Advanced Robotics

ENGG5402 Spring 2023

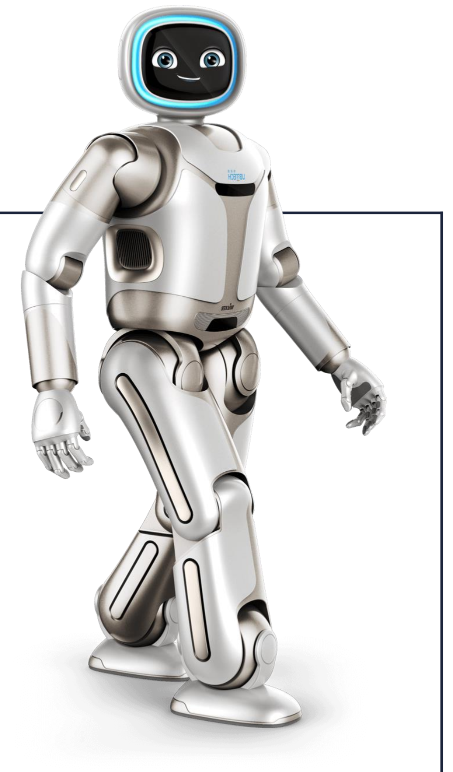
Fei Chen

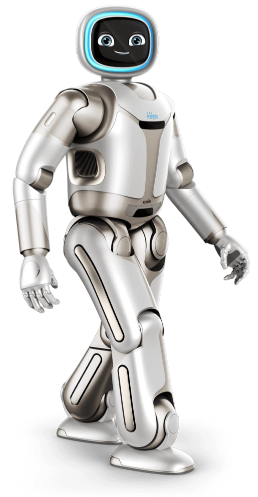
Topics:

- Euler and roll-pitch-yaw angles

Readings:

- Siciliano: Sec. 2.1-2.6, 2.10

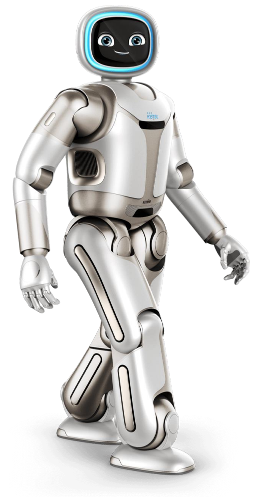




Outline

Euler and roll-pitch-yaw angles

- Basic Definitions



Basic Definitions

"Minimal" representation

- rotation matrices:

direct problem ↑ ↓ inverse problem

	9 elements
–	3 orthogonality relationships
–	3 unitary relationships
=	3 independent variables

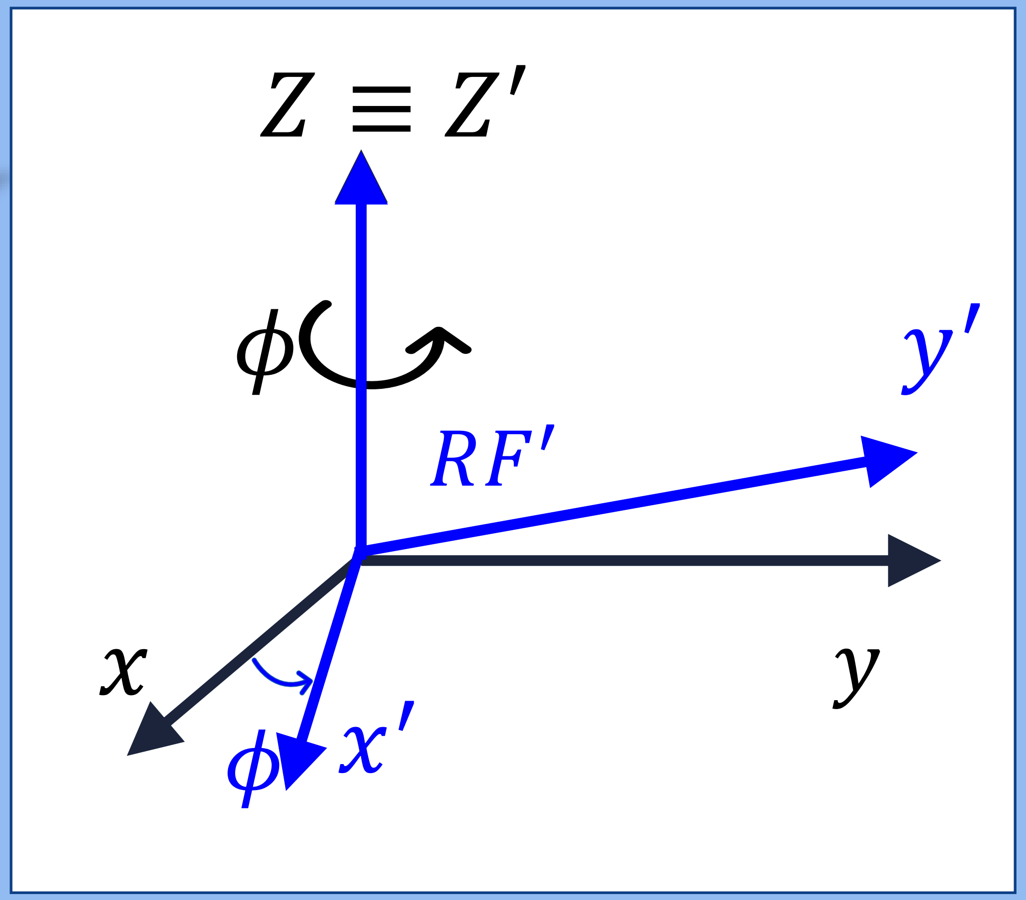
- sequence of 3 rotations w.r.t. independent axes
  - by angles $\alpha_i, i = 1,2,3$, around fixed (a_i) or moving/current (a_i') axes
    - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
  - 12+12 possible different sequences (e.g., ***XYX***)
    - without contiguous repetitions of axes (e.g., no ***XXZ*** nor ***YZ'Z'***)
  - actually, only 12 sequences are different since we shall see that

$$\{(a_1,\alpha_1),(a_2,\alpha_2),(a_3,\alpha_3)\} \equiv \{(a_3',\alpha_3),(a_2',\alpha_2),(a_1',\alpha_1)\}$$

Basic Definitions

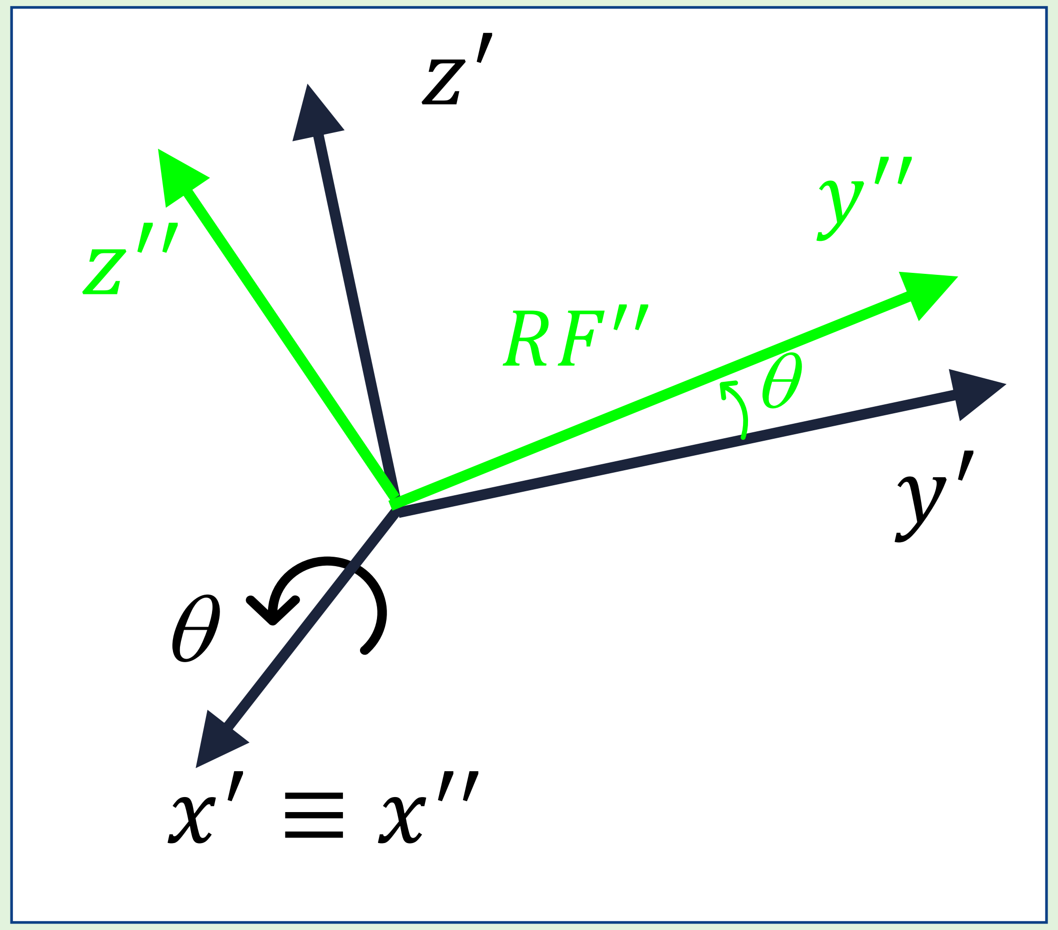
ZX'Z'' Euler angles

1



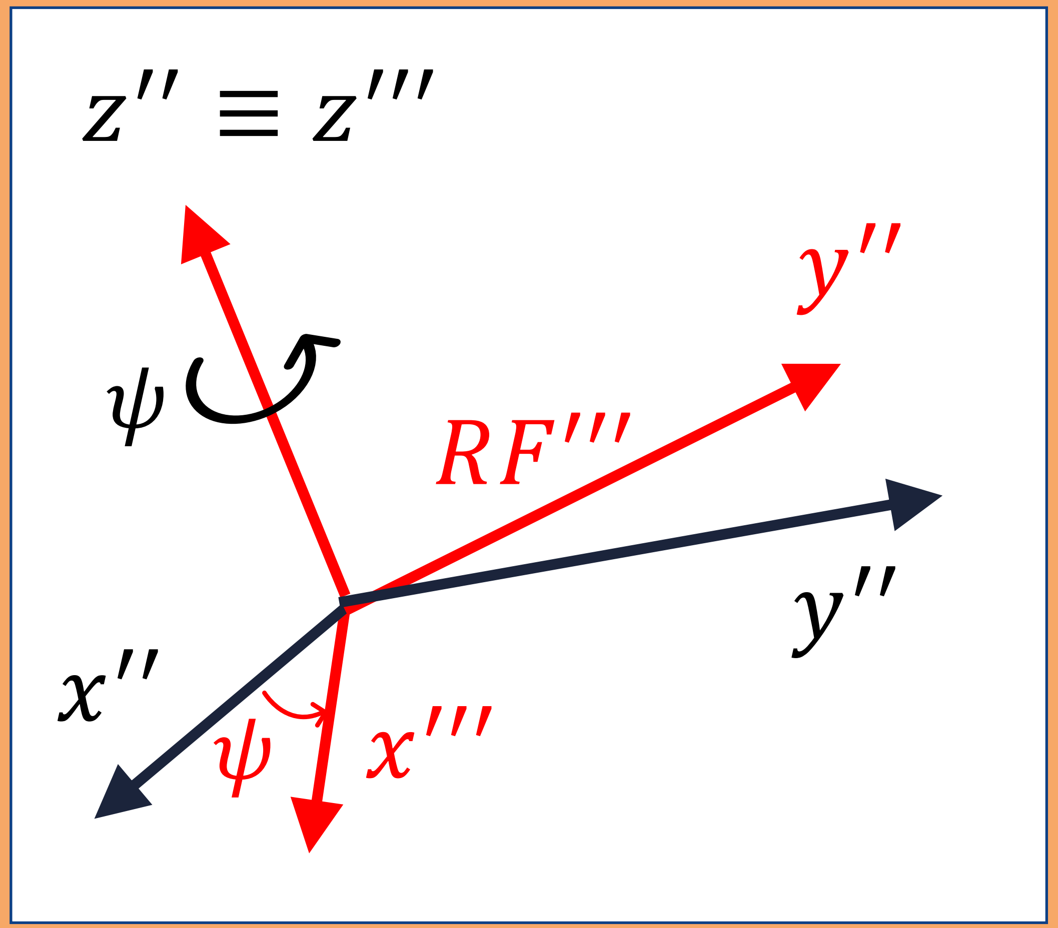
$$R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2

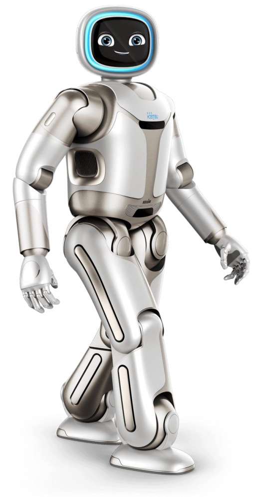


$$R_{X'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3



$$R_{Z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Basic Definitions

ZX'Z'' Euler angles

- direct problem: given ϕ, θ, ψ, find R

$$R_{ZX'Z''}(\phi,\theta,\psi) = R_Z(\phi)R_{X'}(\theta)R_{Z''}(\psi)$$

order of definition
in concatenation

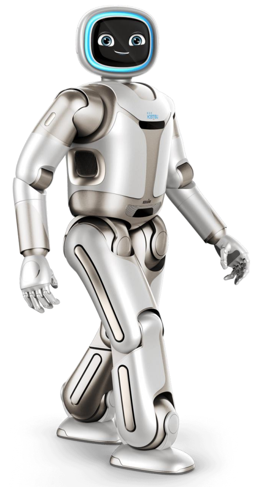
$$= \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- given a vector $v''' = (x''', y''', z''')$ expressed in RF'', its expression in the coordinates of RF is

$$v = R_{ZX'Z''}(\phi,\theta,\psi)v'''$$

- the orientation of RF is the same that would be obtained with the sequence of rotations

ψ around z, θ around x (fixed), ϕ around z (fixed)



Basic Definitions

ZX'Z'' Euler angles

- inverse problem: given $R = \{r_{ij}\}$, find ϕ, θ, ψ

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- if $r_{13}^2 + r_{23}^2 = s^2\theta, \quad r_{33} = c\theta \Rightarrow$

$$\theta = \operatorname{atan2}\left\{\pm\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right\}$$

two values differing just for the sign

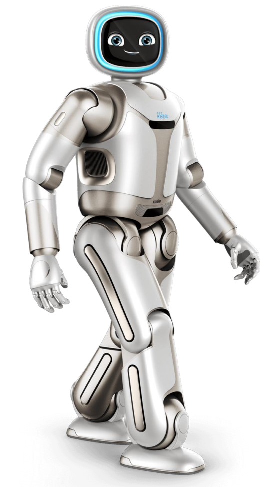
- if $r_{13}^2 + r_{23}^2 \neq 0 (i.e., s\theta \neq 0)$

  $r_{31}/s\theta = s\psi, r_{32}/s\theta = c\psi \Rightarrow$

$$\phi = \operatorname{atan2}\{r_{13}/s\theta, -r_{23}/s\theta\}$$

$$\psi = \operatorname{atan2}\{r_{31}/s\theta, r_{32}/s\theta\}$$

- similarly…
- there is always a pair of solutions in the regular case
- there are always singularities (here $\theta = 0$ *or* $\pm\pi$)$\Rightarrow$ only the sum $\phi + \psi$ or the difference $\phi - \psi$ can be determined

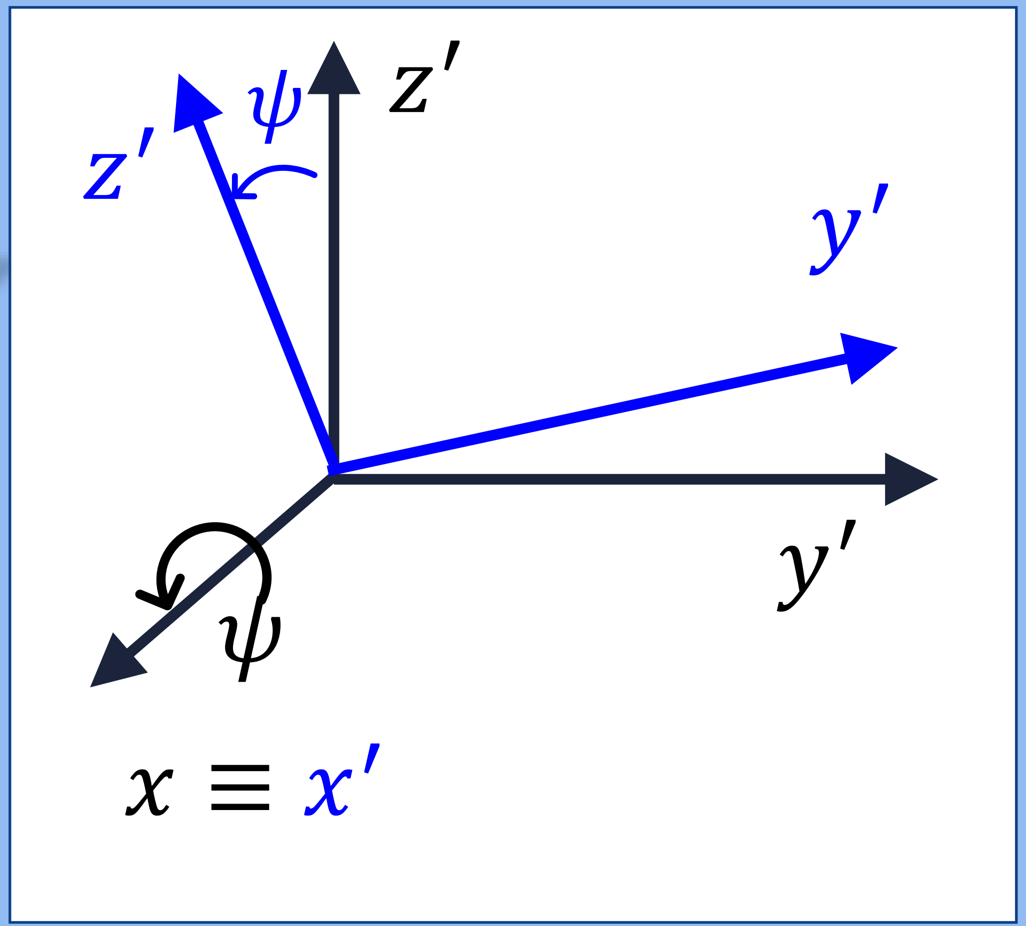


Basic Definitions

Roll-Pitch-Yaw angles(fixed XYZ)

ROLL

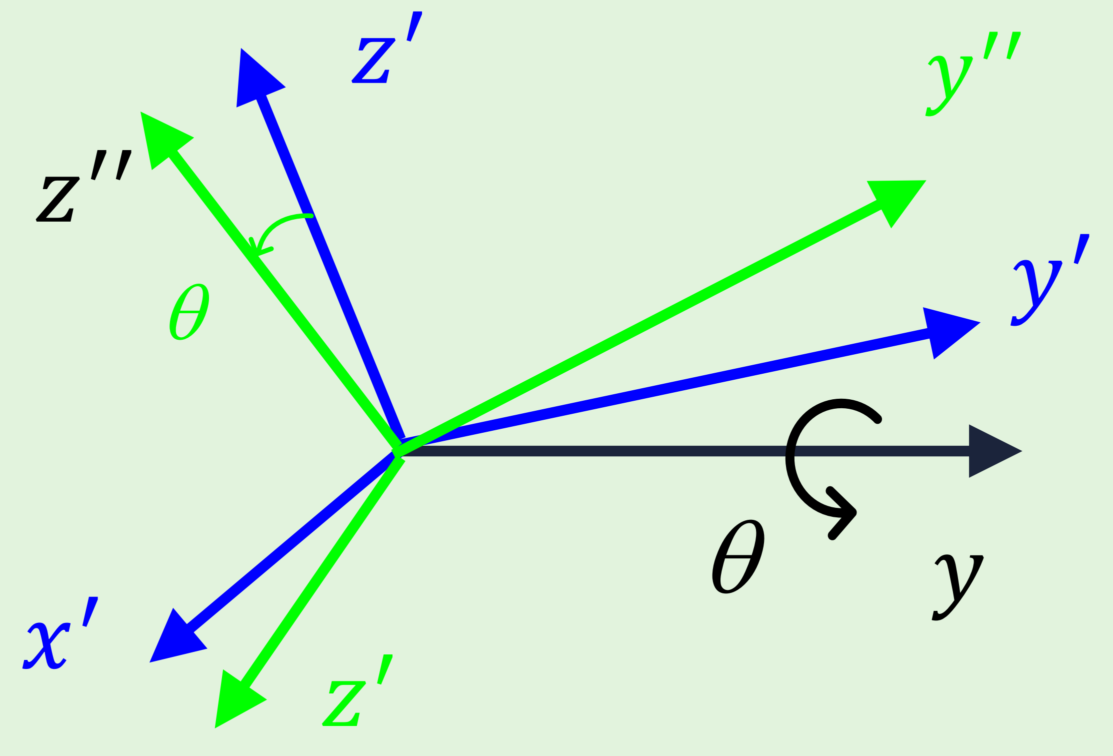
1



$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

PITCH

2



$C_1 R_Y(\theta) C_1^T$ with

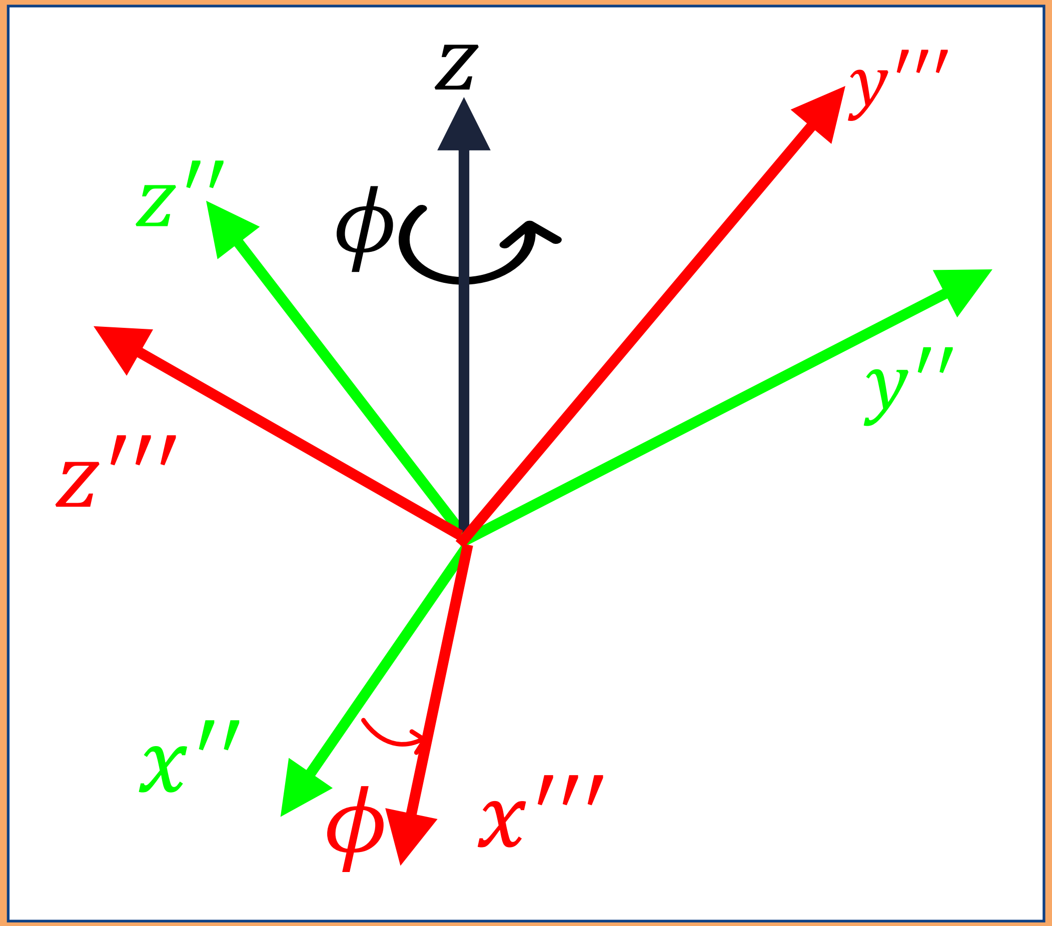
$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

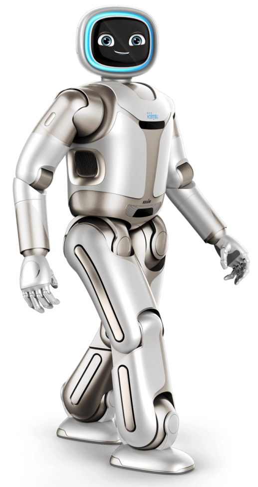
YAW

3

$C_2 R_Z(\phi) C_2^T$

with $R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$





Basic Definitions

Roll-Pitch-Yaw angles (fixed XYZ)

- direct problem: given ϕ, θ, ψ, find R

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi) \Leftarrow$$

note the order of products!

order of definition

$$= \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}$$

- inverse problem: given $R = \{r_{ij}\}$, find ϕ, θ, ψ

- $r_{32}^2 + r_{33}^2 = c^2\theta, r_{31} = -s\theta \Rightarrow$

$$\theta = \text{atan}\,2\left\{-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\right\}$$

for $r_{31} < 0$, two symmetric values $w.r.t.\ \pi/2$

- if $r_{32}^2 + r_{33}^2 \neq 0 (i.e., c\theta \neq 0)$

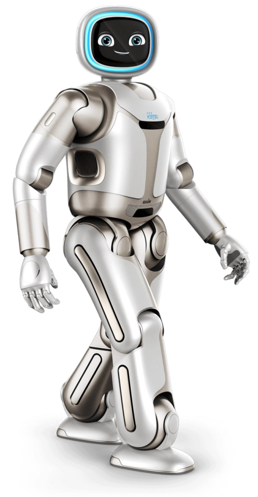
  $r_{31}/s\theta = s\psi, r_{32}/s\theta = c\psi \Rightarrow$

$$\psi = \text{atan}\,2\{r_{32}/c\theta, r_{33}/c\theta\}$$

- similarly…

$$\phi = \text{atan}\,2\{r_{21}/c\theta, -r_{11}/c\theta\}$$

- singularities for $\theta = \pm\pi/2 \Rightarrow$ only $\phi + \psi$ or $\phi - \psi$ are defined



Basic Definitions

… why this order in the product

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi)$$

order of definition

"reverse" order in the product

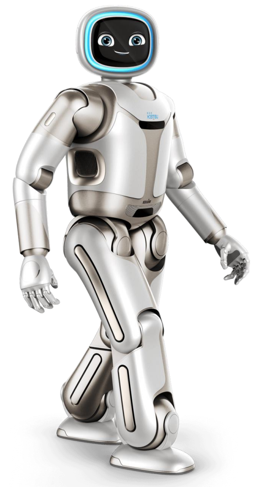
(pre-multiplication…)

- need to refer each rotation in the sequence to one of the original fixed axes
  - use the angle/axis technique for each rotation in the sequence:

  $CR(\alpha)C^T$, with C being the rotation matrix reverting the previously made rotations (= "go back" to the original axes)

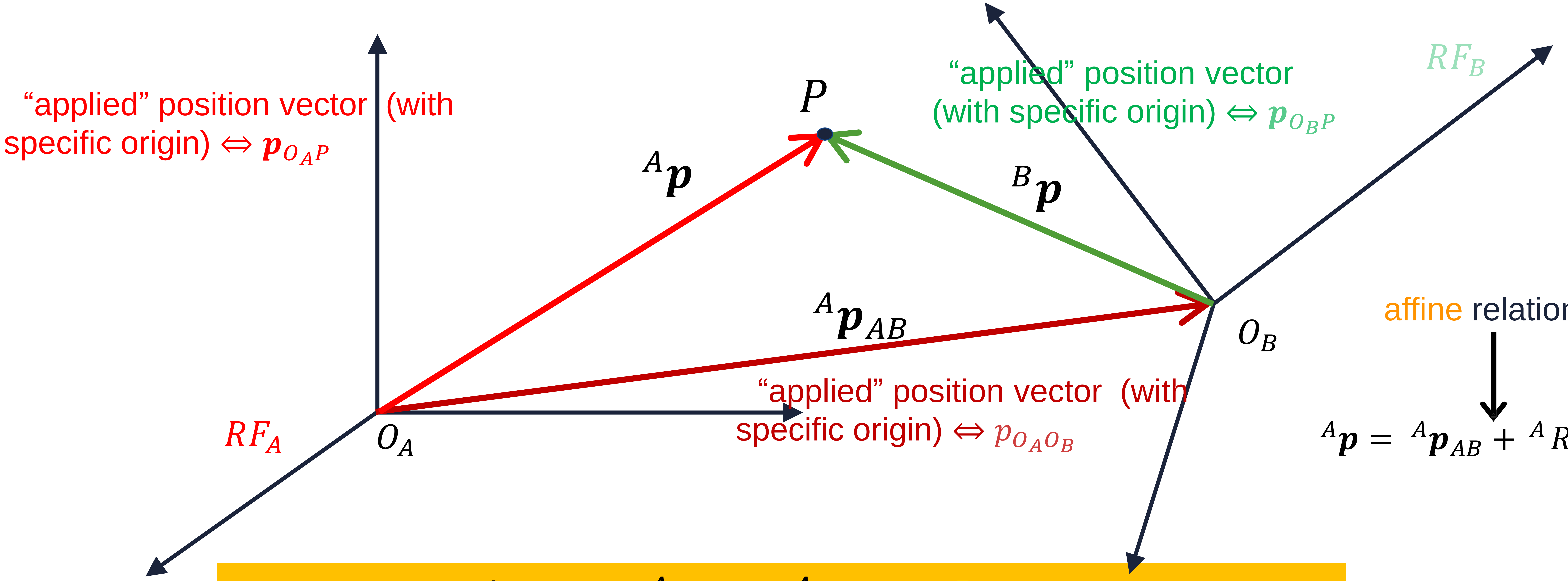
  concatenating three rotations: [ ] [ ] [ ] (post-multiplication…)

$$\begin{aligned} R_{RPY}(\psi,\theta,\phi) \\ &= [R_X(\psi)][R_X^T(\psi)R_Y(\theta)R_X(\psi)][R_X^T(\psi)R_Y^T(\theta)R_Z(\phi)R_Y(\theta)R_X(\psi)] \\ &= R_Z(\phi)R_Y(\theta)R_X(\psi) \end{aligned}$$



Basic Definitions

Homogeneous transformations



affine relationship

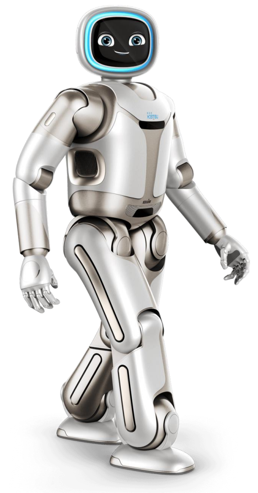
$$^{A}\boldsymbol{p} = {}^{A}\boldsymbol{p}_{AB} + {}^{A}R_B\, {}^{B}\boldsymbol{p}$$

$$^{A}\boldsymbol{p}_{hom} = \begin{bmatrix} ^{A}\boldsymbol{p} \\ 1 \end{bmatrix} = \begin{bmatrix} ^{A}R_B & ^{A}\boldsymbol{p}_{AB} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} ^{B}\boldsymbol{p} \\ 1 \end{bmatrix} = {}^{A}T_B\, {}^{A}\boldsymbol{p}_{hom}$$

linear relationship

vector in homogeneous coordinates

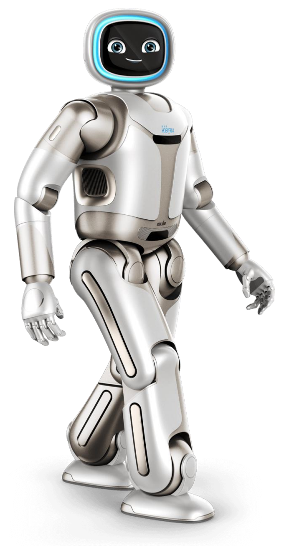
4×4 matrix of homogeneous transformation



Basic Definitions

Use of homogeneous transformation $\boldsymbol{T}$

- describes the relation between two reference frames (relative pose = position & orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from one frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible $({}^{A}T_{B})^{-1} = ({}^{B}T_{A})$
- can be composed, i.e., ${}^{A}T_{B}\,{}^{B}T_{C} = {}^{A}T_{C}$ ⟵ note: it does not commute in general!



Basic Definitions

Inverse of a homogeneous transformation

exchange $A \leftrightarrows B$

…with the original vectors/matrices…

$$ {}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{p}_{AB} + {}^{A}R_{B}\, {}^{B}\boldsymbol{p} $$

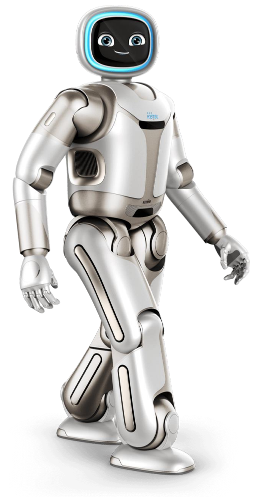
$$ {}^{B}\boldsymbol{p} = {}^{B}\boldsymbol{p}_{BA} + {}^{B}R_{A}\, {}^{A}\boldsymbol{p} = -{}^{A}R_{B}^{T}\, {}^{A}\boldsymbol{p}_{AB} + {}^{A}R_{B}^{T}\, {}^{A}\boldsymbol{p} $$

$$ \left[\begin{array}{ccc:c} & {}^{A}R_{B} & & {}^{A}\boldsymbol{p}_{AB} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{A}T_{B} $$

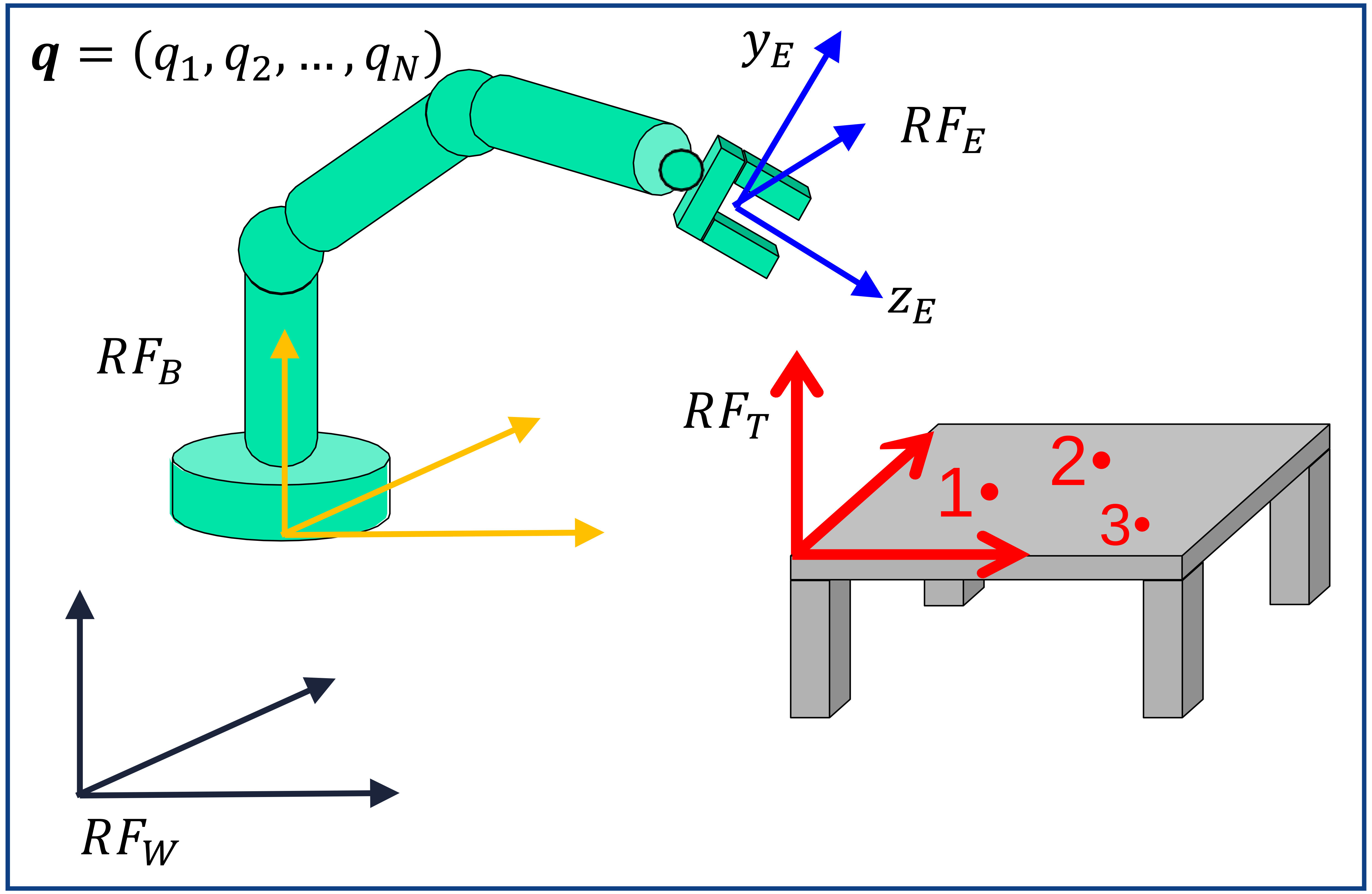
$$ \left[\begin{array}{ccc:c} & {}^{B}R_{A} & & {}^{B}\boldsymbol{p}_{BA} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc:c} & {}^{A}R_{B}^{T} & & -{}^{A}R_{B}^{T}\, {}^{A}\boldsymbol{p}_{AB} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{B}T_{A} \qquad \left({}^{A}T_{B}\right)^{-1} $$



Basic Definitions

Defining a robotic task



absolute definition of task

task definition relative to the robot end-effector

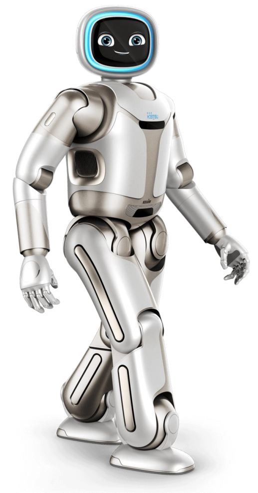
$$^{W}T_T = {}^{W}T_B \, {}^{B}T_E \, {}^{E}T_T$$

known, once the robot is placed

direct kinematics of the robot arm (function of $\boldsymbol{q}$)

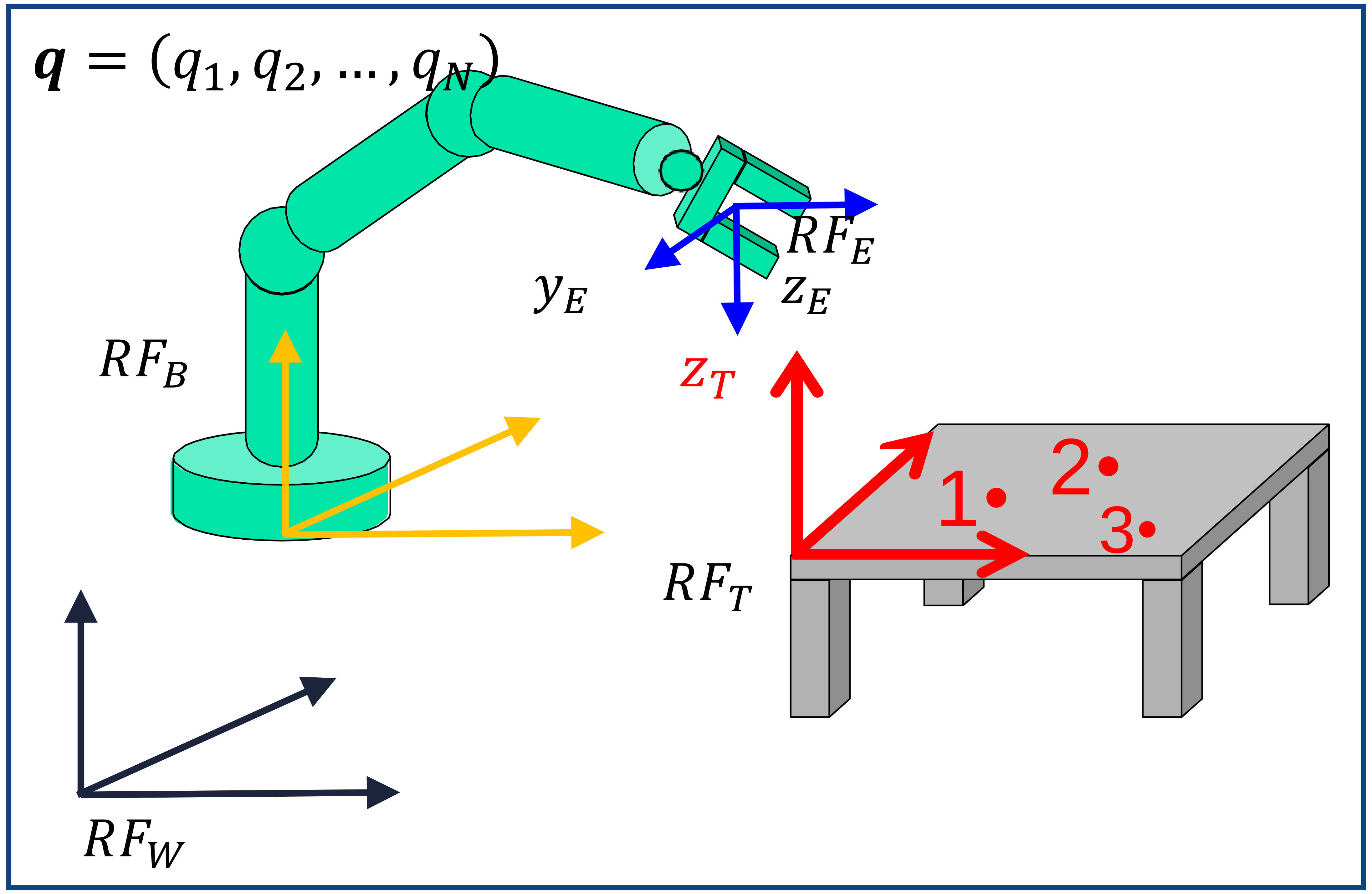
$$^{B}T_E(\boldsymbol{q}) = {}^{W}T_B^{-1} \, {}^{W}T_T \, {}^{E}T_T^{-1} = constant$$

solve for $\boldsymbol{q}$ (inverse kinematics)



Basic Definitions

Example of task definition



- the robot carries a depth camera (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point P on the table, pointing its approach axis z_E downward and being aligned with the table sides

$$ {}^{E}R_T = [\,{}^{E}\mathrm{x}_T \quad {}^{E}\mathrm{y}_T \quad {}^{E}\mathrm{z}_T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} $$

- point $\boldsymbol{P}$ is known in the table frame RF_T

$$ {}^{T}\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} $$

- the robot proceeds by centering point $\boldsymbol{P}$ in its camera image until it senses a depth h from the table (in RF_E)

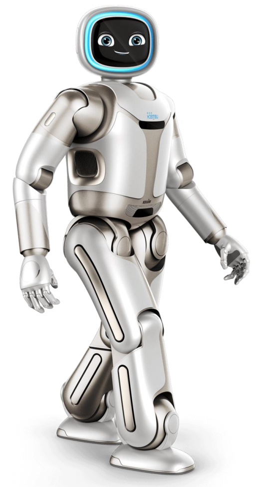
$$ {}^{E}\boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} $$

- Q: where is the EE frame w.r.t. the table frame?

$$ {}^{T}T_E = \begin{bmatrix} {}^{T}R_E & {}^{T}\boldsymbol{p}_{TE} \\ 0^T & 1 \end{bmatrix} = {}^{E}T_T^{-1} $$

- with

$$ {}^{T}R_E = ({}^{E}R_T)^T = {}^{E}R_T \qquad {}^{T}\boldsymbol{p}_{TE} = {}^{T}\boldsymbol{p} - {}^{T}R_E\,{}^{E}\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ h \end{bmatrix} $$



Basic Definitions

Remarks on homogeneous matrices

- the main tool used for computing the direct kinematics of robot manipulators
- relevant in many other applications (in robotics and beyond)
  - in positioning/orienting a vision camera (matrix ${}^{b}T_{c}$ with extrinsic parameters of the camera pose
  - in computer graphics, for the real-time visualization of 3D solid objects when changing the observation point

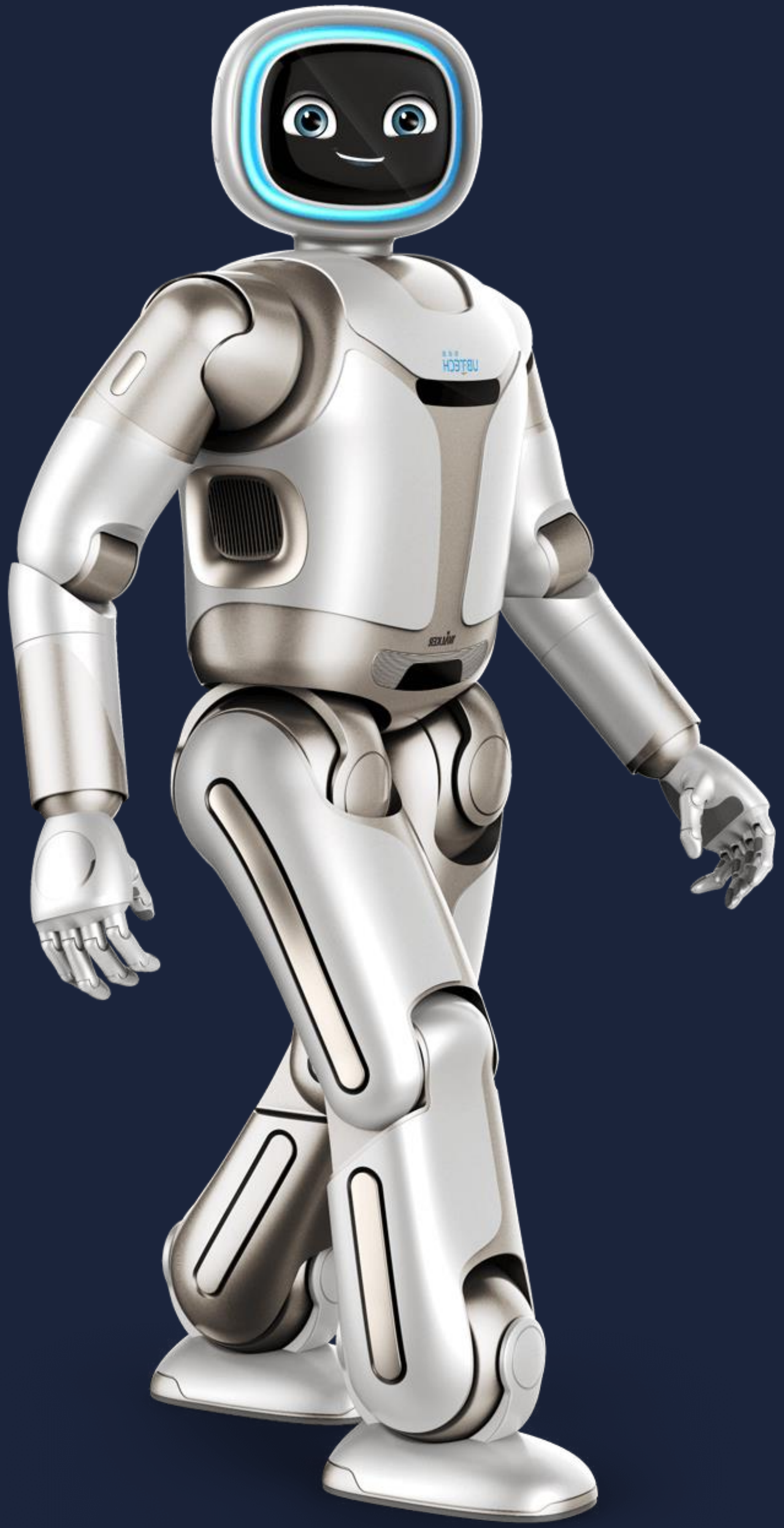
$$ {}^{A}T_{B} = \begin{bmatrix} & {}^{A}R_{B} & & {}^{A}p_{AB} \\ \alpha_x & \alpha_y & \alpha_z & \sigma \end{bmatrix} $$

all zero in robotics

coefficients of perspective deformation

scaling coefficient

always unitary in robotics



Q&A