Advanced Robotics

ENGG5402 Spring 2023

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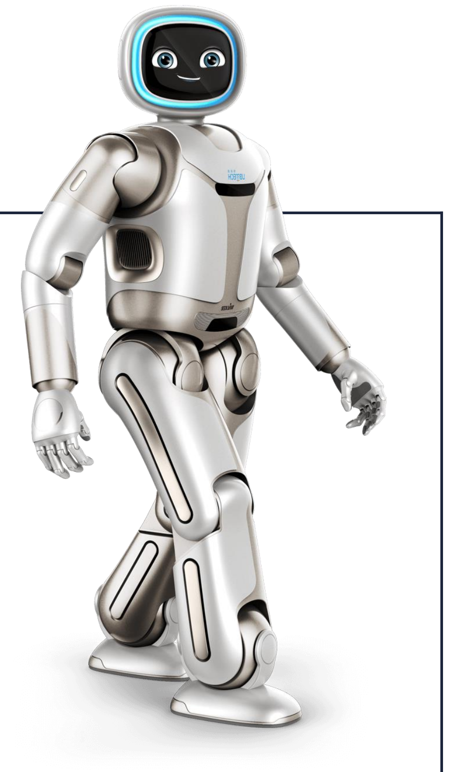
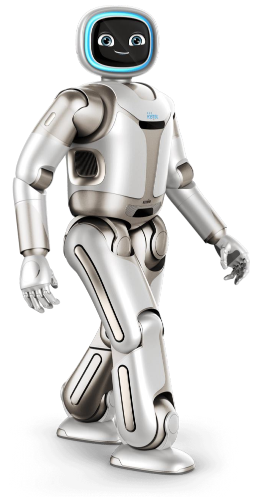


Topics:

- Differential Kinematics

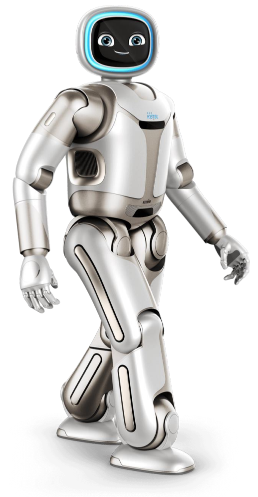
Readings:

- Siciliano: Sec. 3

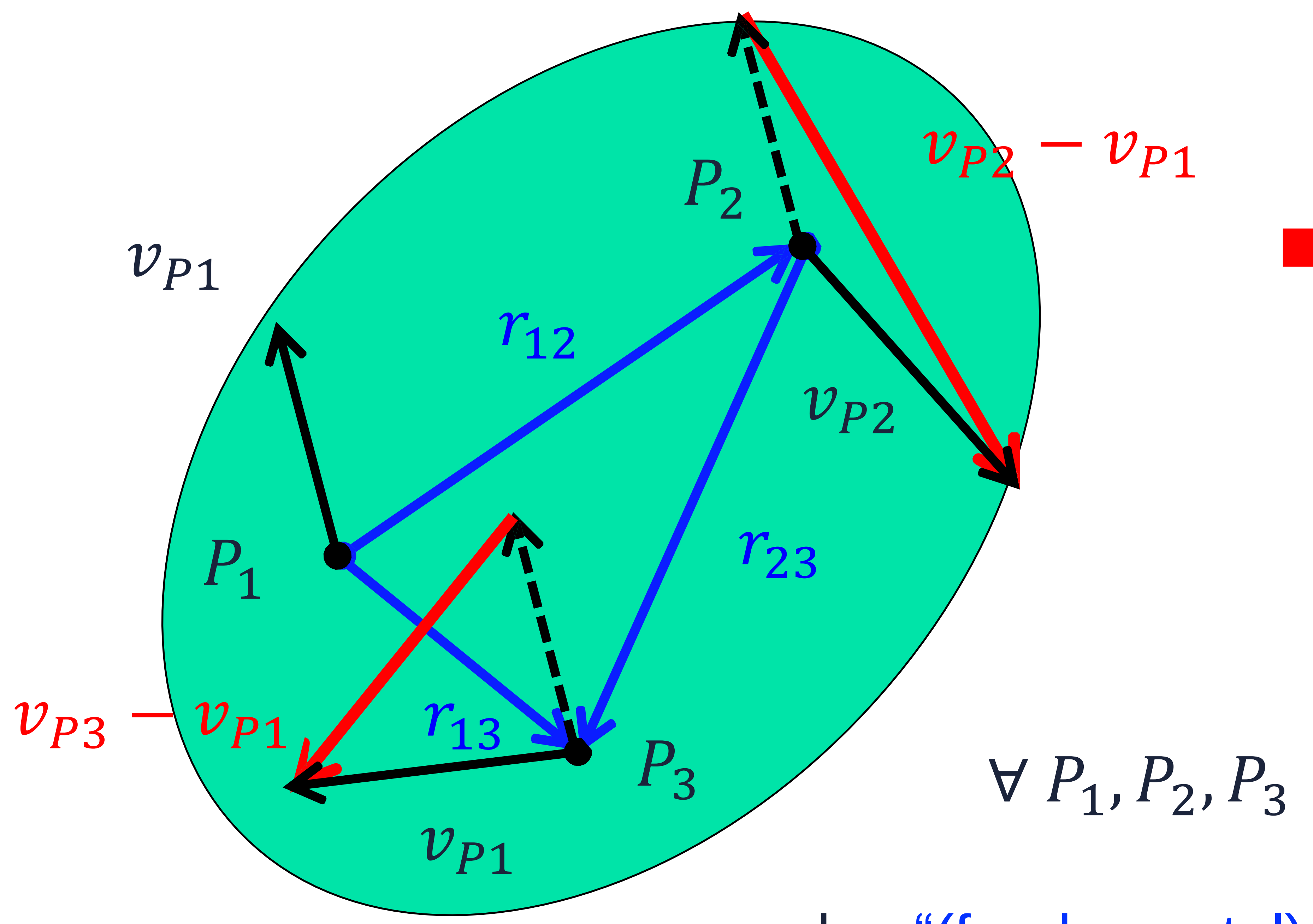


Differential Kinematics

- relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)
- instantaneous velocity mappings can be obtained through time differentiation of the direct kinematics or in a geometric way, directly at the differential level
  - different treatments arise for rotational quantities
  - establish the relation between angular velocity and
    - time derivative of a rotation matrix
    - time derivative of the angles in a minimal representation of orientation



Angular Velocity of Rigid Body



$\forall\, P_1, P_2, P_3$

"rigidity" constraint on distances among points:

$$\|r_{ij}\| = constant$$

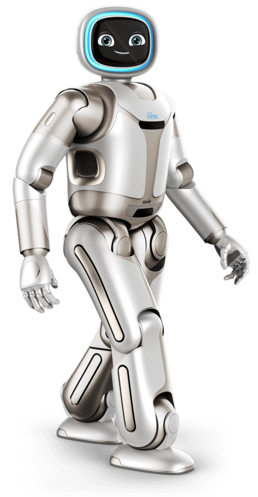
$\Rightarrow$ $v_{Pi} - v_{Pj}$ orthogonal to r_{ij}

1	$v_{P2} - v_{P1} = \omega_1 \times r_{12}$
2	$v_{P3} - v_{P1} = \omega_1 \times r_{13}$
3	$v_{P3} - v_{P2} = \omega_2 \times r_{23}$
2-1=3	$\Rightarrow$ $\omega_1 = \omega_2 = \omega$

aka, "(fundamental) kinematic equation" of rigid bodies

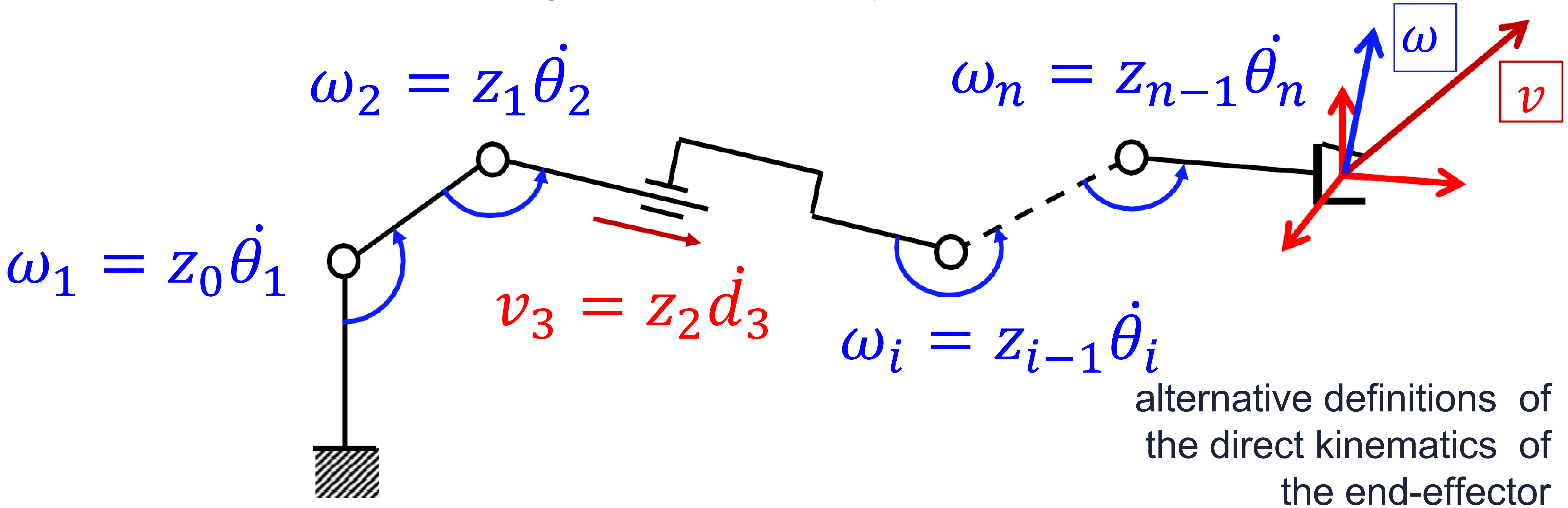
$$\boxed{v_{Pj} = v_{Pi} + \omega \times r_{ij} = v_{Pi} + S(\omega) r_{ij}} \Leftrightarrow \boxed{\dot{r}_{ij} = \omega \times r_{ij}}$$

- the angular velocity ω is associated to the whole body (**not** to a point)
- if $\exists P_1, P_2$: $v_{P1} = v_{P2} = 0 \Rightarrow$ pure rotation (circular motion of all $P_j \notin$ line P_1P_2)
- $\omega = 0 \Rightarrow$ pure translation (**all** points have the same velocity v_P)



Velocity of End-Effector

Linear and angular velocity of the robot end-effector



alternative definitions of the direct kinematics of the end-effector

$$T = \begin{bmatrix} R & p \\ 0^T & 1 \end{bmatrix}$$
$$r = (p, \phi)$$

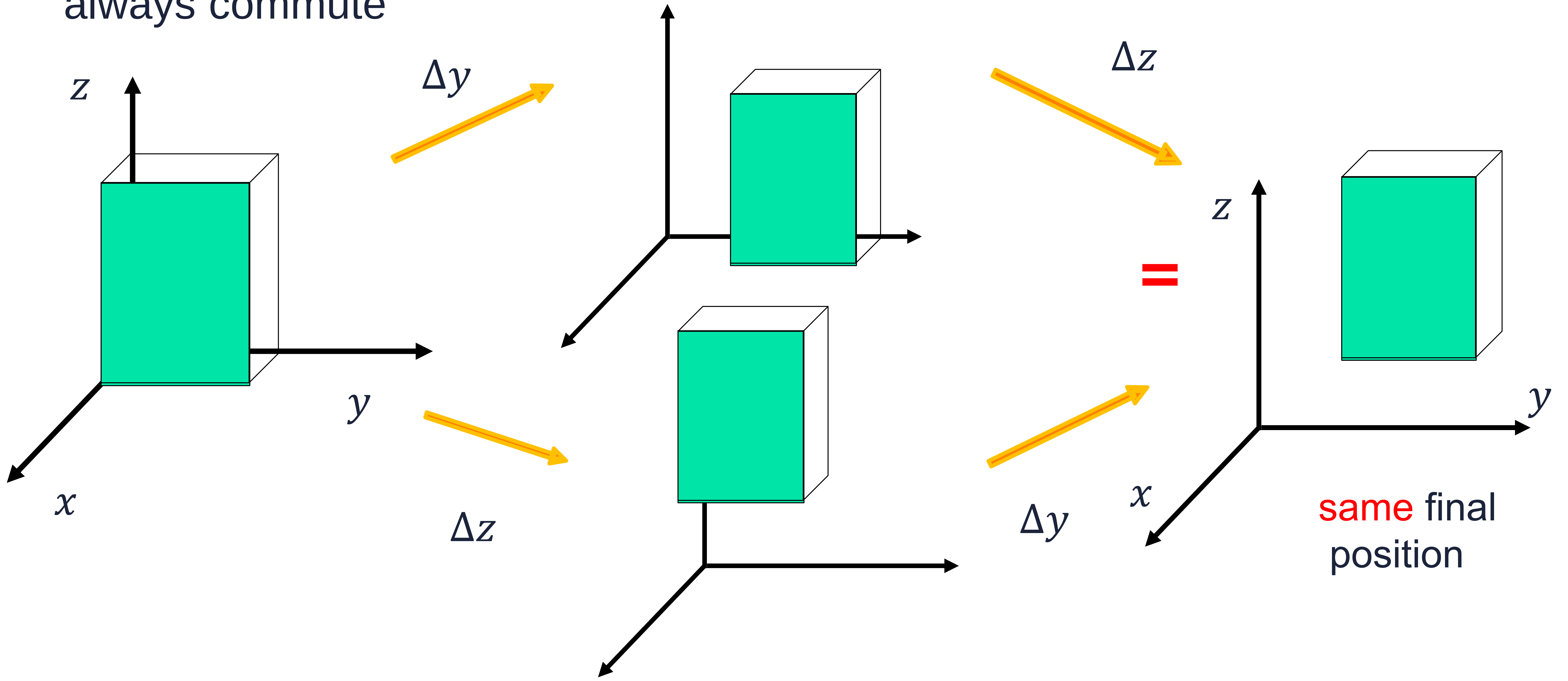
- v and ω are "vectors", namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - such contributions will be given by the single (linear or angular) joint velocities
- on the other hand, ϕ (and $\dot{\phi}$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

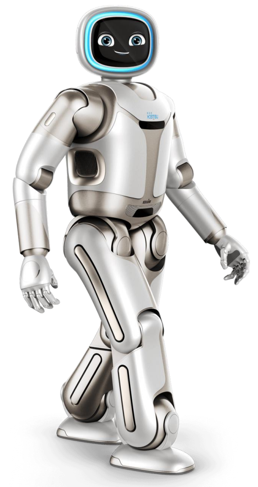
in general, $\omega \neq \dot{\phi}$

Translations

Finite and infinitesimal translations

- finite $\Delta x, \Delta y, \Delta z$ or infinitesimal dx, dy, dz translations (linear displacements) always commute



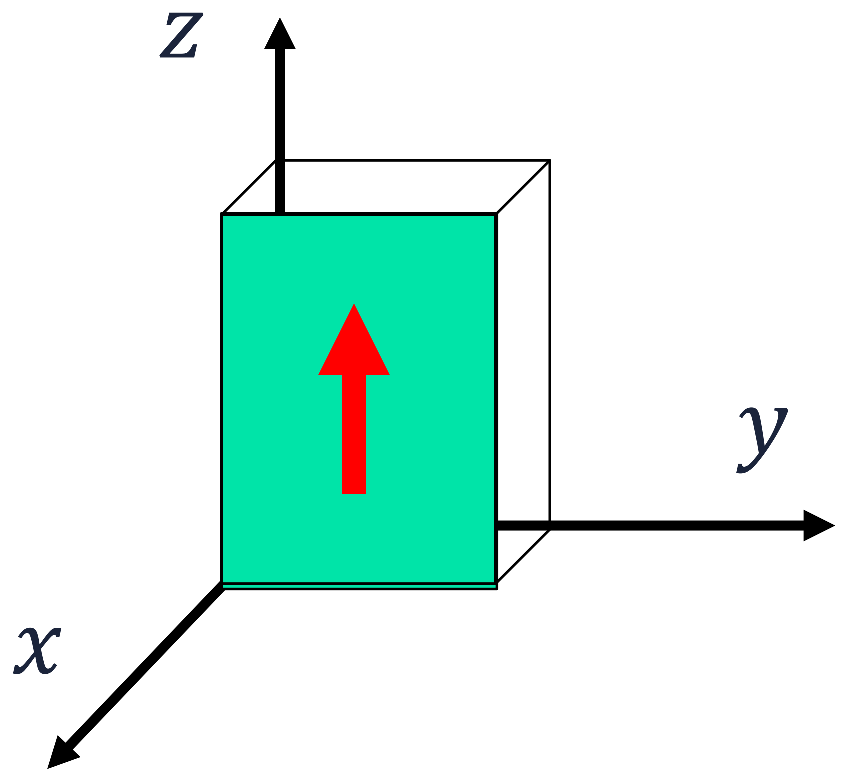


Translations

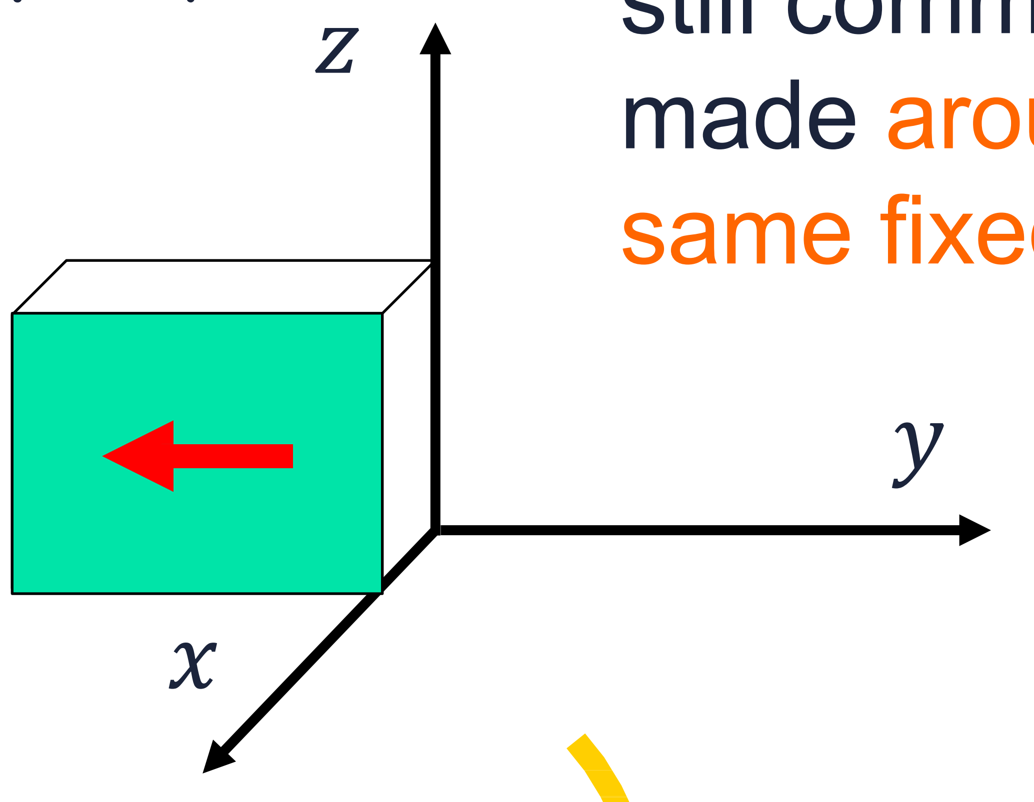
Finite rotations do not commute (example)

note: finite rotations still commute when made around the same fixed axis

initial orientation

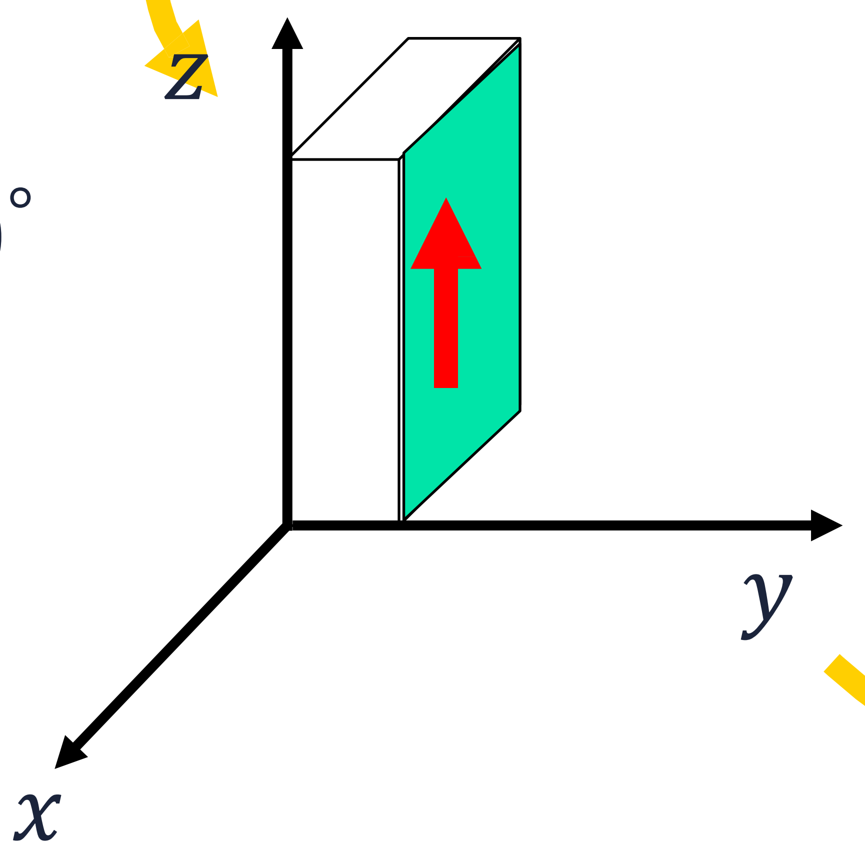


$\phi_x = 90^\circ$

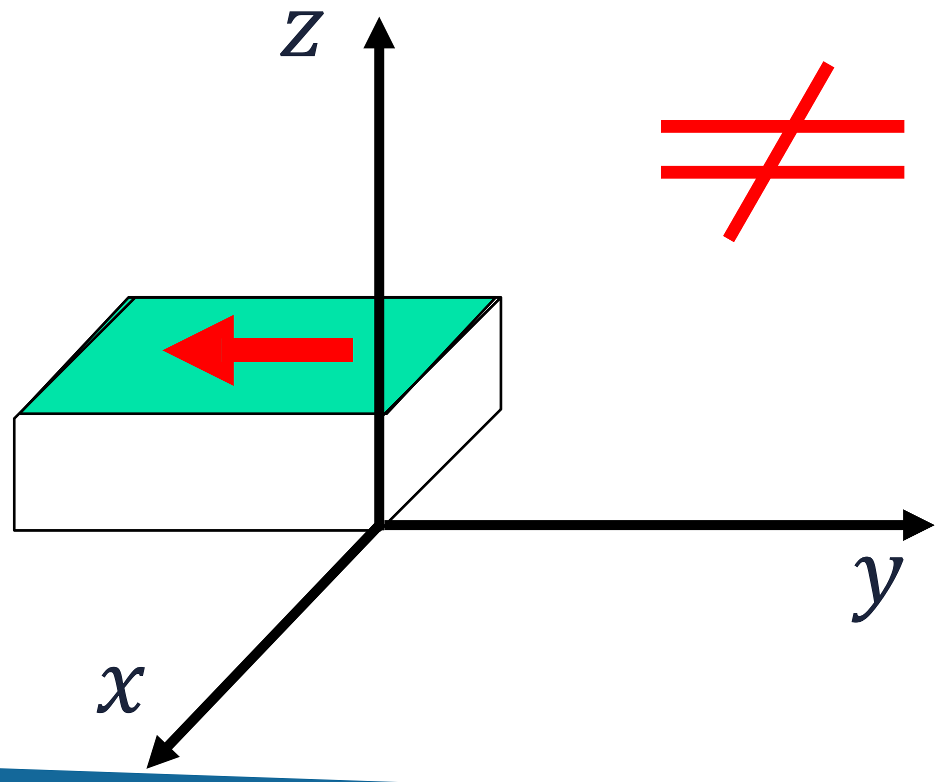


$\phi_z = 90^\circ$

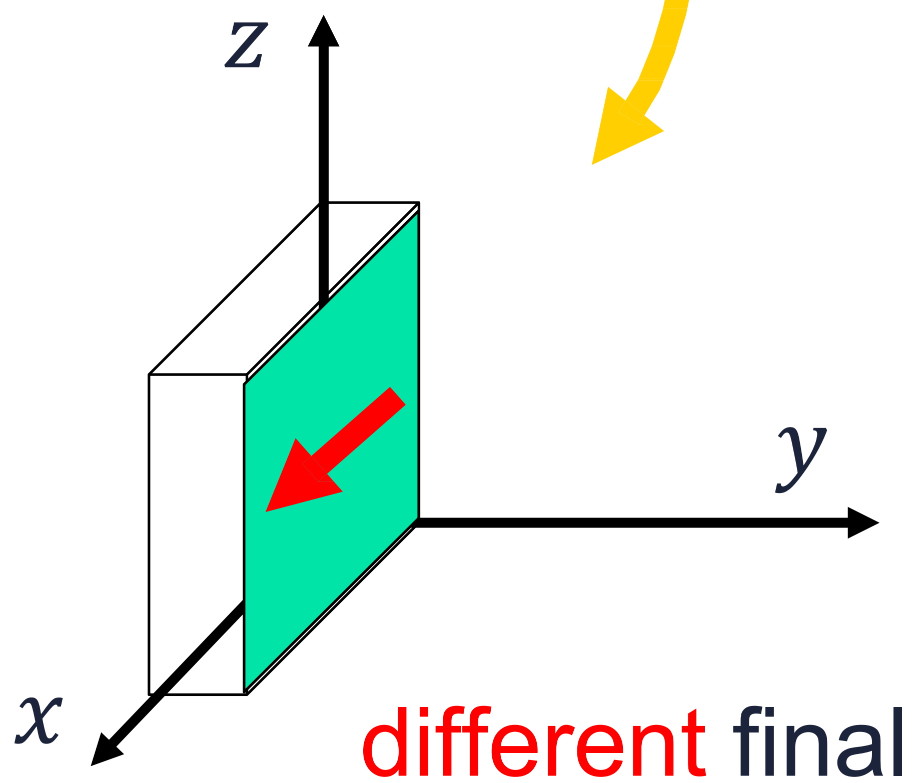
$\phi_z = 90^\circ$



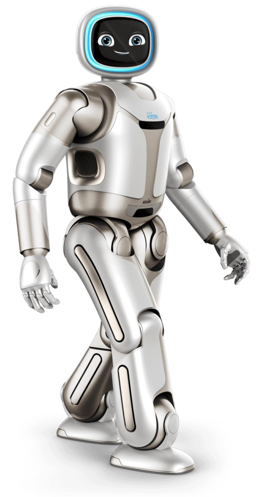
mathematical fact: ω is NOT an exact differential form (the integral of ω over time depends on the integration path!)



$\neq$



different final orientations!



Infinitesimal Rotations

Infinitesimal rotations commute!

- infinitesimal rotations $d\phi_X, d\phi_Y, d\phi_Z$ around x, y, z axes

$$R_X(\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_X & -\sin\phi_X \\ 0 & \sin\phi_X & \cos\phi_X \end{bmatrix} \Rightarrow R_X(d\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_X \\ 0 & d\phi_X & 1 \end{bmatrix}$$

$$R_y(\phi_y) = \begin{bmatrix} \cos\phi_y & 0 & \sin\phi_y \\ 0 & 1 & 0 \\ -\sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \Rightarrow R_Y(d\phi_Y) = \begin{bmatrix} 1 & 0 & d\phi_Y \\ 0 & 1 & 0 \\ -d\phi_Y & 0 & 1 \end{bmatrix}$$

$$R_z(\phi_z) = \begin{bmatrix} \cos\phi_z & -\sin\phi_z & 0 \\ \sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_Z(d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & 0 \\ d\phi_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(d\phi) = R(d\phi_X, d\phi_Y, d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & d\phi_Y \\ d\phi_Z & 1 & -d\phi_X \\ -d\phi_Y & d\phi_X & 1 \end{bmatrix} \leftarrow$$

neglecting second- and third-order (infinitesimal) terms

in any order

$$= I + S(d\phi)$$

Time Derivative of R

- let $R = R(t)$ be a rotation matrix, given as a function of time
- since $I = R(t)R^T(t)$ taking the time derivative of both sides yields

$$0 = \frac{d\left(R(t)R^T(t)\right)}{dt} = \left(\frac{dR(t)}{dt}\right)R^T(t) + R(t)\left(\frac{dR^T(t)}{dt}\right)$$

$$= \left(\frac{dR(t)}{dt}\right)R^T(t) + ((dR(t)/dt)R^T(t))^T$$

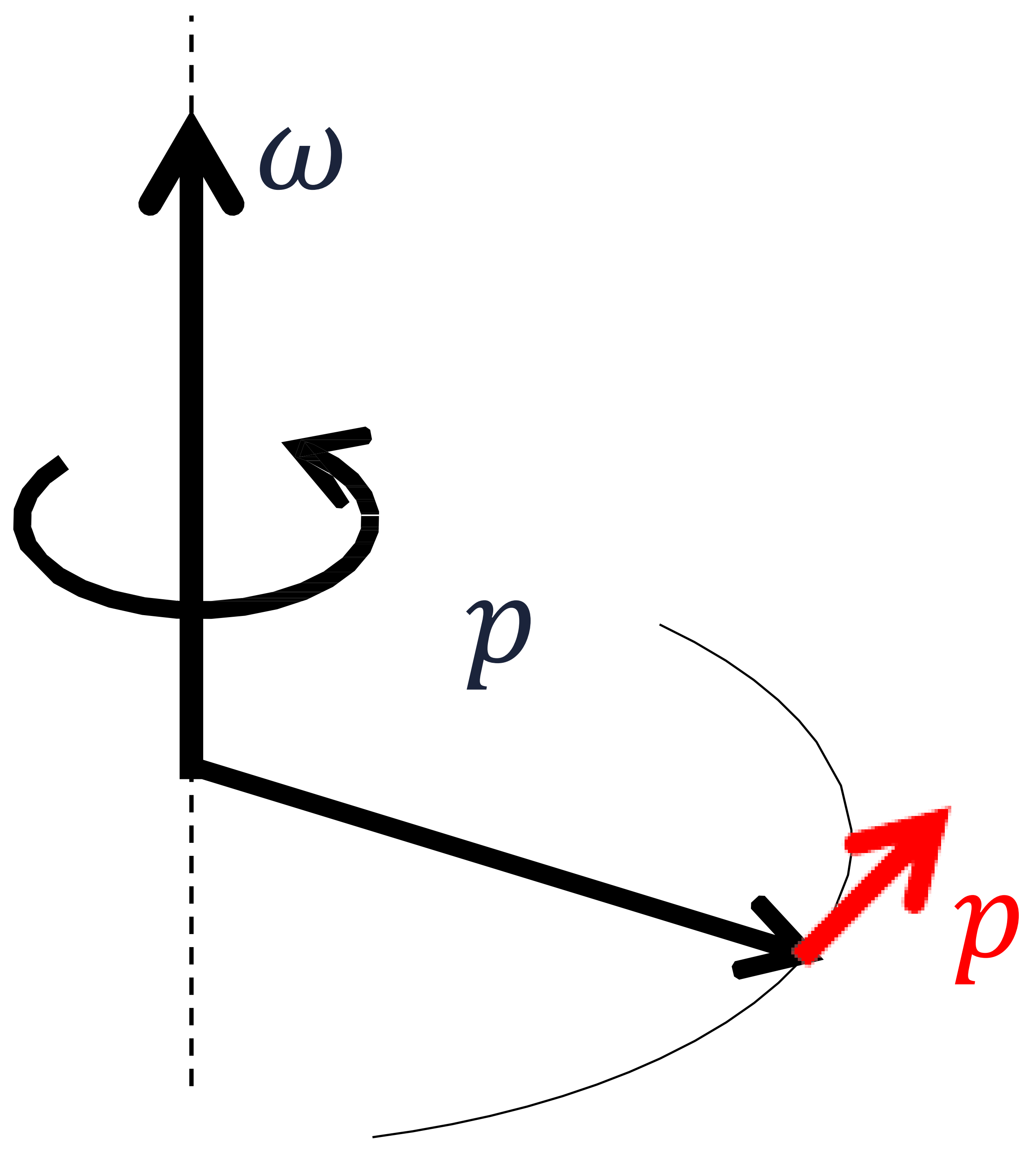
  - thus $\left(\frac{\mathrm{dR(t)}}{\mathrm{dt}}\right)R^T\ (t)\ = S(t)$ is a skew-symmetric matrix
- let $p(t) = R(t)p'$ a vector (with constant norm) rotated over time
- comparing

$$\dot{p}(t) = (dR(t)/dt)p' = S(t)R(t)p' = S(t)p(t)$$

$$\dot{p}(t) = \omega(t) \times p(t) = S(\omega(t))p(t)$$

- we get $S = S(\omega)$

$$\boxed{\dot{R} = S(\omega)R} \Longleftrightarrow \boxed{S(\omega) = \dot{R}R^T}$$



Time Derivative of R

Example (Time derivative of an elementary rotation matrix)

$$R_X(\phi(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi\,(t) & -\sin\phi\,(t) \\ 0 & \sin\phi\,(t) & \cos\phi\,(t) \end{bmatrix}$$

$$\dot{R}_X(\phi)R_X^T(\phi) = \dot{\phi}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{bmatrix} = S(\omega)$$

$$\Rightarrow \quad \omega = \omega_X = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

more in general, for the axis/angle rotation matrix

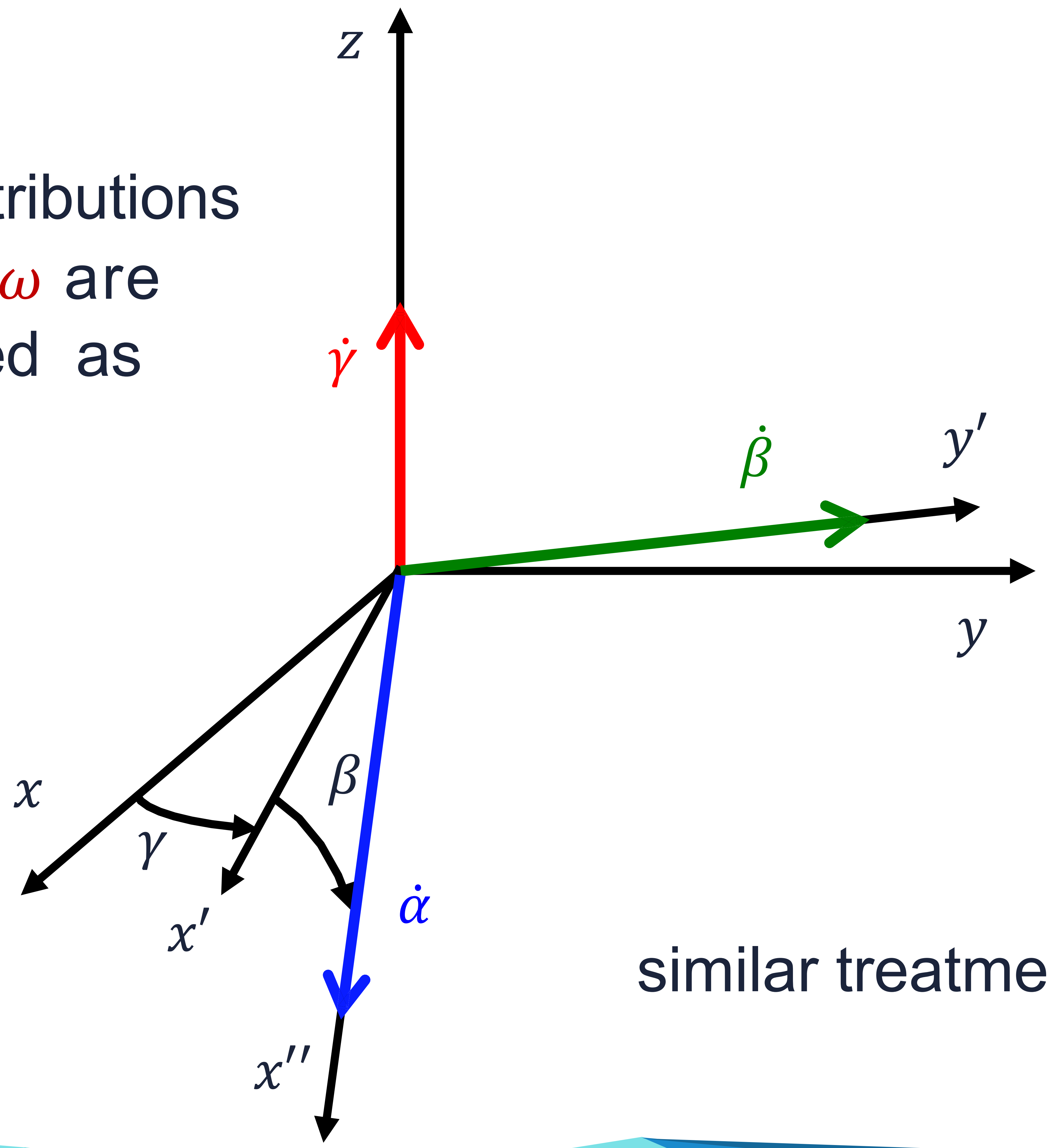
$$R(r,\theta(t)) \Longrightarrow \dot{R}(r,\theta)R^T(r,\theta) = S(\omega) \quad \Rightarrow \quad \omega = \omega_r = \dot{\theta}r = \dot{\theta}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

RPY Angles and $\boldsymbol{\omega}$

Time derivative of RPY angles and $\boldsymbol{\omega}$

$$R_{RPY}(\alpha_X, \beta_Y, \gamma_Z) = R_{ZY'X''}(\gamma_Z, \beta_Y, \alpha_X) = R_Z(\gamma) R_{Y'}(\beta) R_{X''}(\alpha)$$

the three contributions $\dot{\gamma} Z, \dot{\beta} Y', \dot{\alpha} X''$ to ω are simply summed as vectors



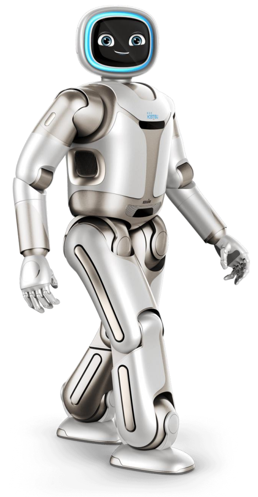
$$\omega = \overbrace{\begin{bmatrix} c\beta c\gamma & -s\gamma & 0 \\ c\beta s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \end{bmatrix}}^{T_{RPY}(\beta,\gamma)} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

$$\begin{array}{ccc} X'' & Y'' & Z \\ \uparrow & \uparrow & \end{array}$$

1st col in $R_Z(\gamma) R_{Y'}(\beta)$

2nd col in $R_Z(\gamma)$

similar treatment for the other 11 minimal representations...



Jacobian

Robot Jacobian matrices

- analytical Jacobian (obtained by time differentiation)

$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} = f_r(q) \quad \Rightarrow \quad \dot{r} = \begin{pmatrix} \dot{p} \\ \dot{\phi} \end{pmatrix} = \frac{\partial f_r(q)}{\partial q} \dot{q} = J_r(q)\dot{q}$$

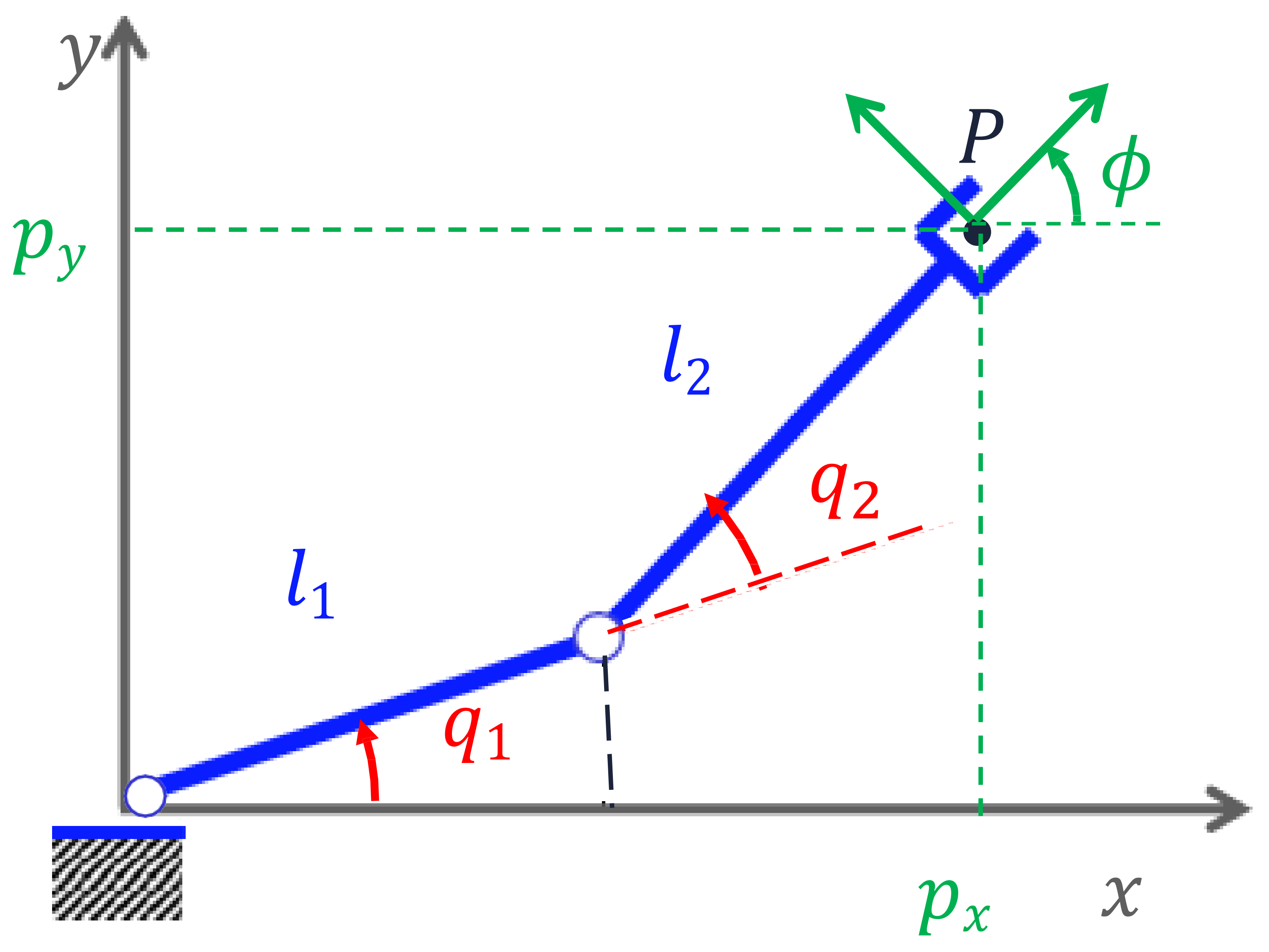
- geometric or basic Jacobian (no derivatives)

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = J(q)\dot{q}$$

- in both cases, the Jacobian matrix depends on the (current) configuration of the robot

Analytical Jacobian of Planar 2R Robot

Equivalent interpretations of a rotation matrix



direct kinematics

$$\begin{cases} p_x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ p_y = l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \\ \phi = q_1 + q_2 \end{cases}$$

$$\dot{p}_x = -l_1 s_1 \dot{q}_1 - l_2 s_{12}(\dot{q}_1 + \dot{q}_2)$$

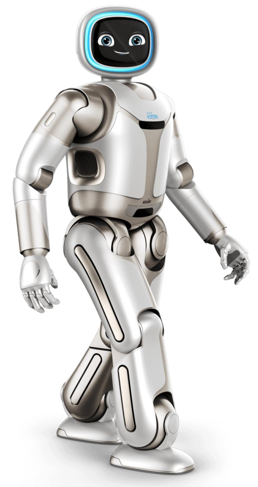
$$\dot{p}_y = l_1 c_1 \dot{q}_1 + l_2 c_{12}(\dot{q}_1 + \dot{q}_2)$$

$$\dot{\phi} = \omega_z = \dot{q}_1 + \dot{q}_2$$

$$J_r(q) = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{pmatrix}$$

given r, this is a 3×2 matrix

here, all rotations occur around the same fixed axis z (normal to the plane of motion)



Geometric Jacobian

Geometric Jacobian

always a $6 \times n$ matrix

end-effector
instantaneous velocity

$$\begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \cdots & J_{Ln}(q) \\ J_{A1}(q) & \cdots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

superposition of effects

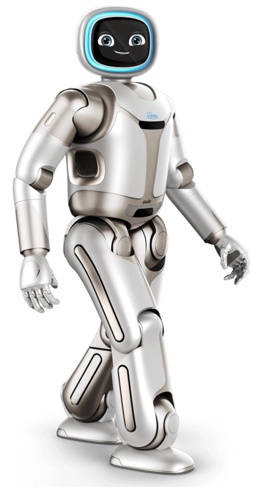
$$v_E = \underbrace{J_{L1}(q)\dot{q}_1}_{} + \cdots + J_{Ln}(q)\dot{q}_n$$

$$\omega_E = \underbrace{J_{A1}(q)\dot{q}_1}_{} + \cdots + J_{An}(q)\dot{q}_n$$

contribution to the linear e-e velocity due to $\dot{q}_1$

contribution to the angular e-e velocity due to $\dot{q}_1$

linear and angular velocity belong to (linear) vector spaces in $\mathbb{R}^3$

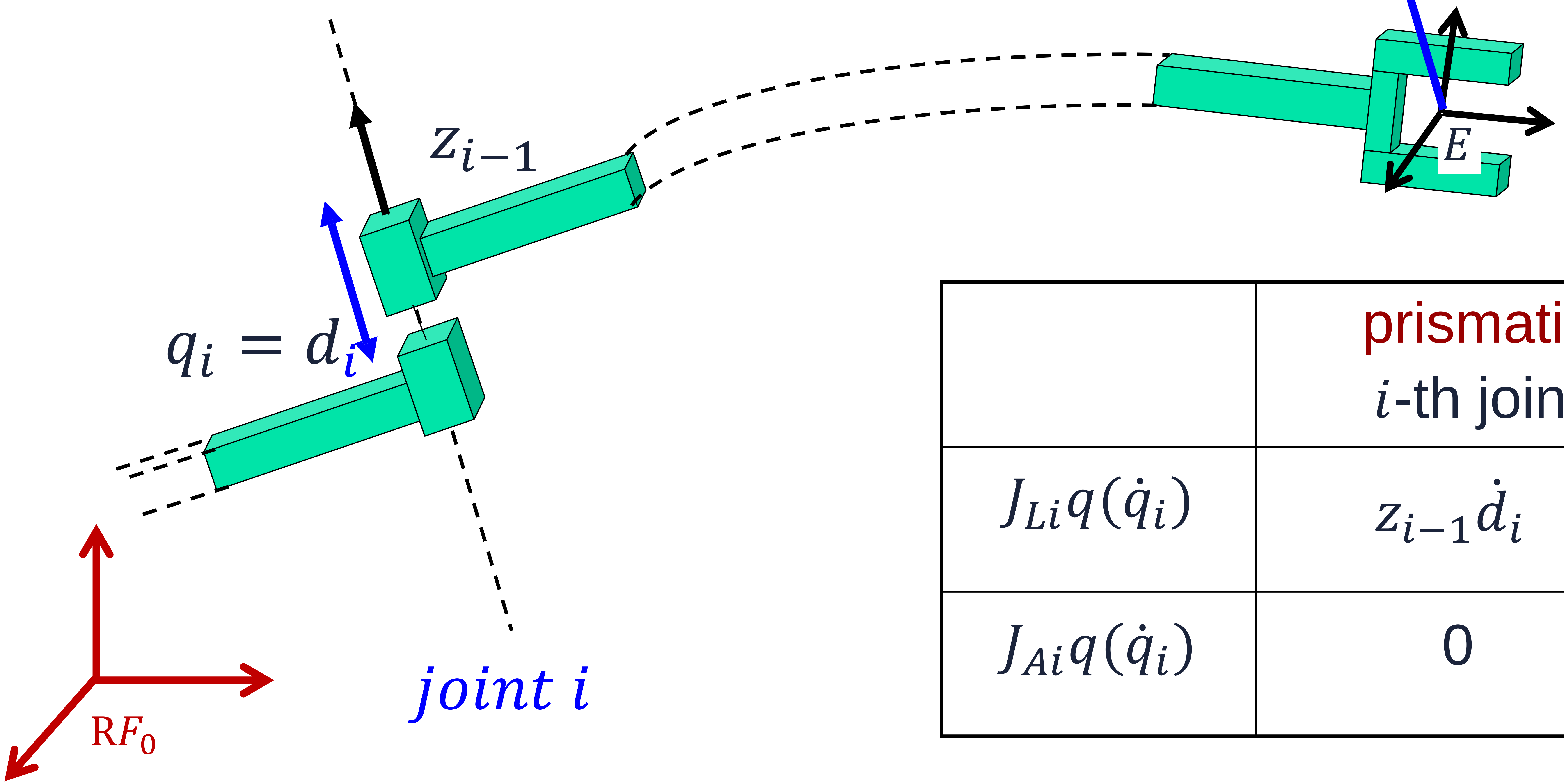


Prismatic Joint

Contribution of a prismatic joint

note: joints beyond the i-th one are considered to be "frozen", so that the distal part of the robot is a single rigid body

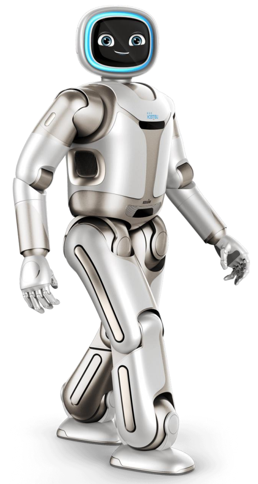
$$J_{Li}q(\dot{q}_i) = d_i z_{i-1}\dot{d}_i$$

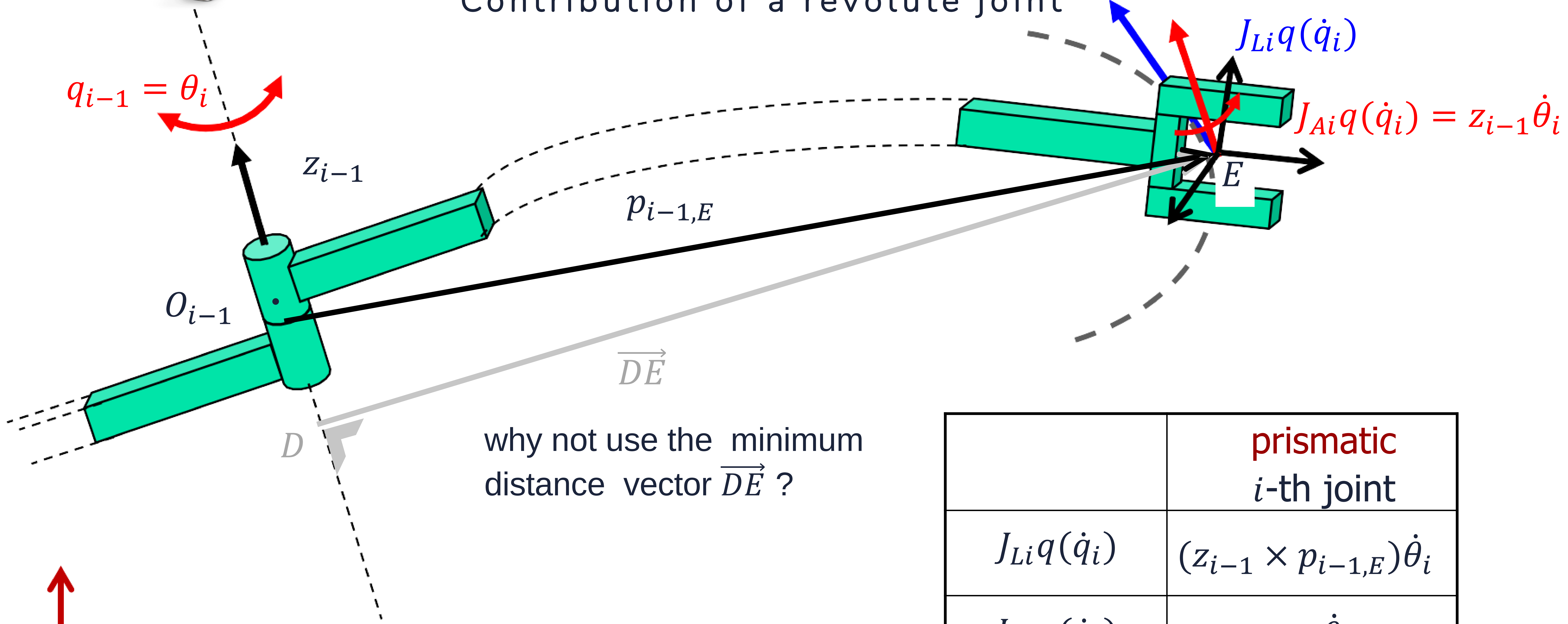


	prismatic i-th joint
$J_{Li}q(\dot{q}_i)$	$z_{i-1}\dot{d}_i$
$J_{Ai}q(\dot{q}_i)$	0

Revolute Joint

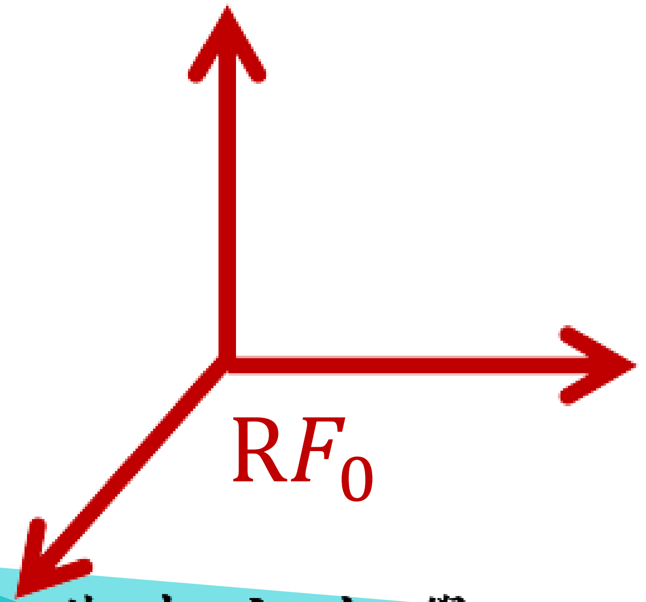
Contribution of a revolute joint

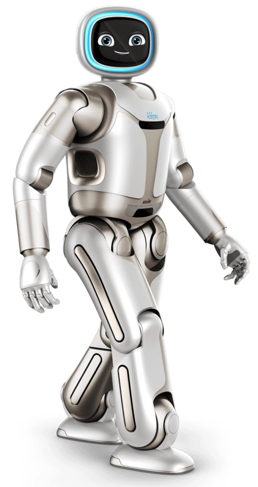




why not use the minimum distance vector $\overrightarrow{DE}$?

	prismatic i-th joint
$J_{Li}q(\dot{q}_i)$	$(z_{i-1} \times p_{i-1,E})\dot{\theta}_i$
$J_{Ai}q(\dot{q}_i)$	$z_{i-1}\dot{\theta}_i$





Geometric Jacobian

Expression of geometric Jacobian

$$\begin{pmatrix}\dot{p}_{0,E}\\ \omega_E\end{pmatrix} = \begin{pmatrix}v_E\\ \omega_E\end{pmatrix} = \begin{pmatrix}J_L(q)\\ J_A(q)\end{pmatrix}\dot{q} = \begin{pmatrix}J_{L1}(q) & \cdots & J_{Ln}(q)\\ J_{A1}(q) & \cdots & J_{An}(q)\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{pmatrix}$$

	prismatic i-th joint	revolute i-th joint
$J_{Li}q$	z_{i-1}	$z_{i-1} \times p_{i-1,E}$
$J_{Ai}q$	0	z_{i-1}

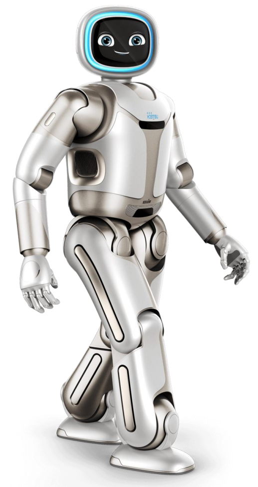
* $z_{i-1} = {}^0R_1(q_1)\cdots{}^{i-2}R_{i-1}(q_{i-1})\,{}^{i-1}z_{i-1} \longleftarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$

* $p_{i-1,E} = p_{0,E}(q_1,\cdots,q_n) - p_{0,i-1}(q_1,\cdots,q_{i-1})$

complete kinematics for e-e position

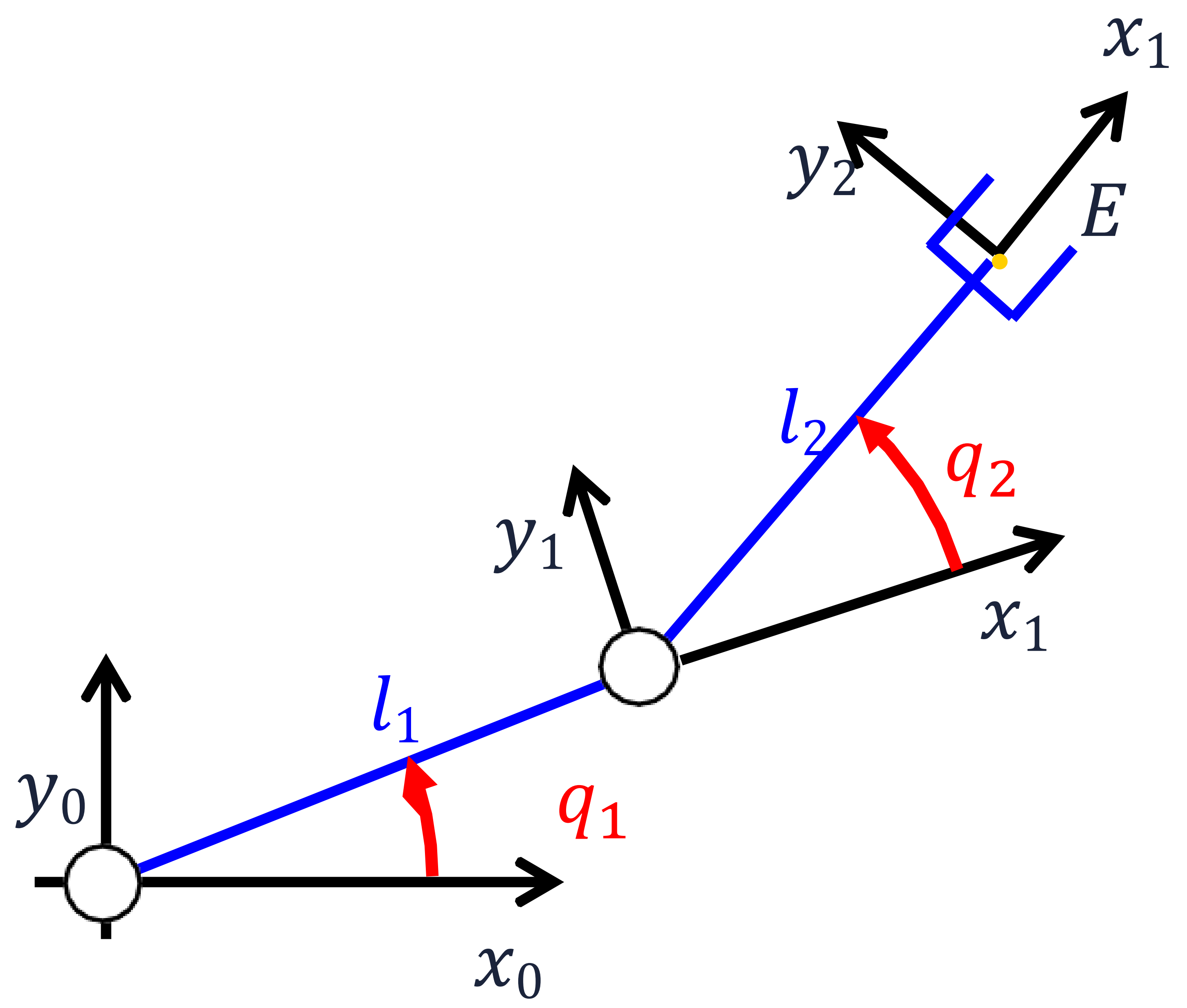
partial kinematics for O_{i-1} position

$\longleftarrow$ all vectors should be expressed in the same reference frame (here, the base frame RF_0)



Geometric Jacobian

Geometric Jacobian of planar 2R arm



Denavit-Hartenberg table

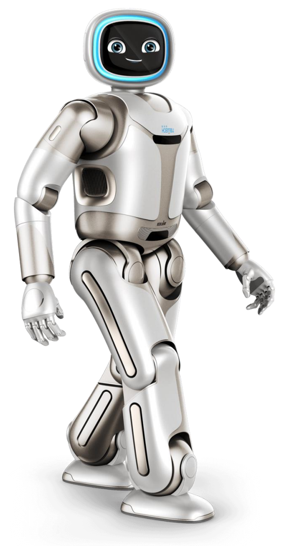
joint	α_i	d_i	a_i	θ_i
1	0	0	l_1	q_1
2	0	0	l_2	q_2

$$J(q) = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

* $z_0 = z_1 = z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
* $p_{1,E} = p_{0,E} - p_{0,1}$

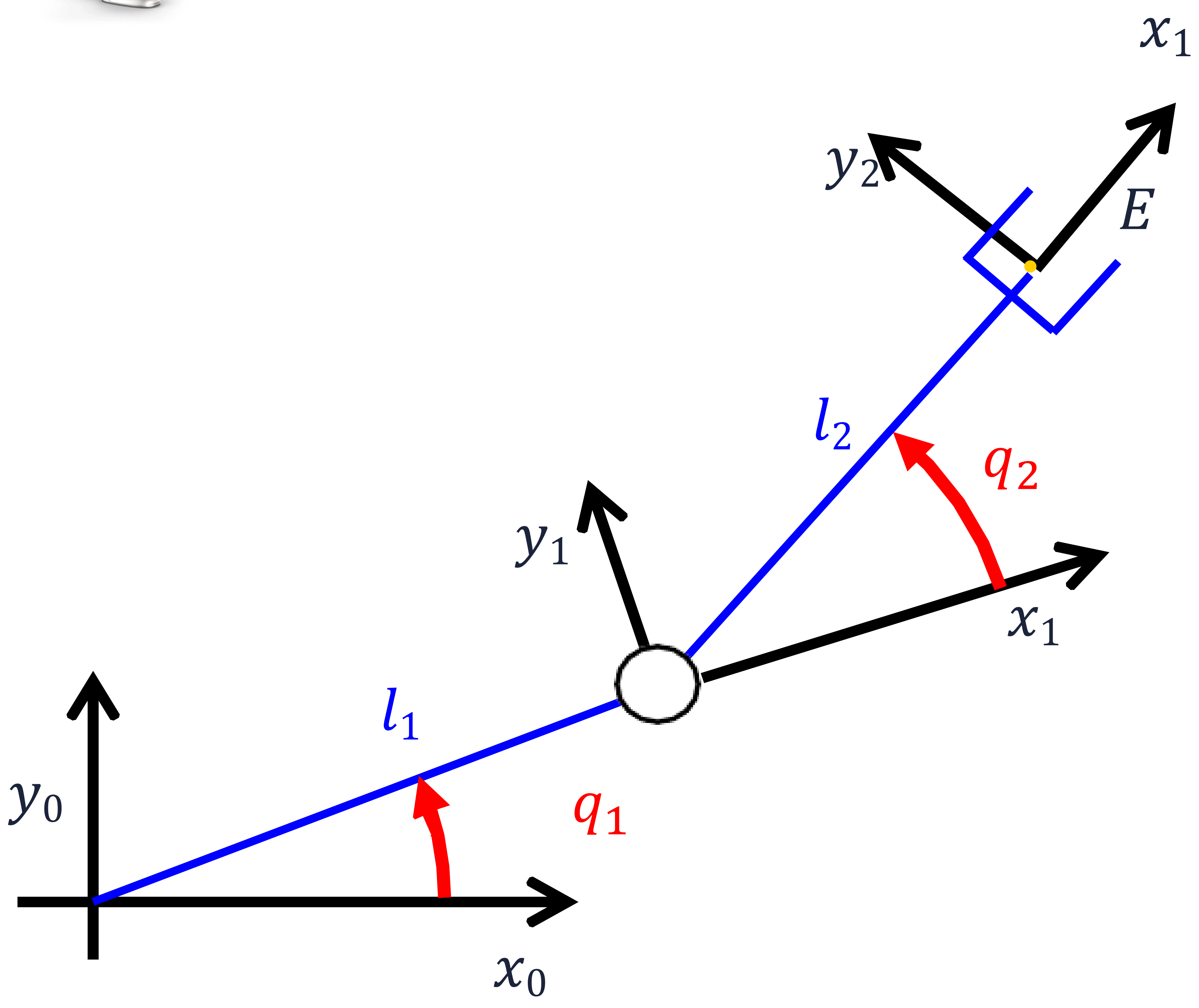
$${}^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,1}$$

$${}^0A_2 = \begin{pmatrix} c_{12} & -s_{12} & 0 & c_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,E}$$



Geometric Jacobian

Geometric Jacobian of planar 2R arm

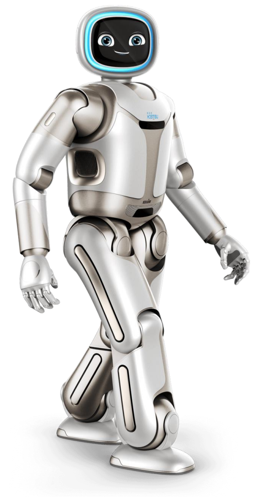


$$J(q) = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & l_2 s_{12} \\ l_1 c_1 - l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

compare rows 1, 2, and 6 with the analytical Jacobian in previous slide.

$$J_r(q) = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ \hline 1 & 1 \end{pmatrix}$$

note: the Jacobian is here a 6×2 mat thus its maximum rank is 2 ➡ at most 2 components of the linear/angular end-effector velocity can be independently assigned



Summary

Summary of differential relations

$\dot p \rightleftarrows v$ $\qquad \dot p = v$

$\dot R \rightleftarrows \omega$ $\qquad \dot R = S(\omega)R \qquad S(\omega) = \dot R R^T \quad \Longleftrightarrow \quad$ for each (unit) column r_i of R (a frame): $\dot r_i = \omega \times r_i$

$\dot\phi \rightleftarrows \omega$

$$\omega = \omega_{\dot\phi_1} + \omega_{\dot\phi_2} + \omega_{\dot\phi_2} = a_1\dot\phi_1 + a_2(\phi_1)\dot\phi_2 + a_3(\phi_1,\phi_2)\dot\phi_3 = T(\phi)\dot\phi$$

(moving) axes of definition for the sequence of rotations $\phi_i, i = 1,2,3$

$$\omega = \overbrace{\begin{bmatrix} c\beta c\gamma & -s\gamma & 0 \\ c\beta s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \end{bmatrix}}^{T_{RPY}(\beta,\gamma)} \begin{bmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{bmatrix}$$

if the task vector r is

$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} \quad \Rightarrow \quad J_r(q) = \begin{pmatrix} I & 0 \\ 0 & T^{-1}(\phi) \end{pmatrix} J(q) \quad \Longleftrightarrow \quad J(q) = \begin{pmatrix} I & 0 \\ 0 & T(\phi) \end{pmatrix} J_r(q)$$

$T(\phi)$ has always a singularity $\Longleftrightarrow$ singularity of the specific minima representation of orientation

Primer on Linear Algebra

given a matrix J: $m \times n$ (m rows, n columns)

- rank $\rho(J) = \max$ # of rows or columns that are linearly independent
  - $\rho(J) \leq \min(m, n)$ $\Leftarrow$ if equality holds, J has full rank
  - if $m = n$ and J has full rank, J is nonsingular and the inverse J^{-1} exists
  - $\rho(J) =$ dimension of the largest nonsingular square submatrix of J
- range space $\mathcal{R}(J)$ =subspace of all linear combinations of the columns of J

$$\mathcal{R}(J) = \{v \in \mathbb{R}^m : \exists \xi \in \mathbb{R}^n, v = J\xi\}$$

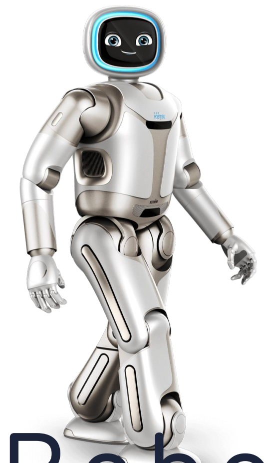
$\longleftarrow$ also called image of J

  - dim $\mathcal{R}(J) = \rho(J)$
- null space $\mathcal{N}(J) =$ subspace of all vectors that are zeroed by matrix J

$$\mathcal{N}(J) = \{\xi \in \mathbb{R}^n : J\xi = 0 \in \mathbb{R}^m\}$$

$\longleftarrow$ also called kernel of J

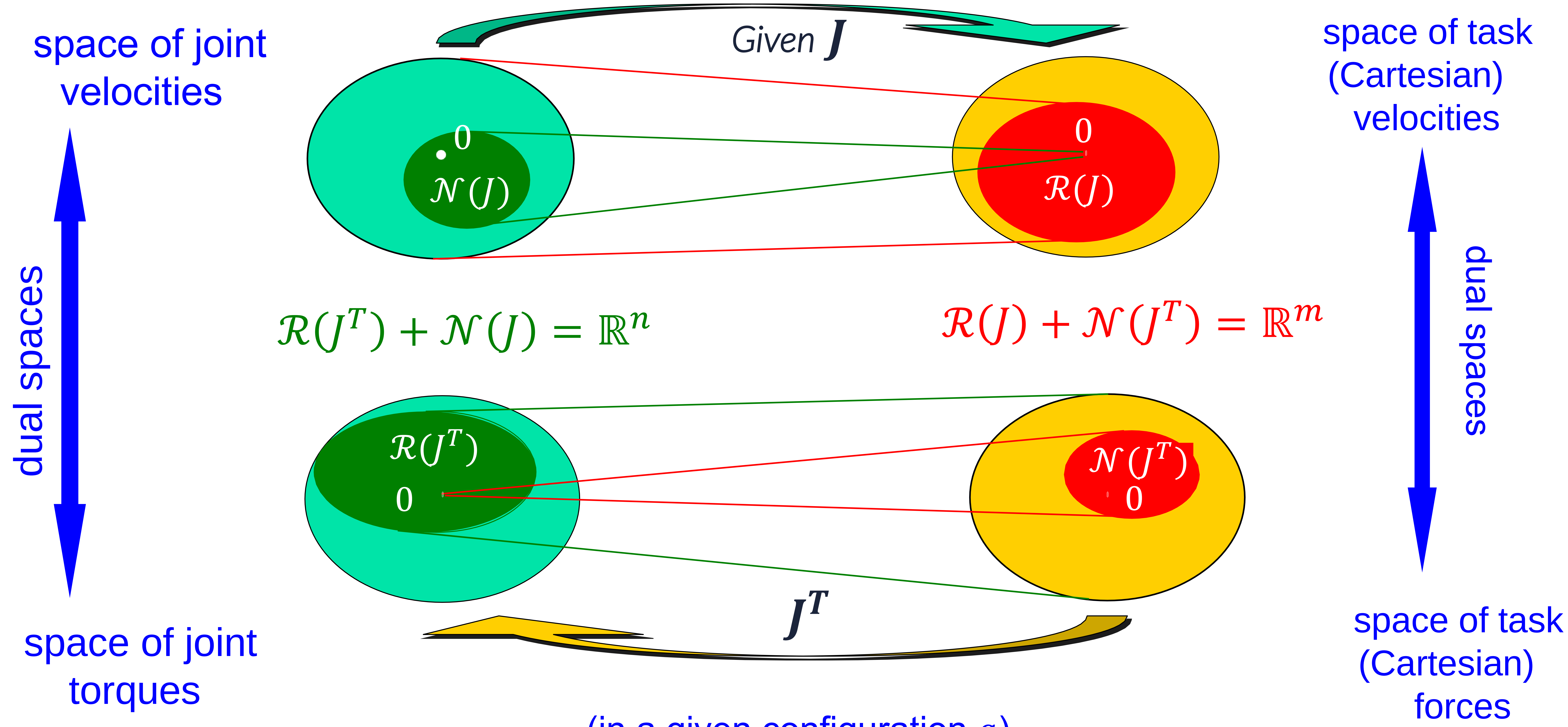
  - dim $\mathcal{N}(J)$= $n - \rho(J)$
- $\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^m$ and $\mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^n$ (sum of vector subspaces)



Robot Jacobian

Bruno: page 122

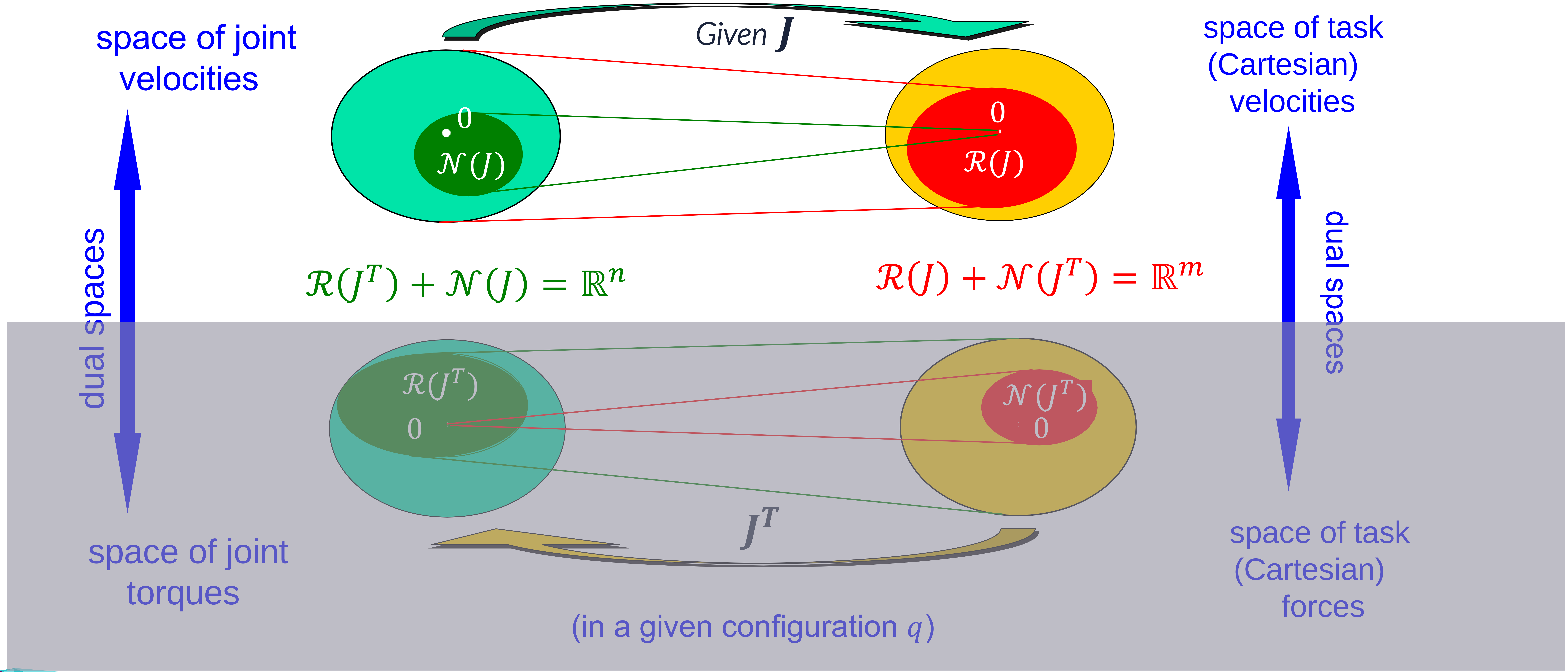
Robot Jacobian (decomposition in linear subspaces and duality)

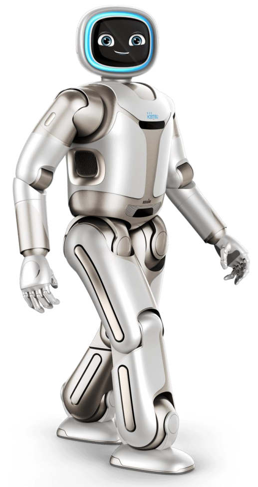


(in a given configuration q)

Robot Jacobian

Robot Jacobian (decomposition in linear subspaces and duality)

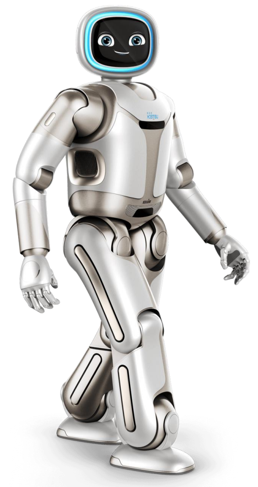




Mobility Analysis

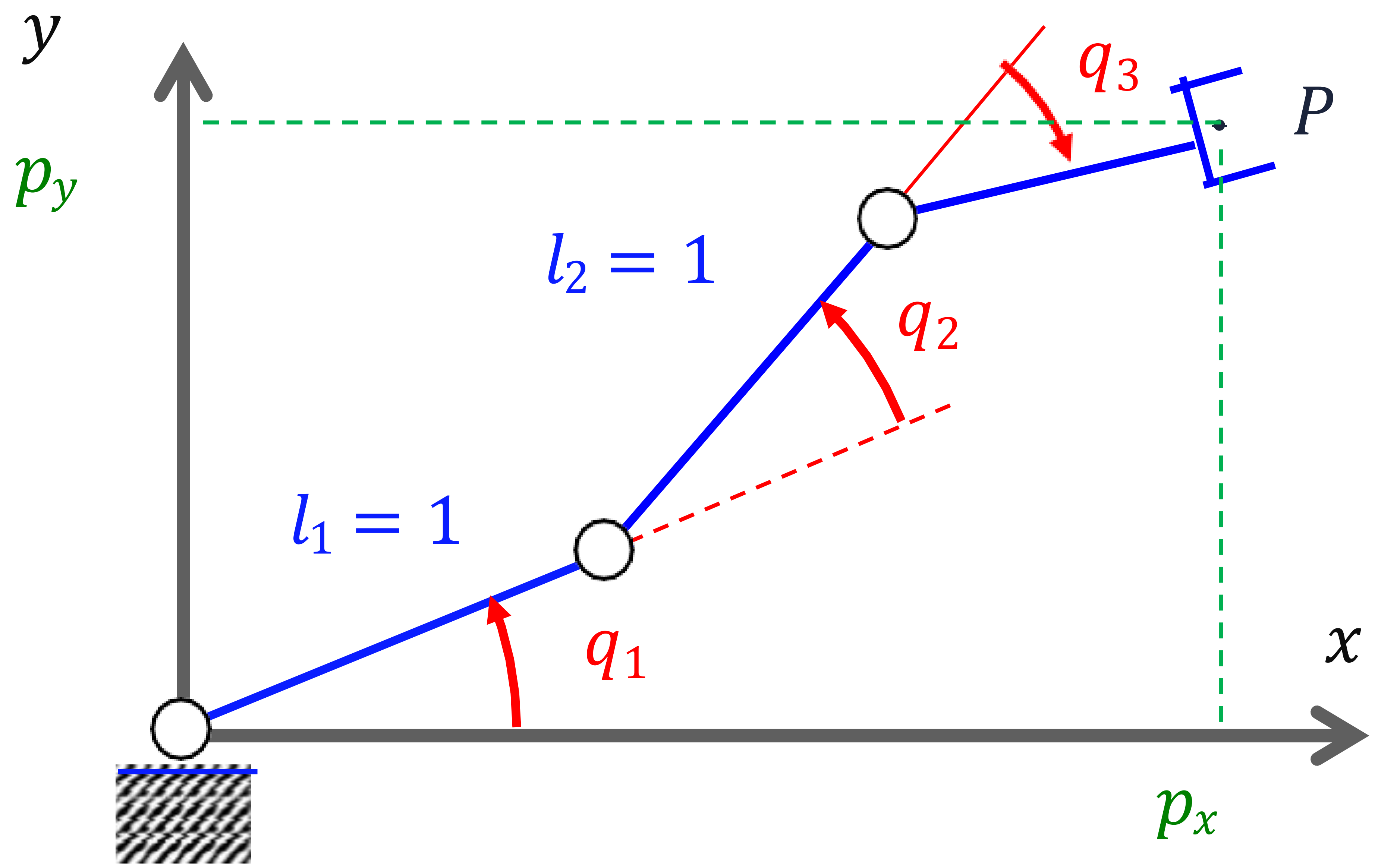
Mobility analysis in the task space

- $\rho(J) = \rho\big(J(q)\big), \mathcal{R}(J) = \mathcal{R}\left(J(q)\right), \mathcal{N}(J^T) = \mathcal{N}(J^T(q))$, etc. are locally defined, i.e., they depend on the current configuration q
- $\mathcal{R}\left(J(q)\right)$ is the is the subspace of all "generalized" velocities (with linear and/or angular components) that can be instantaneously realized by the robot end-effector when varying the joint velocities $\dot{q}$ at the current q
- if $\rho\big(J(q)\big) = m$ at q ($J(q)$ has max rank, with $m \leq n$), the end-effector can be moved in any direction of the task space $\mathbb{R}^m$
- if $\rho\big(J(q)\big) < m$, there are directions in $\mathbb{R}^m$ in which the end-effector cannot move (at least, not instantaneously!)
  - these directions $\in \mathcal{N}(J^T(q))$, the complement of $\mathcal{R}(J(q))$ to task space $\mathbb{R}^m$, which is of dimension $m - \rho(J(q))$
- if $\mathcal{N}\big(J(q)\big) \neq \{0\}$, there are non-zero joint velocities $\dot{q}$ that produce zero end-effector velocity ("self motions")
  - this happens always for $m < n$, i.e., when the robot is redundant for the task



Mobility Analysis

Mobility analysis for a planar 3R robot



$l_1 = l_2 = l_3 = 1 \qquad n = 3, \quad m = 2$

$WS_1 = \{p \in \mathbb{R}^2 : \|p\| \le 3\} \subset \mathbb{R}^2$

$WS_2 = \{p \in \mathbb{R}^2 : \|p\| \le 1\} \subset \mathbb{R}^2$

$$p = \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix}$$

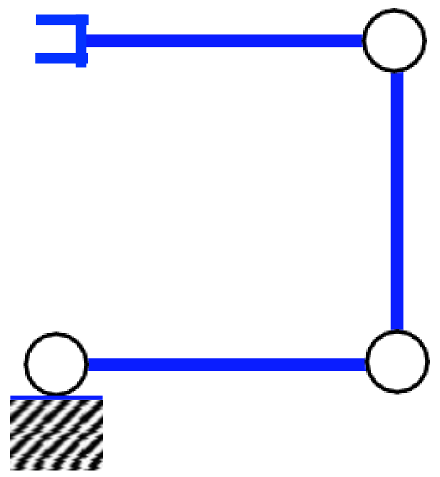
in $\mathbb{R}^2$ in $\mathbb{R}^3$

$$v = \dot{p} = \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} \dot{q}$$
$$= J(q)\dot{q}$$

case 1)

$q = (0, \pi/2, \pi/2)$

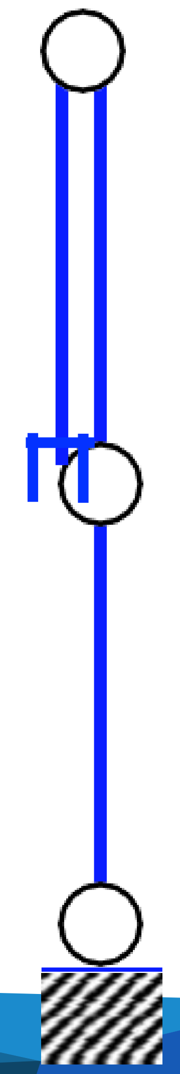
$$J = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

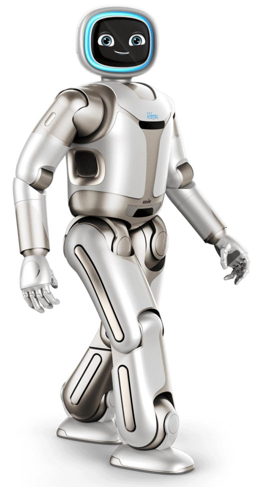


case 2)

$q = (\pi/2\,, 0, \pi)$

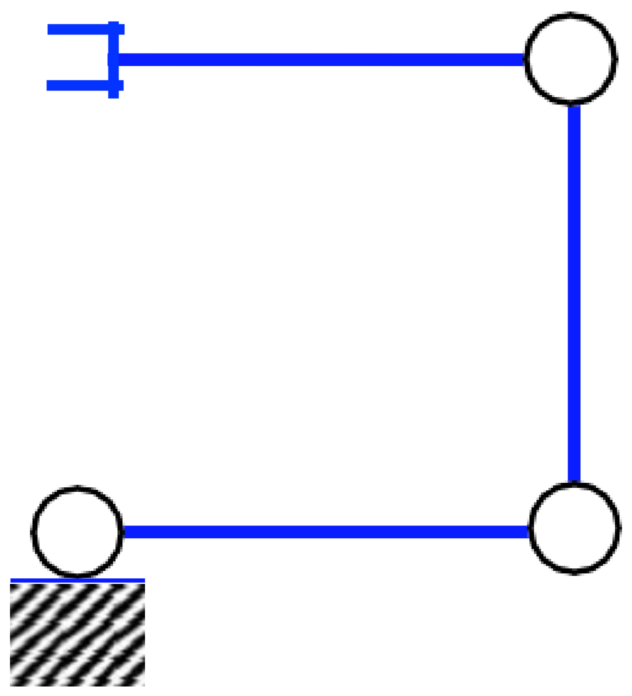
$$J = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$





Mobility Analysis

Mobility analysis for a planar 3R robot



case 1)

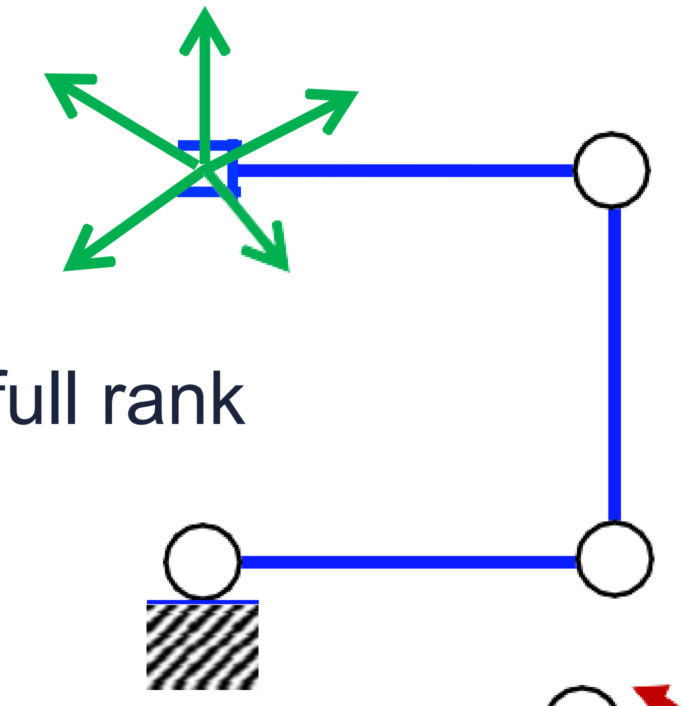
$$q = (0, \pi/2, \pi/2)$$

$$J = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad J^T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\rho(J) = 2 = m \qquad \rho(J^T) = \rho(J) = 2$$

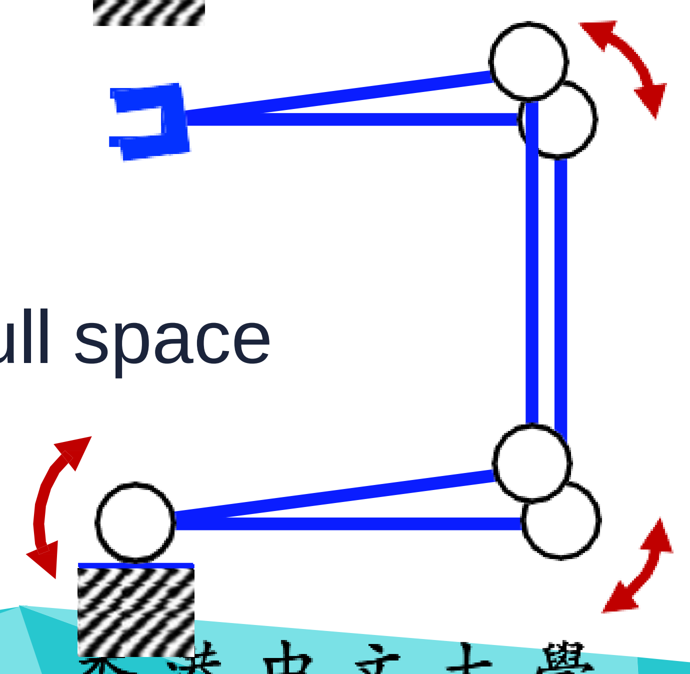
full rank, non-singular case

$$\mathcal{R}(J) = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2 \qquad \mathcal{N}(J) = \left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\} \qquad \dim\ \mathcal{N}(J) = 1 = n - \rho(J) = n - m$$



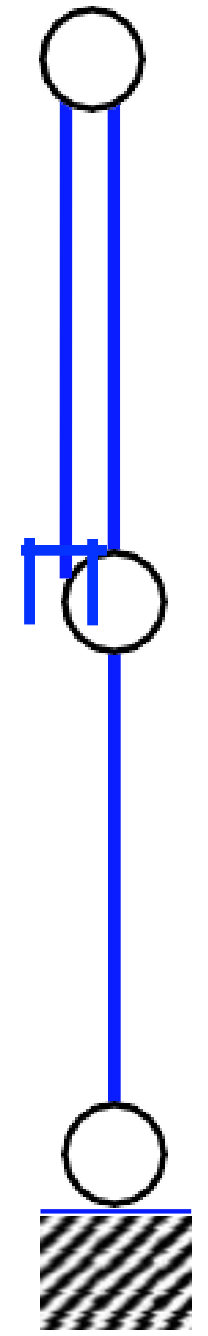
If full rank

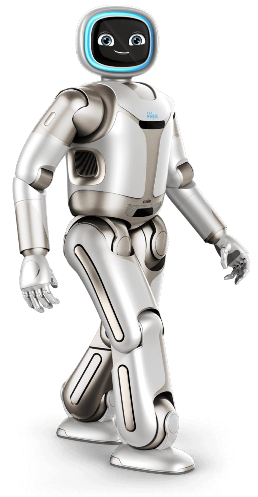
$$\mathcal{R}(J^T) = \left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\} \qquad \mathcal{N}(J^T) = 0 \qquad \dim\ \mathcal{R}(J^T) = 2 = m$$



If null space

$$\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^2$$
$$\mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^3$$





Mobility Analysis

Mobility analysis for a planar 3R robot

case 2)

$q = (\pi/2, 0, \pi)$

$$J = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad J^T = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

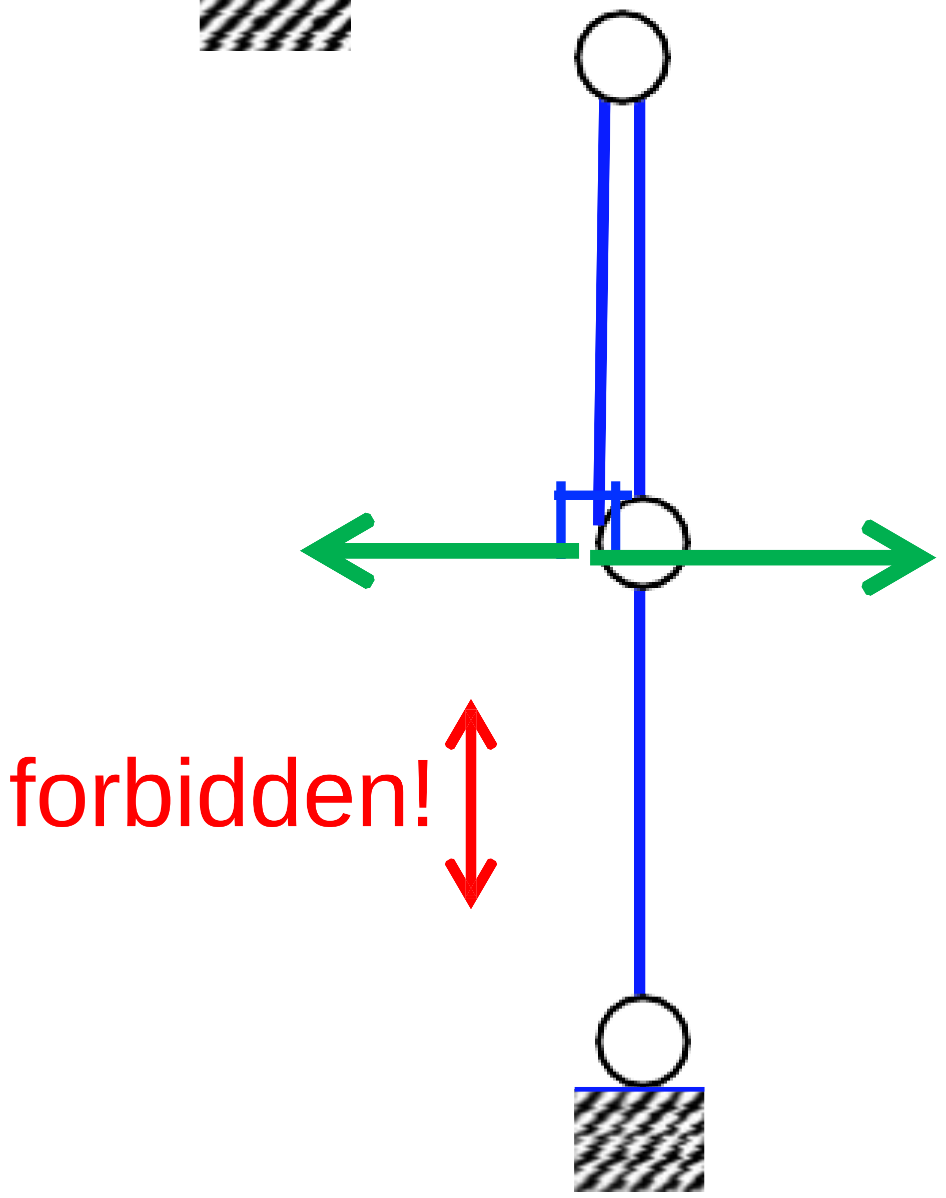
$$\rho(J) = 1 < m \qquad \rho(J^T) = \rho(J) = 1$$

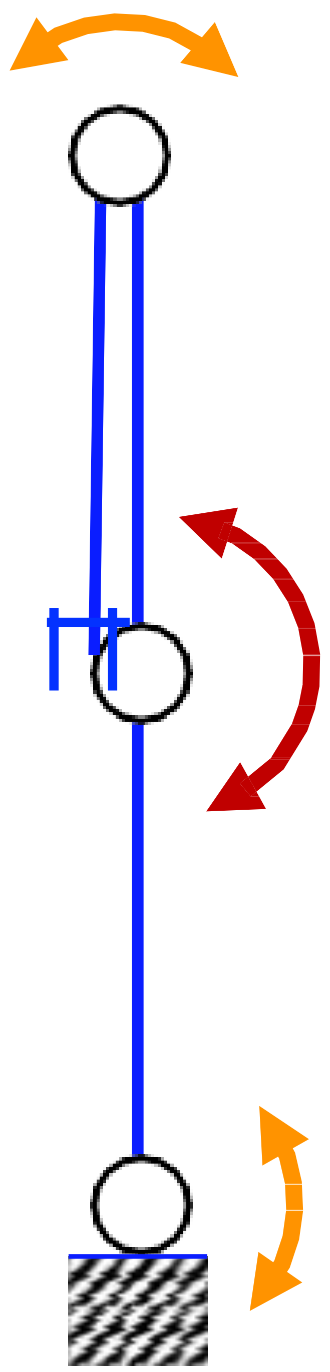
$$\mathcal{R}(J) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \qquad \mathcal{N}(J) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim \ \mathcal{R}(J) = 1 = \rho(J)$$
$$\dim \ \mathcal{N}(J) = 2 = n - \rho(J)$$

$$\mathcal{R}(J^T) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{N}(J^T) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim \ \mathcal{R}(J^T) = 1 = m - \rho(J)$$
$$\dim \ \mathcal{N}(J^T) = 1 = n - \rho(J)$$





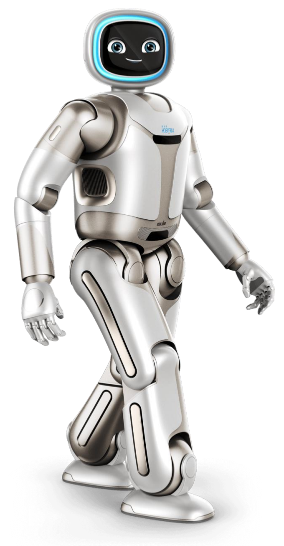
$$\Downarrow$$

$$\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^2$$
$$\mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^3$$

Kinematic Singularities

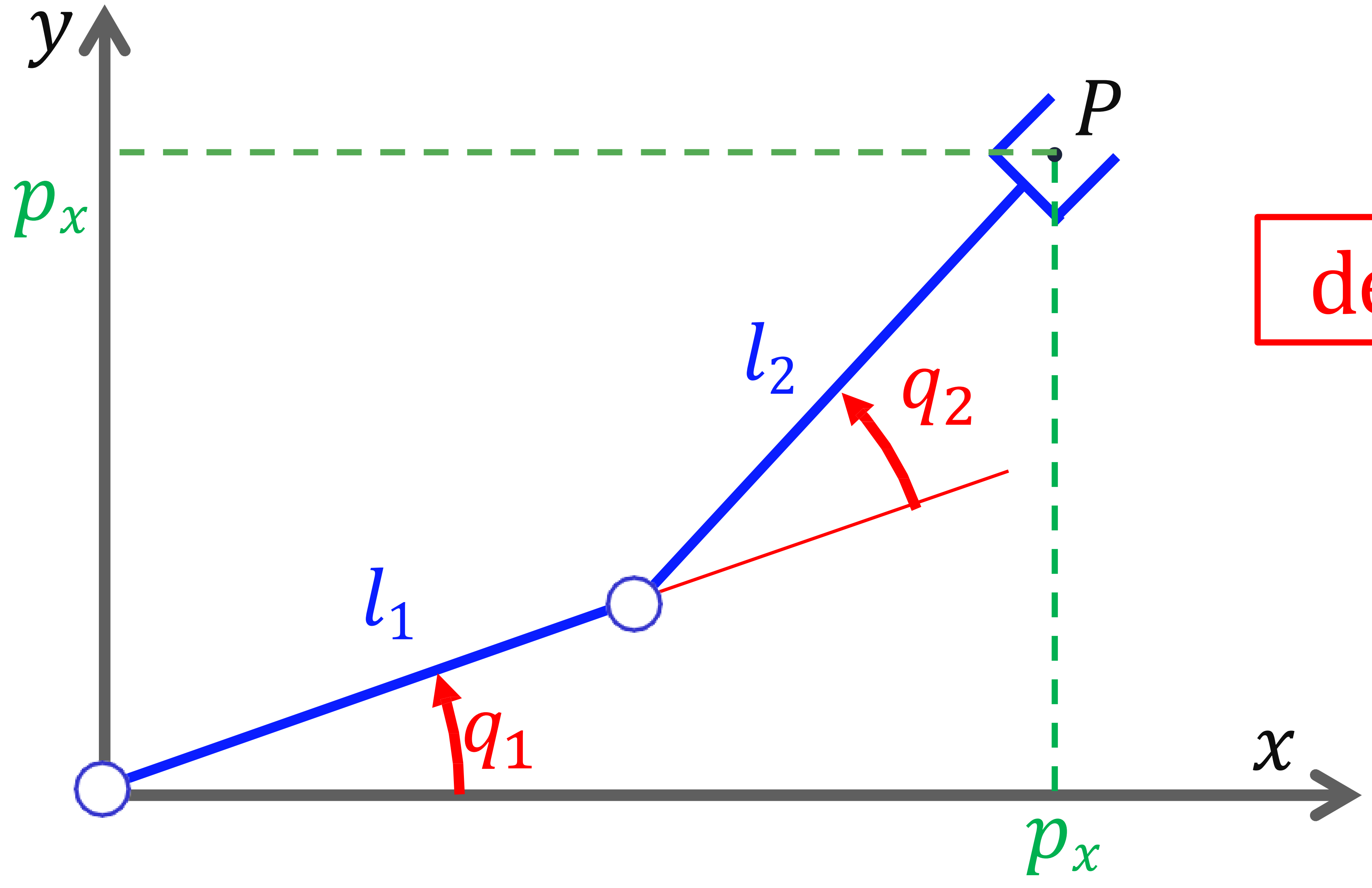
- **configurations where the Jacobian loses rank**

  $\Longleftrightarrow$ **loss of instantaneous mobility** of the robot end-effector
- for $m = n$, they correspond to Cartesian poses at which the number of solutions of the inverse kinematics problem differs from the generic case
- "in" a singular configuration, we cannot find any joint velocity that realizes a desired end-effector velocity in some directions of the task space
- "close" to a singularity, large joint velocities may be needed to realize even a small velocity of the end-effector in some directions of the task space
- finding and analyzing in advance the mobility of a robot helps in singularity avoidance during trajectory planning and motion control
  - when $m = n$: find the configurations q such that $\det J(q) = 0$
  - when $m < n$: find the configurations q such that all $m \times m$ minors of $J(q)$ are singular (or, equivalently, such that $\det(J(q)J^T(q)) = 0$)
- finding all singular configurations of a robot with a large number of joints, or the actual "distance" from a singularity, is a complex computational task



Kinematic Singularities

Singularities on planar 2R robot



$$\det J(q) = l_1 l_2 s_2$$

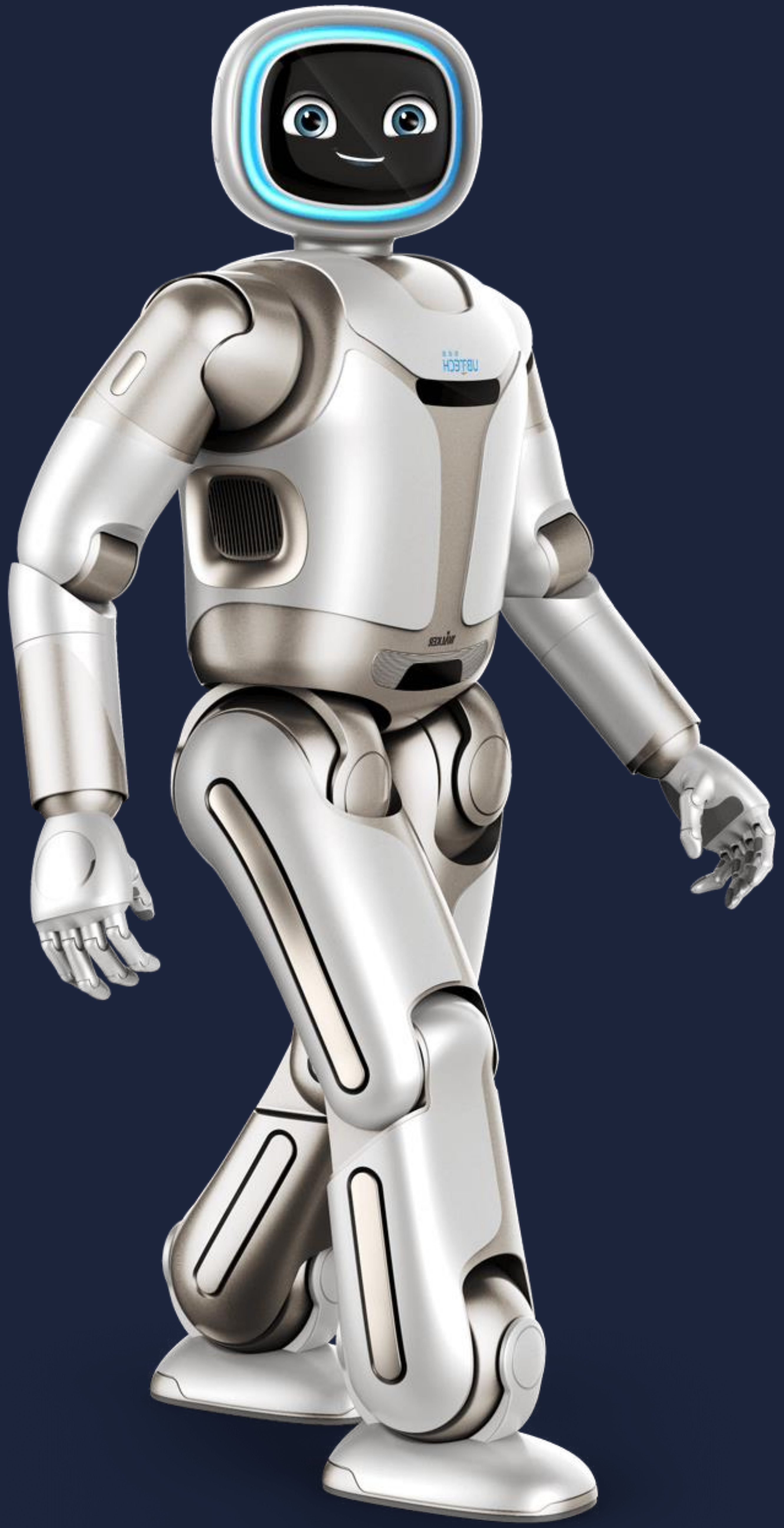
direct kinematics

$$p_x = l_1 c_1 + l_2 c_{12}$$
$$p_y = l_1 s_1 + l_2 s_{12}$$

analytical Jacobian

$$\dot{p} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \dot{q} = J(q)\dot{q}$$

- singularities: robot arm is stretched ($q_2 = 0$) or folded ($q_2 = \pi$)
- singular configurations correspond here to Cartesian points that are on the boundary of the primary workspace
- here, and in many cases, singularities separate configuration space regions with distinct inverse kinematic solutions (e.g., elbow “up” or “down”)



Q&A