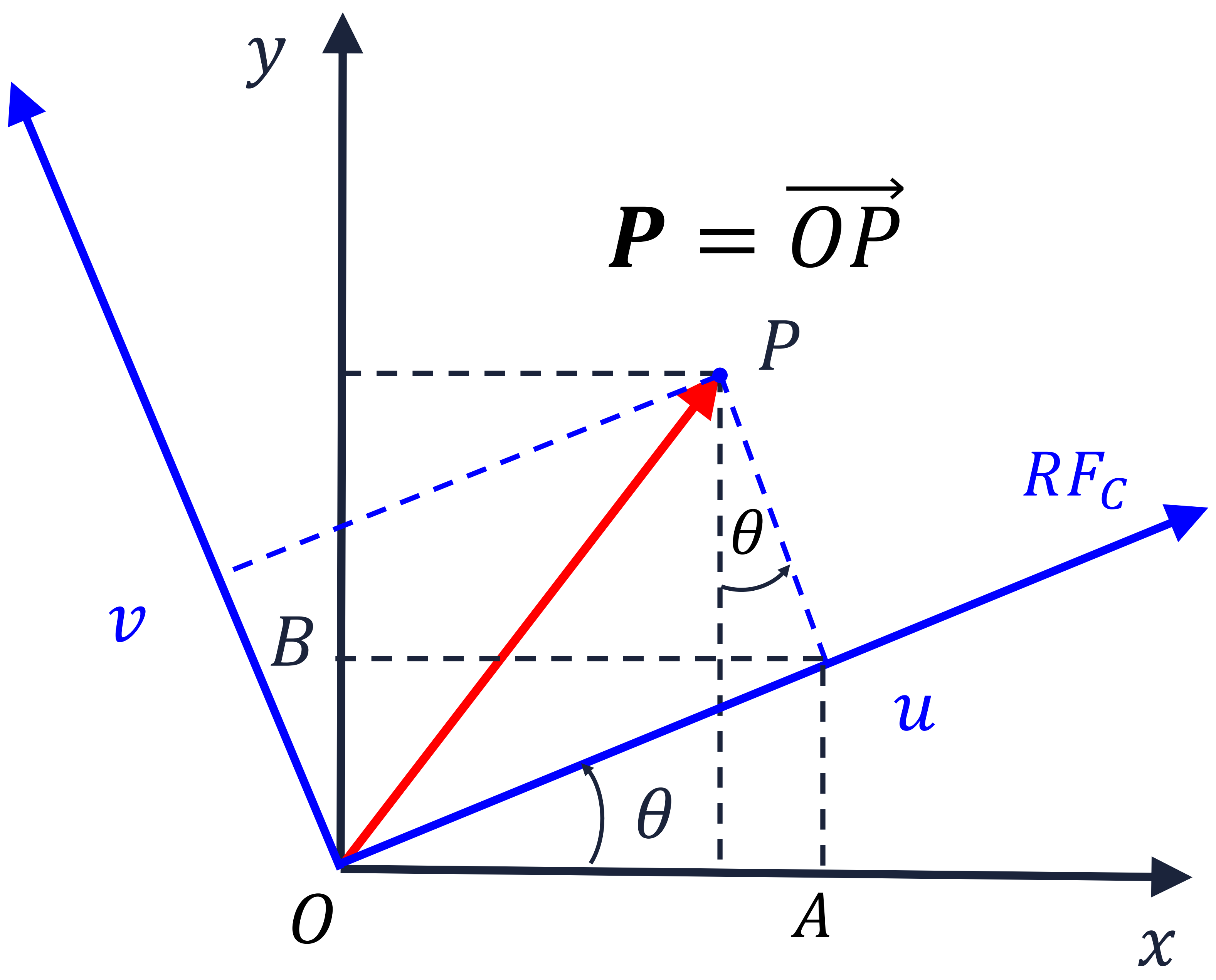



- $\boldsymbol{x}_0$ is aligned with $\boldsymbol{p} = \overrightarrow{OP}$
- $y_0$ is completes a right-handed frame
- $\boldsymbol{z}_0$ is orthogonal to $\boldsymbol{y}_1 (\boldsymbol{z}_0^T \boldsymbol{y}_1 = \boldsymbol{0})$ and is positive on $\boldsymbol{x}_1 (\boldsymbol{z}_0^T \boldsymbol{x}_1 = {}^{1}/_{\sqrt{2}})$



# Orientation of Frames

Orientation of frames in a plane
(elementary rotation around $z$-axis as example)




$${}^{O}\boldsymbol{p} \rightarrow \begin{aligned} x &= OA - xA = u\cos\theta - v\sin\theta \\ y &= OB + By = u\sin\theta + v\cos\theta \\ z &= w \end{aligned}$$

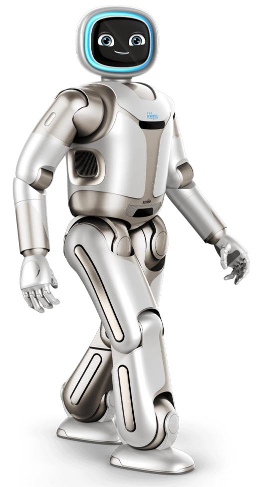
**or…**

$$\begin{array}{cccccc} & {}^{O}\boldsymbol{x}_C \downarrow & {}^{O}\boldsymbol{y}_C \downarrow & {}^{O}\boldsymbol{z}_C \downarrow & & {}^{C}\boldsymbol{p} \downarrow \end{array}$$

$${}^{O}\boldsymbol{p} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_z(\theta) \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

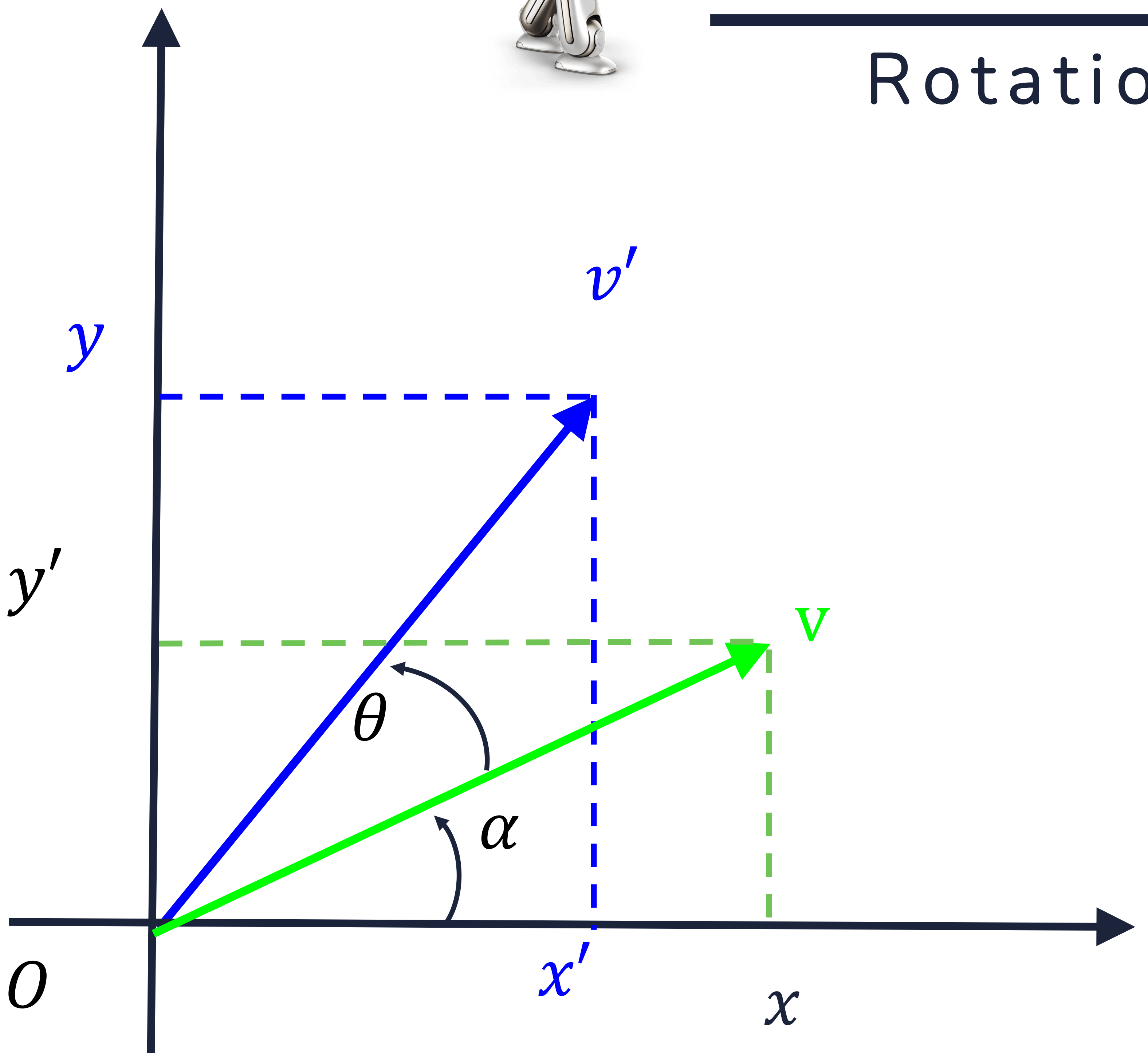
$R_z(-\theta) = R_Z^T(\theta)$

similarly: $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ $R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$



# Rotation of a Vector

Rotation of a vector around $z$ as an example



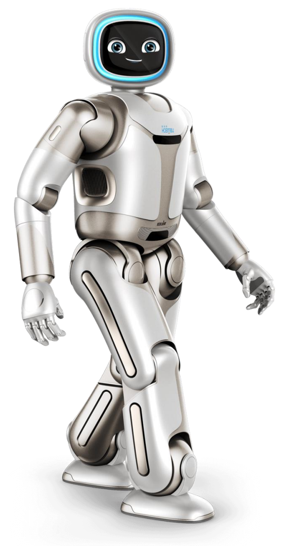

$$x = \|v\| \cos\alpha$$
$$y = \|v\| \sin\alpha$$

$$x' = \|v\| \cos(\alpha + \theta) = \|v\|(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$
$$= x\cos\theta - v\sin\theta$$
$$y' = \|v\| \sin(\alpha + \theta) = \|v\|(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$
$$= x\sin\theta + y\cos\theta$$
$$z' = z$$

or…

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

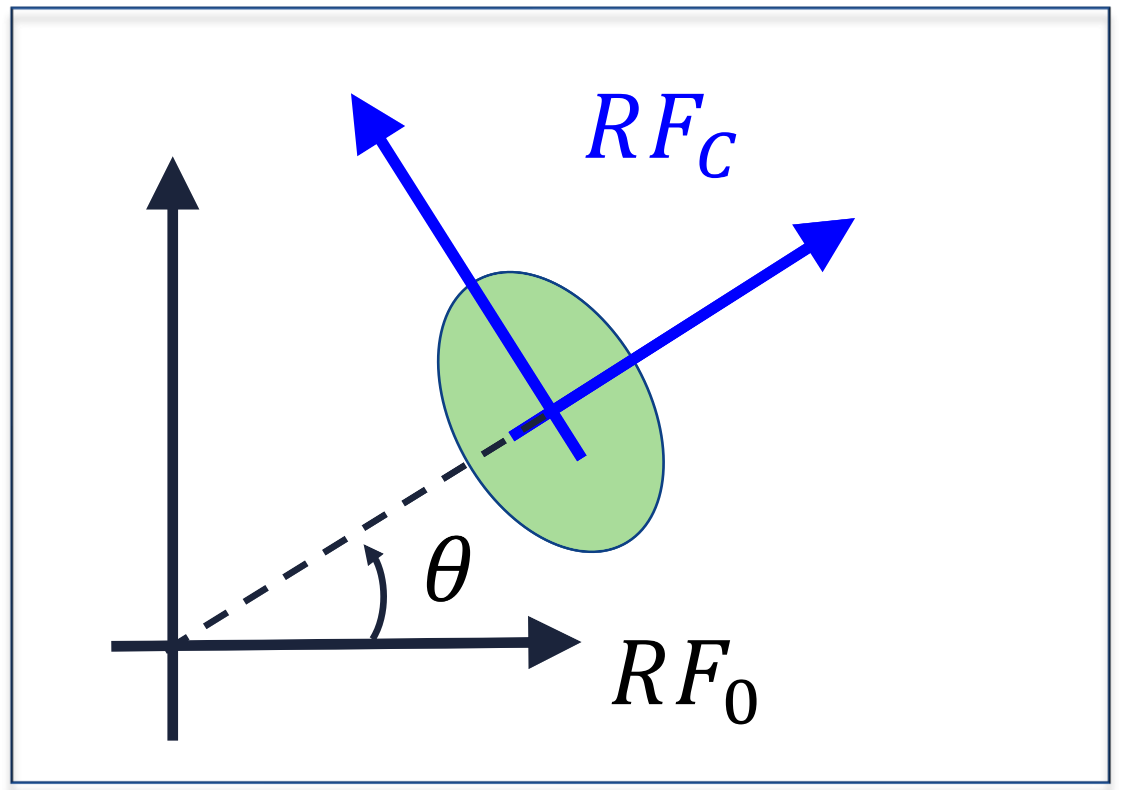
… same as before!

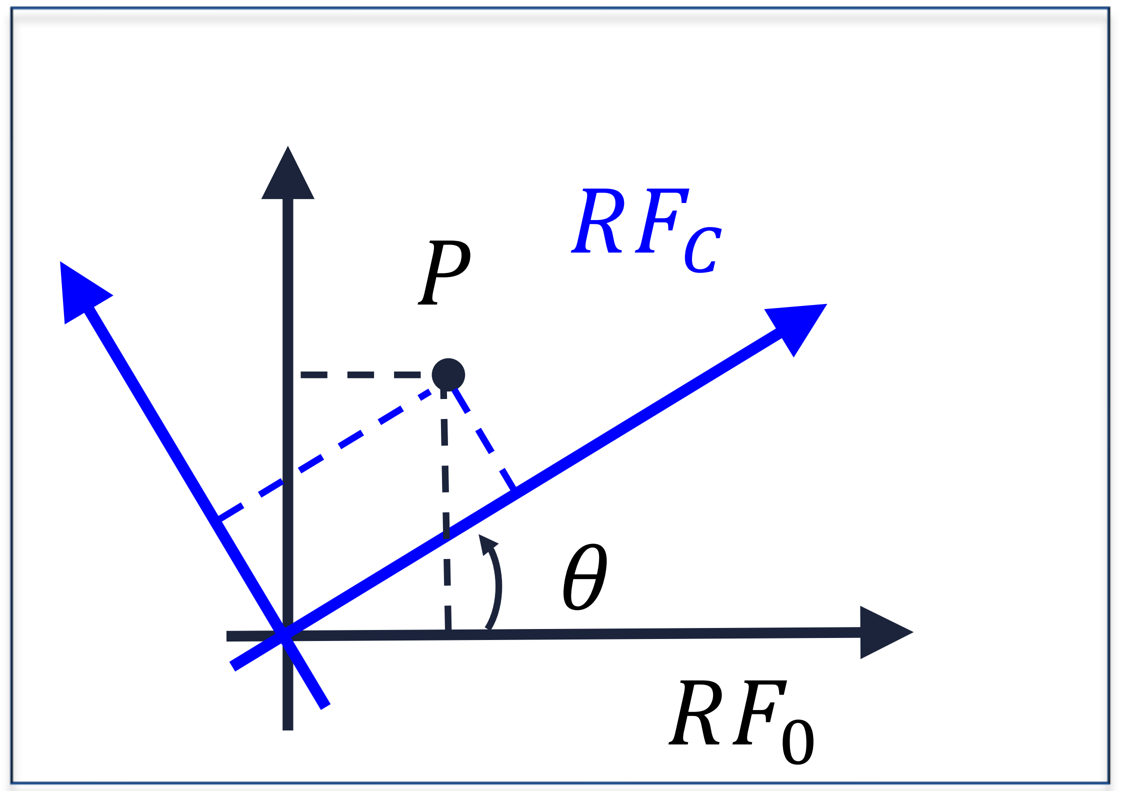


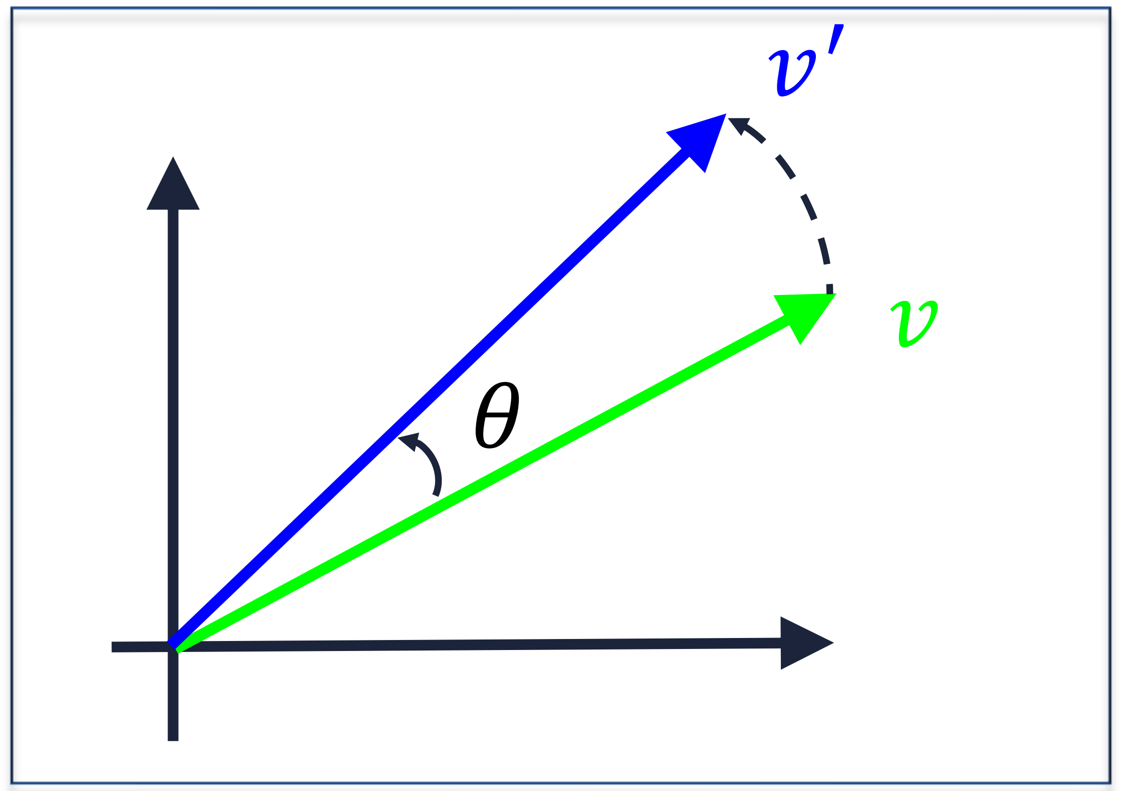
# Equivalent Interpretations !!

Equivalent interpretations of a rotation matrix

- the same rotation matrix (e.g., $R_Z(\theta)$) may represent




the orientation of a rigid body with respect to a reference frame $RF_0$

e.g.,$[\,{}^0\boldsymbol{x}_c \;\; {}^0\boldsymbol{y}_c \;\; {}^0\boldsymbol{z}_c] = R_Z(\theta)$

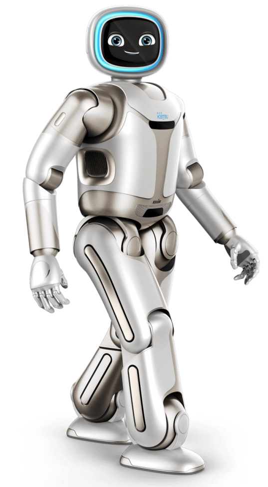
the change of coordinates from $RF_C$ to $RF_0$

e.g., ${}^0\boldsymbol{p} = R_Z(\theta)\,{}^C\boldsymbol{p}$

the rotation operator on vectors

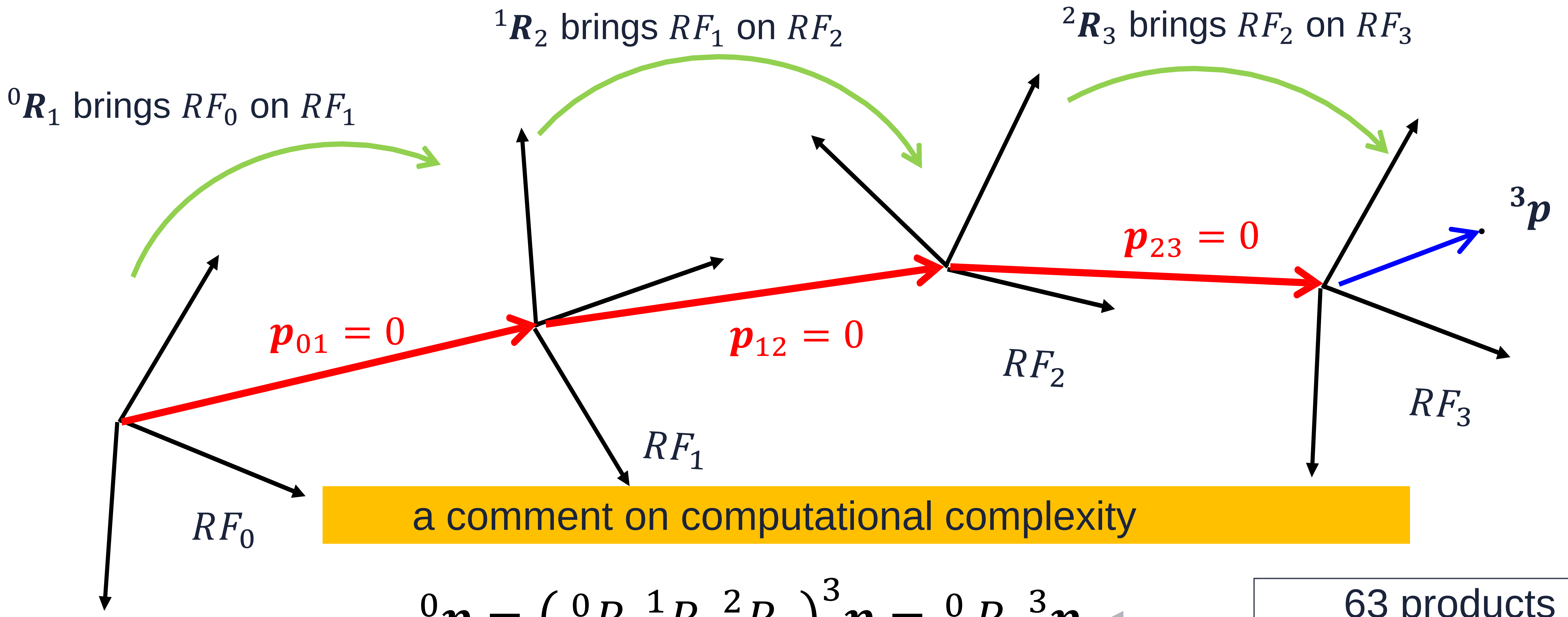
e.g.,$\boldsymbol{v}' = R_Z(\theta)\boldsymbol{v}$

the rotation matrix ${}^0\boldsymbol{R}_c$ is an operator superposing frame $RF_0$ to frame $RF_C$



# Composition of Rotation

A small extension of knowledge




a comment on computational complexity

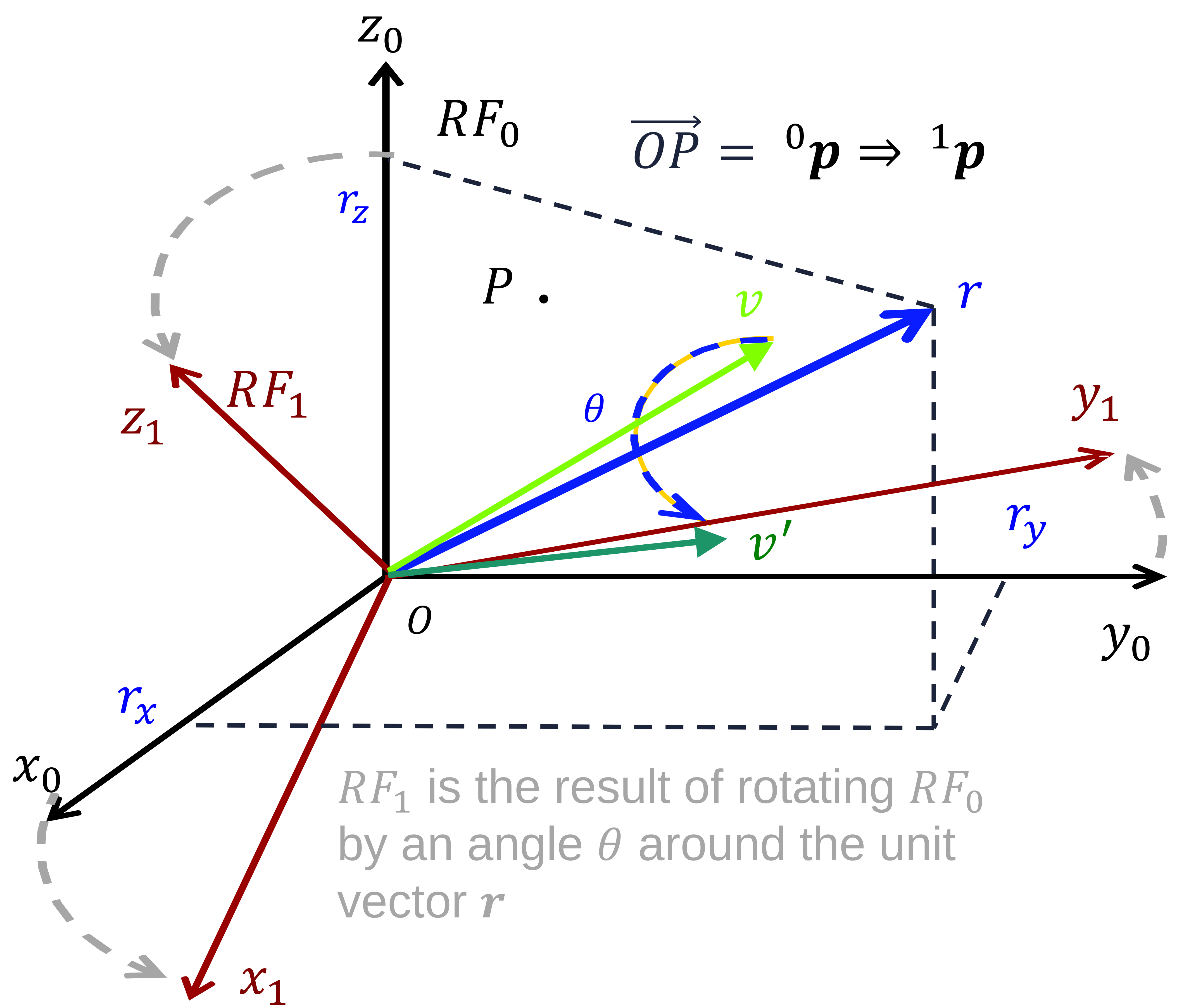
$${}^{0}\boldsymbol{p} = \left({}^{0}R_1\,{}^{1}R_2\,{}^{2}R_3\right)^{3}\boldsymbol{p} = {}^{0}R_3\,{}^{3}\boldsymbol{p}$$

63 products 42 summations

$${}^{0}\boldsymbol{p} = {}^{0}R_1 \underbrace{\left({}^{1}R_2 \left(\underbrace{{}^{2}R_3\,{}^{3}\boldsymbol{p}}_{{}^{2}\boldsymbol{p}}\right)\right)}_{{}^{1}\boldsymbol{p}}$$

27 products 18 summations

# Axis/angle Representation




$RF_1$ is the result of rotating $RF_0$ by an angle $\theta$ around the unit vector $\boldsymbol{r}$

**DATA**

- axis $\boldsymbol{r}$ (unit vector in $\mathbb{R}^3$, $\|\boldsymbol{r}\| = 1$)
- angle $\theta$, positive counterclockwise (as seen from an "observer" oriented like $\boldsymbol{r}$ with the head placed on the arrow, looking down to her/his feet)

**DIRECT PROBLEM**

find a rotation matrix $R(\theta, \boldsymbol{r})$

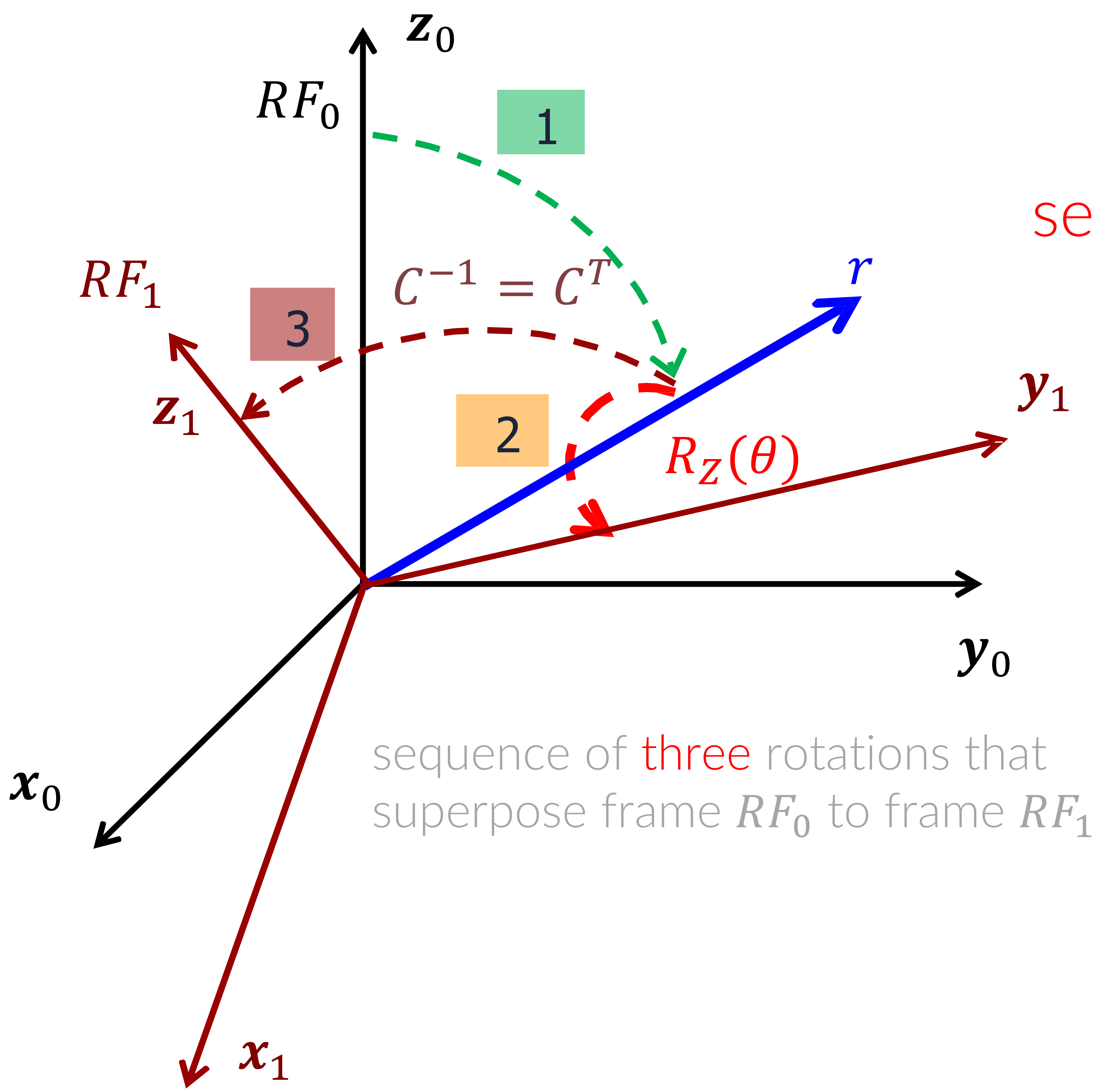
$$R(\theta, \boldsymbol{r}) = [{}^0\boldsymbol{x}_1 \ {}^0\boldsymbol{y}_1 \ {}^0\boldsymbol{z}_1]$$

such that

$${}^0\boldsymbol{p} = R(\theta, \mathrm{r})^1\boldsymbol{p} \qquad {}^0\boldsymbol{v}' = R(\theta, \mathrm{r})^0\boldsymbol{v}$$

# Axis/angle Representation

## Axis/angle: Direct problem




sequence of three rotations that superpose frame $RF_0$ to frame $RF_1$

$$R(\theta, \mathrm{r}) = CR_z(\theta)C^T$$

sequence of three rotations (one of which is elementary)

head placed after the first rotation the $\boldsymbol{z}$-axis coincides with $\boldsymbol{r}$

$$C = \begin{bmatrix} \mathrm{n} & \mathrm{s} & \mathrm{r} \end{bmatrix}$$

$\boldsymbol{n}$ and $\boldsymbol{s}$ are orthogonal unit vectors such that $\boldsymbol{n} \times \boldsymbol{s} = \boldsymbol{r}$

# Axis/angle Representation

Axis/angle: Direct problem (solution)

$$R(\theta,\boldsymbol{r}) = CR_z(\theta)C^T$$

taking into account (details in textbook):

$$CC^T = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

$$\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$$

$$R(\theta,\boldsymbol{r}) = [\boldsymbol{n} \quad \boldsymbol{s} \quad \boldsymbol{r}]\begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix}$$
$$= \boldsymbol{r}\boldsymbol{r}^T + (\boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T)^T c\theta + (\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T)s\theta$$

depends only on $\boldsymbol{r}$ and $\theta$ !

$$R(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)c\theta + S(\boldsymbol{r})s\theta$$

# Axis/angle Representation

Final expression of $\boldsymbol{R}(\boldsymbol{\theta}, \boldsymbol{r})$

developing computations…

$$R(\theta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)-r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

note that

$$R(\theta, \boldsymbol{r}) = R(-\theta, -\boldsymbol{r}) = R^T(-\theta, \boldsymbol{r})$$

# Axis/angle Representation

A simple example

$$R(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)c\theta + S(\boldsymbol{r})s\theta$$

$$\boldsymbol{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathrm{z}_0$$

$$R(\theta,\boldsymbol{r})$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} c\theta + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s\theta$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_z(\theta)$$

# Axis/angle Representation

Axis/angle: Rodriguez formula

$$\boldsymbol{v}' = R(\theta, \boldsymbol{r})\boldsymbol{v}$$

$$\boldsymbol{v}' = \boldsymbol{v}\cos\theta + (\boldsymbol{r} \times \boldsymbol{v})\sin\theta + (1 - \cos\theta)\left(\boldsymbol{r}^{\mathrm{T}}\boldsymbol{v}\right)\boldsymbol{r}$$

proof

$$\boxed{\begin{aligned} R(\theta, \boldsymbol{r})\boldsymbol{v} &= (\boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)\cos\theta + S(\boldsymbol{r})\sin\theta)\boldsymbol{v} \\ &= \boldsymbol{r}\boldsymbol{r}^T\boldsymbol{v}(1 - \cos\theta) + \boldsymbol{v}\cos\theta + (\boldsymbol{r} \times \boldsymbol{v})\sin\theta \end{aligned}}$$

q.e.d

# Axis/angle Representation

Properties of $\boldsymbol{R}(\boldsymbol{\theta}, \boldsymbol{r})$

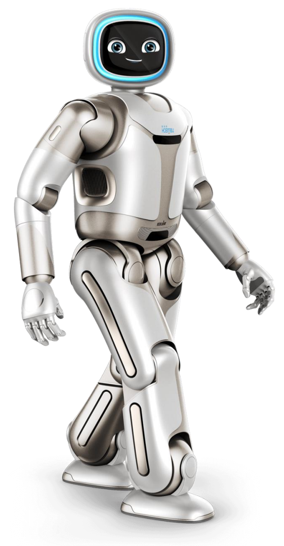
- $R(\theta, \boldsymbol{r})\boldsymbol{r} = \boldsymbol{r}$ ($\boldsymbol{r}$ is the invariant axis in this rotation)
- when r is one of the coordinate axes, R boils down to one of the known elementary rotation matrices
- $(\theta, r) \rightarrow R$ is not an injective map: $R(\theta, \boldsymbol{r})\boldsymbol{r} = R(-\theta, -\boldsymbol{r})$
- $\det(R) = +1 = \prod_i \lambda_i$(eigenvalues)
- $tr(R) = tr(rr^T) + (I - rr^T)c\theta = 1 + 2c\theta = \sum_i \lambda_i$

1\. $\Rightarrow \lambda_1 = 1$

4\. & 5. $\Rightarrow$ $\lambda_2 + \lambda_3 = 2c\theta \Rightarrow \lambda^2 - 2c\theta\lambda + 1 = 0$

$$\Rightarrow \lambda_{2,3} = c\theta \pm \sqrt{c^2\theta - 1} = c\theta \pm is\theta = e^{\pm i\theta}$$

all eigenvalues $\lambda$ have unitary module ($\Leftarrow R$ orthonormal)



# Axis/angle Representation

## Axis/angle: Inverse problem

GIVEN a rotation matrix $\boldsymbol{R}$,
FIND a unit vector $\boldsymbol{r}$ and an angle $\theta$ such that

$$R = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)\cos\theta + S(\boldsymbol{r})\sin\theta = R(\theta, \boldsymbol{r})$$

note first that $tr(R) = R_{11} + R_{22} + R_{33} = 1 + 2\cos\theta$; so, one could solve

$$\theta = \arccos\frac{R_{11} + R_{22} + R_{33} - 1}{2}$$

but

- this formula provides only values in $[0, \pi]$ (thus, never negative angles $\theta$)
- loss of numerical accuracy for $\theta \longrightarrow 0$ (sensitivity of $\cos\theta$ is low around 0)

# Axis/angle Representation

Axis/angle: Inverse problem (solution)

from the data ⟷ from $R(\theta, \boldsymbol{r})$

Skew-sym / Skew-sym

$$R - R^T = \begin{bmatrix} 0 & R_{12} - R_{21} & R_{13} - R_{31} \\ R_{21} - R_{12} & 0 & R_{23} - R_{32} \\ R_{31} - R_{13} & R_{32} - R_{23} & 0 \end{bmatrix} = 2\sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

it follows $\|\boldsymbol{r}\| = 1 \Rightarrow \sin\theta = \pm\frac{1}{2}\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}$ $(*)$

thus $\theta = \operatorname{atan2}\left\{\pm\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1\right\}$ $(**)$

see next side

$$\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

can be used only if $\boxed{\sin\theta \neq 0}$

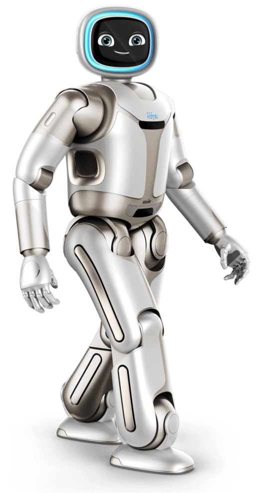
this is made on (*) using the data $\{R_{ij}\}$

# Axis/angle Representation

atan2 function

- arctangent with output values “in the four quadrants”
  - two input arguments
  - takes values in $[-\pi, \pi]$
  - undefined only for $(0,0)$
- uses the sign of both arguments to define the output quadrant
- based on arctan function with output values in $[-\frac{\pi}{2}, +\frac{\pi}{2}]$
- available in main languages (C++, Matlab, …)

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & y \geq 0, x < 0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \mathit{undefined} & y = 0, x = 0 \end{cases}$$



# Unit Quaternion

- to eliminate non-uniqueness and singular cases of the axis/angle $(\theta, r)$ representation, the unit quaternion can be used

$$Q = \{\eta, \boldsymbol{\epsilon}\} = \{\cos(\theta/2), \sin(\theta/2)\mathrm{r}\}$$

a scalar

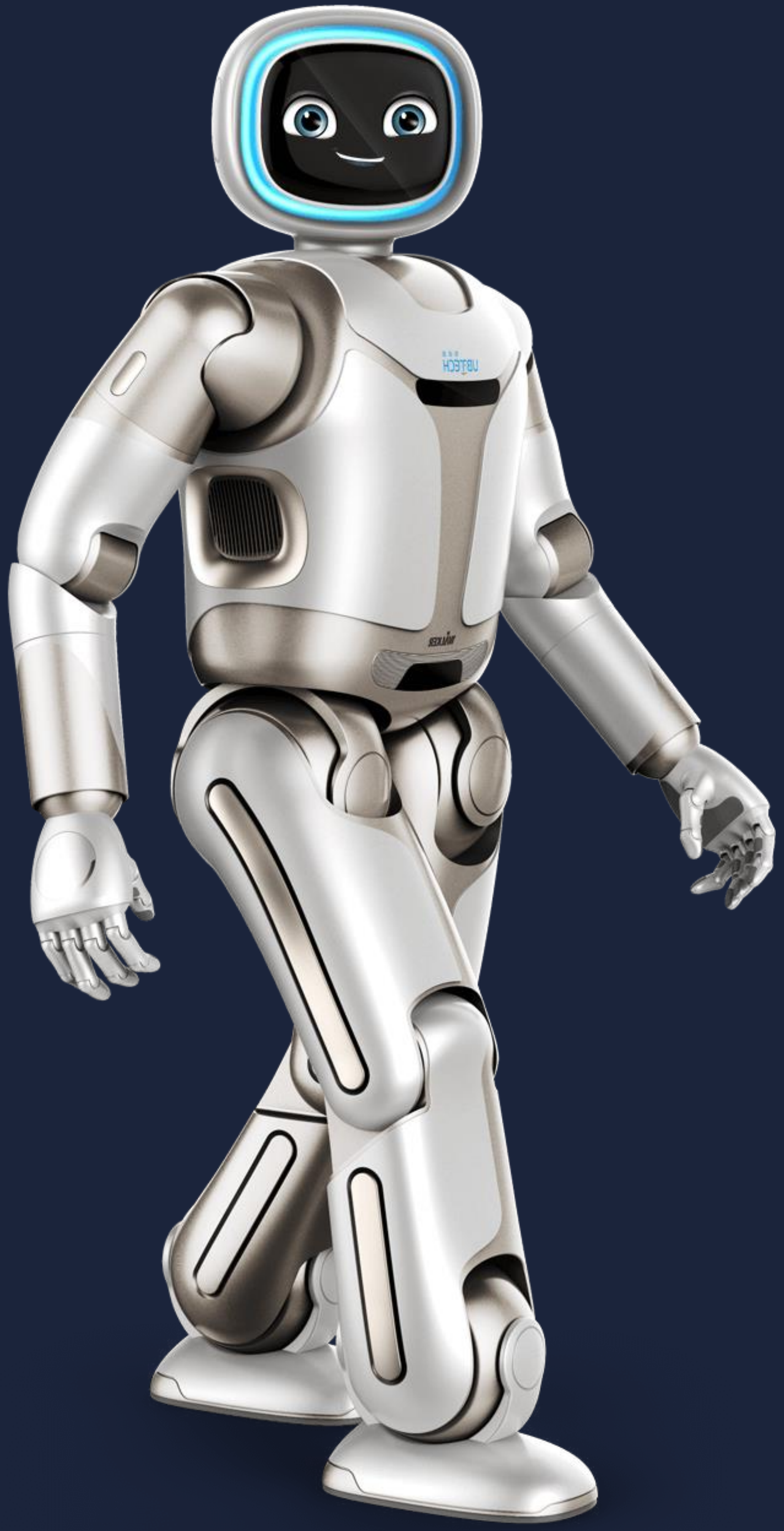
3-dim vector

- $\eta^2 + \|\epsilon\|^2 = 1$ (thus, “unit…”)
- $(\theta, r)$ and $(-\theta, -r)$ are associated to the same quaternion $Q$
- the rotation matrix $R$ associated to a given quaternion $Q$ is

$$R(\eta, \epsilon) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

- no rotation is $Q = \{1, 0\}$, while the inverse rotation is $Q = \{\eta, -\epsilon\}$
- unit quaternions are composed with special rules

$$Q_1 * Q_2 = \{\eta_1\eta_2 - \epsilon_1^T\epsilon_2, \eta_1\epsilon_2 + \eta_2\epsilon_1 + \epsilon_1 \times \epsilon_2\}$$



# Q&A