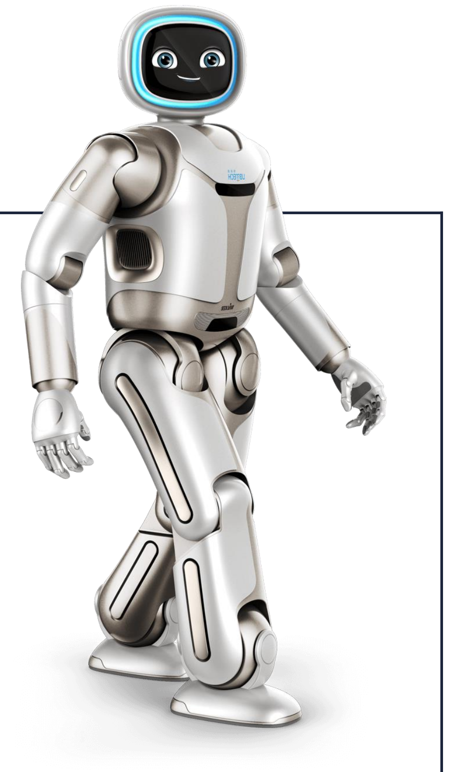
Advanced Robotics

ENGG5402 Spring 2023

Fei Chen

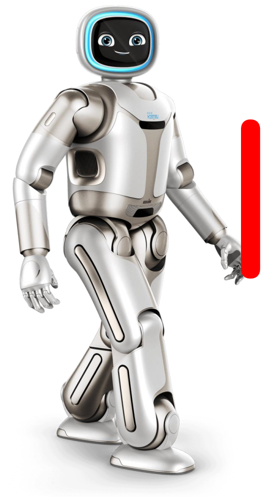


Topics:

- Inverse differential kinematics Statics and force transformations

Readings:

- Siciliano: Sec. 3.5, 3.7-3.9



Inversion of Differential Kinematics

- find the joint velocity vector that realizes a desired task/ end-effector velocity ("generalized" = linear and/or angular)

generalized velocity

J square and non-singular at q

$$v = J(q)\dot{q} \quad \longrightarrow \quad \dot{q} = J^{-1}(q)v$$

- problems
  - near a singularity of the Jacobian matrix (too high $\dot{q}$)
  - for redundant robots (no standard "inverse" of a rectangular matrix)

in these cases, more robust inversion methods are needed

Incremental Solution

Incremental solution (to inverse kinematics problems)

- joint velocity inversion can be used also to solve on-line and incrementally a "sequence" of inverse kinematics problems
- each problem differs by a small amount dr from previous one

$$r = f_r(q)$$

direct kinematics

$$dr = \frac{\partial f_r(q)}{\partial q} dq = J_r(q) dq$$

differential kinematics
(here with a square, analytic Jacobian)

$$r \longrightarrow r + dr$$

$$r + dr = f_r(q)$$

first, increment the desired task variables

$$q = f_r^{-1}(r + dr)$$

then, solve the inverse kinematics problem

$$dq = J_r^{-1}(q) dr$$

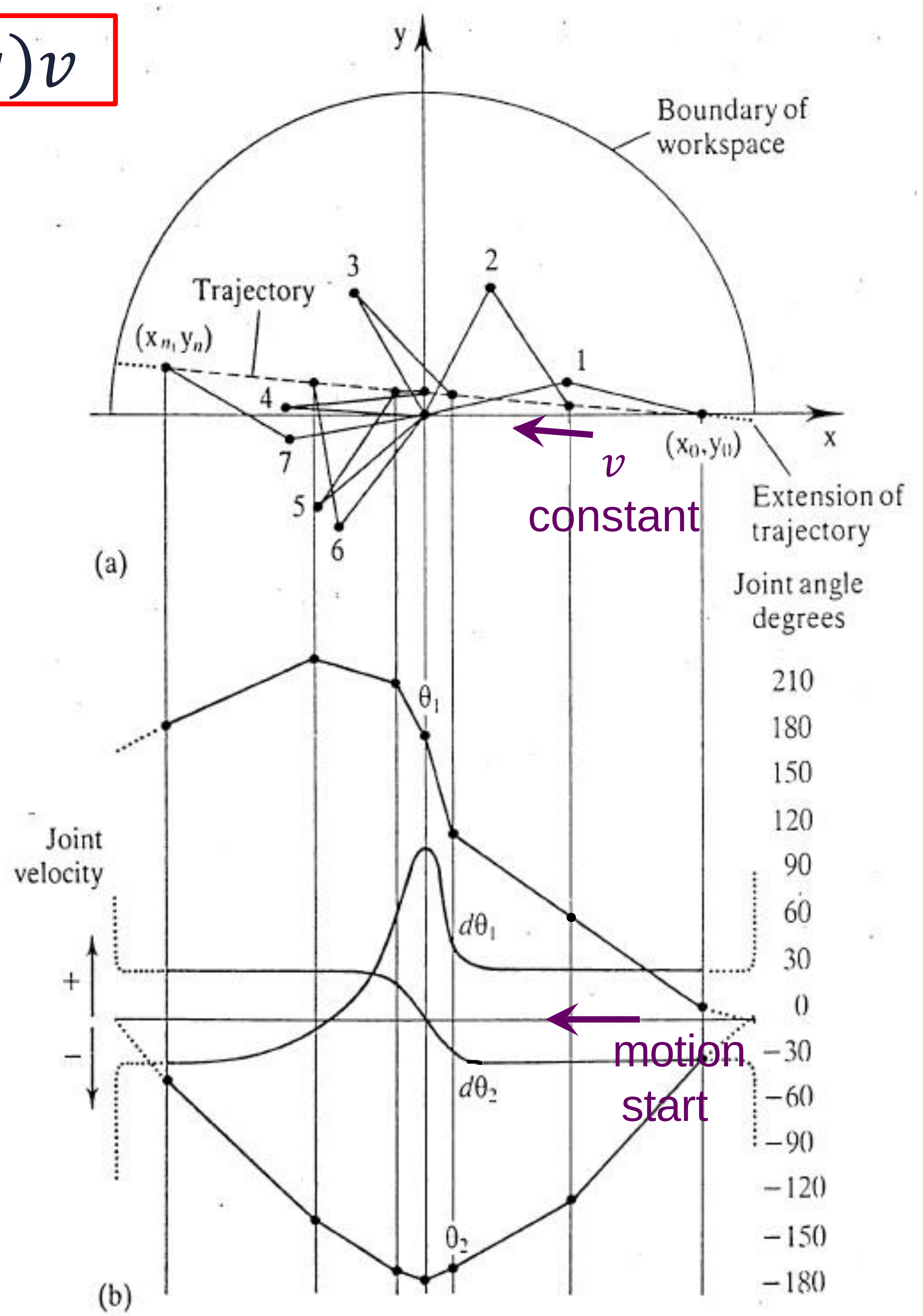
first, solve the inverse differential kinematics problem

$$q \longrightarrow q + dq$$

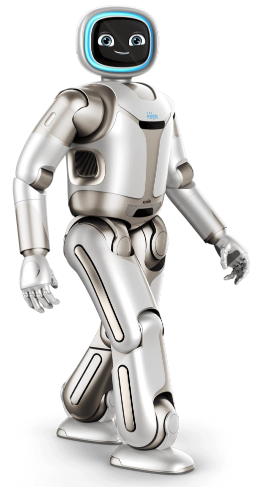
then, increment the original joint variables

Behavior Near a Singularity

$\dot{q} = J^{-1}(q)v$



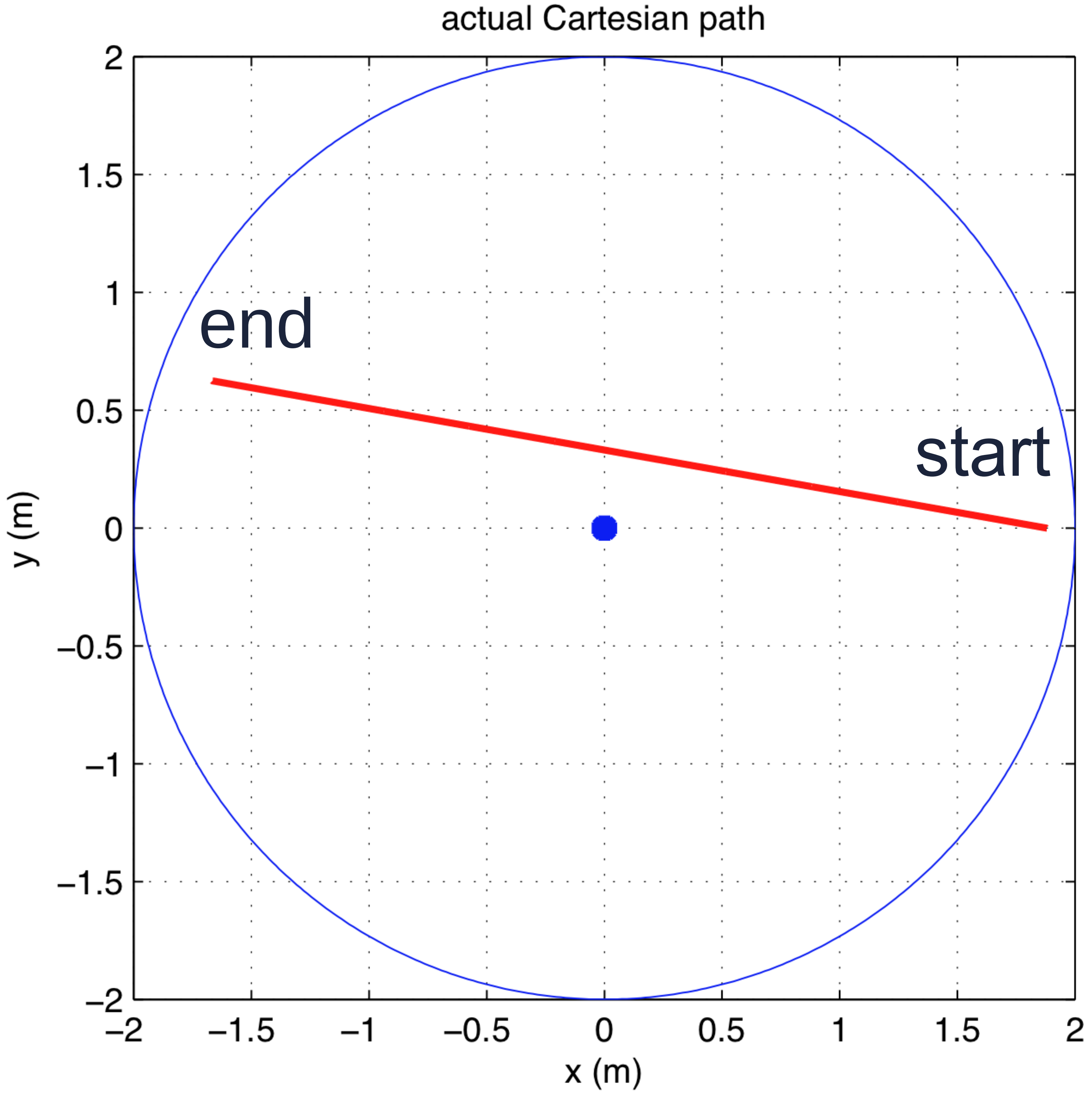
- problems arise only when commanding joint motion by inversion of a given Cartesian motion task
- here, a linear Cartesian trajectory for a planar 2R robot
- there is a sudden increase of the displacement/velocity of the first joint near $\theta_2 = -\pi$ (end- effector close to the origin), despite the required Cartesian displacement is small



Simulation

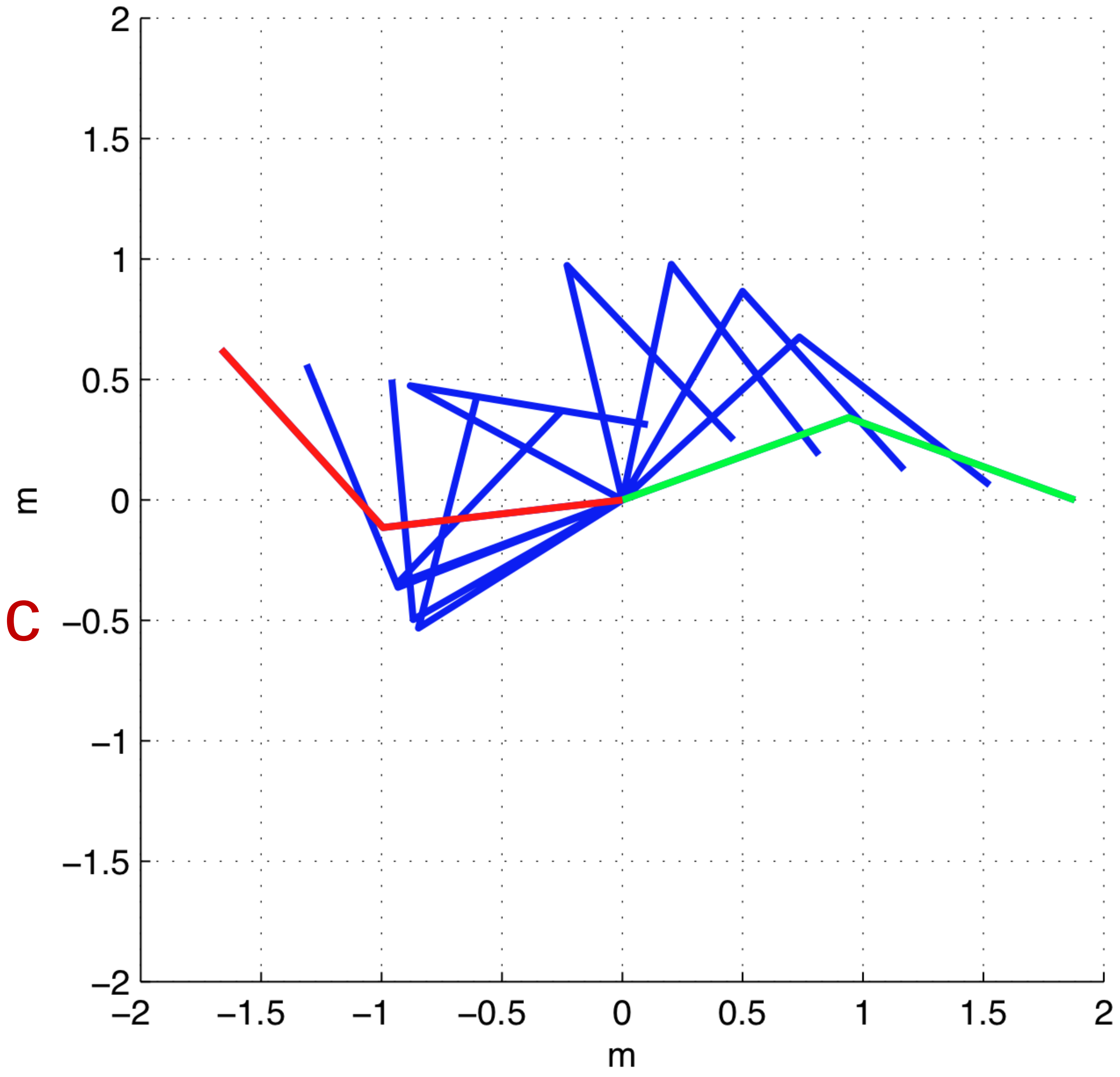
Simulation results
(planar 2R robot in straight line Cartesian motion)

$\dot{q} = J^{-1}(q)v$

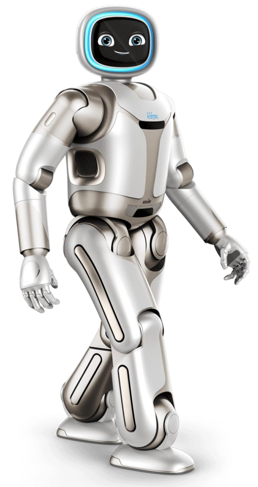


regular case

stroboscopic view



a line from right to left, at $\alpha = 170°$ angle with x-axis,
executed at constant speed $v = 0.6\ m/s$ for $T = 6\ s$

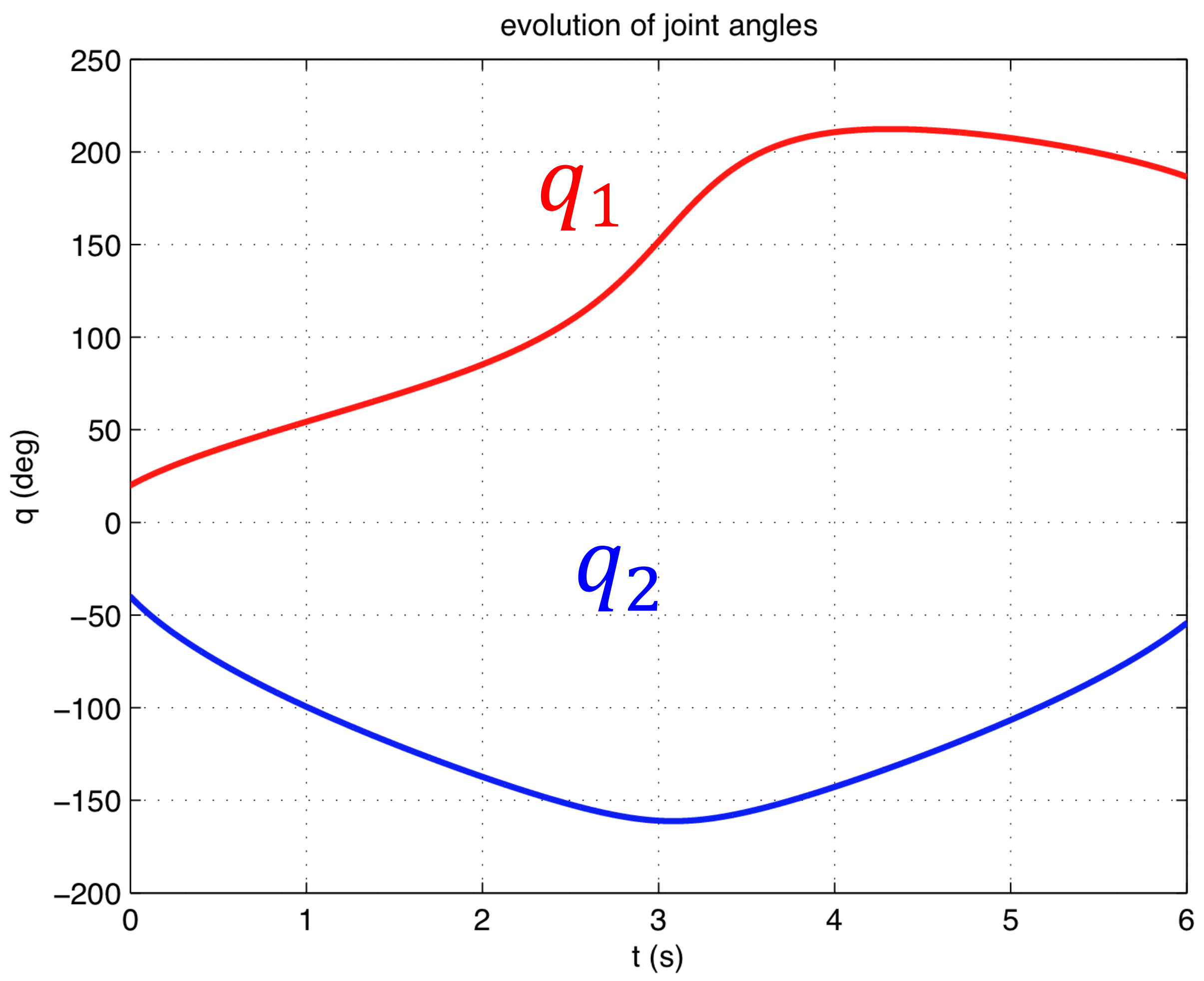


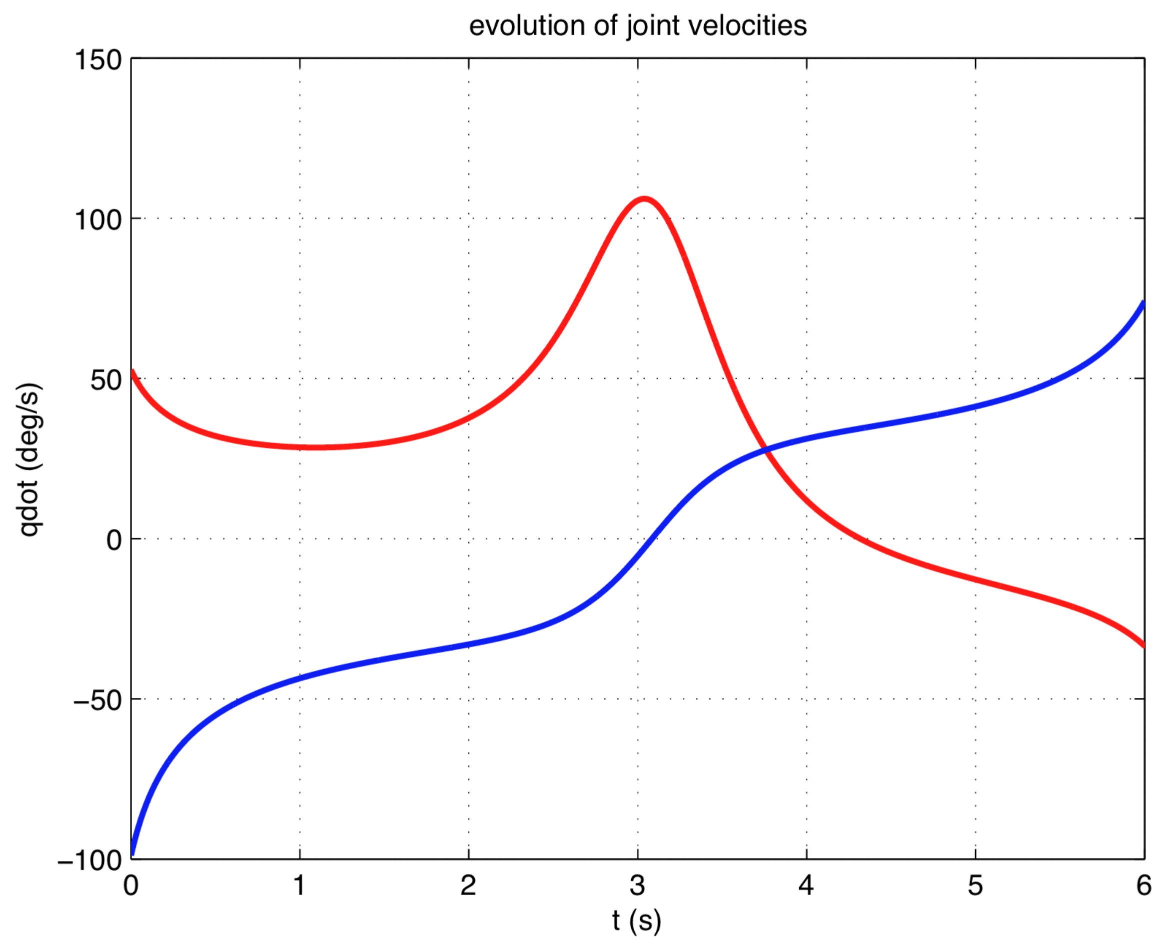
Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

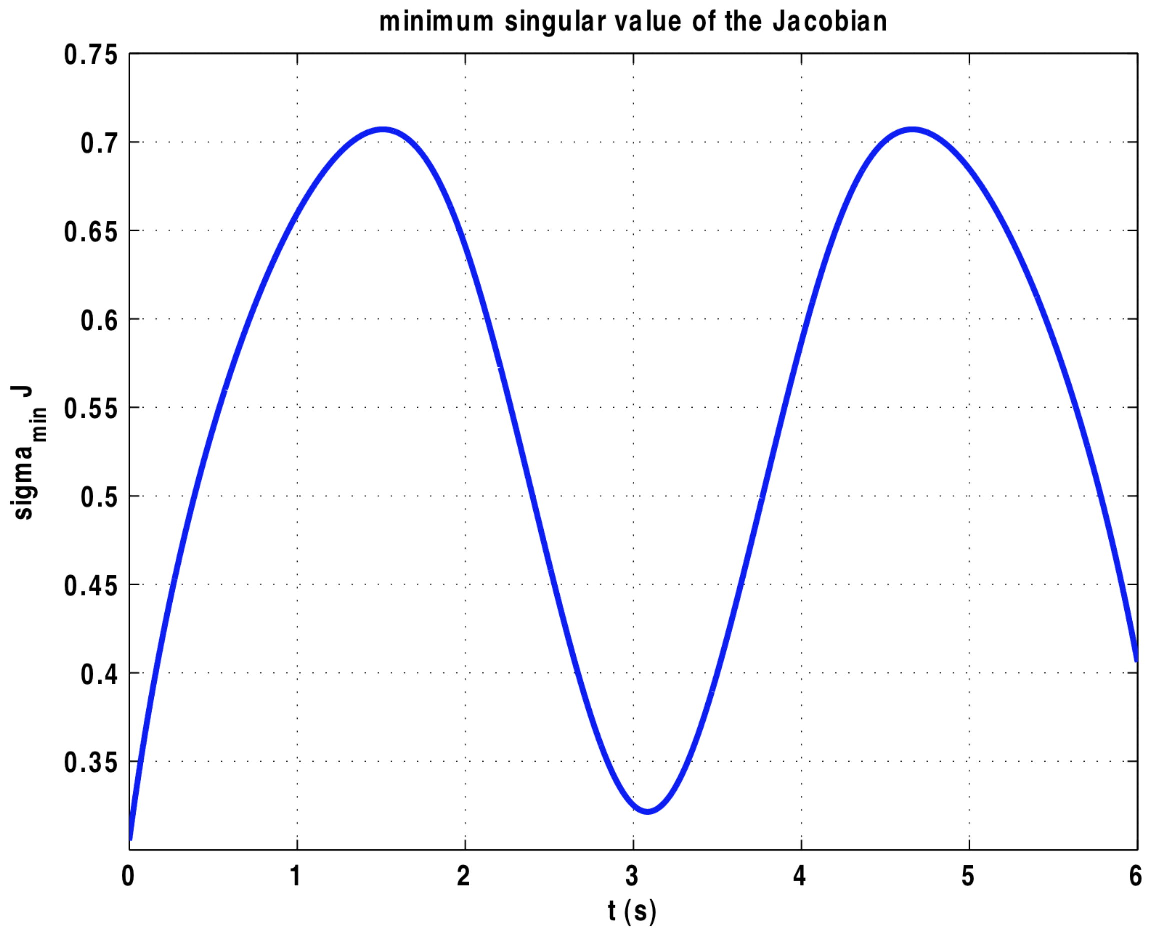
path at $\alpha = 170°$

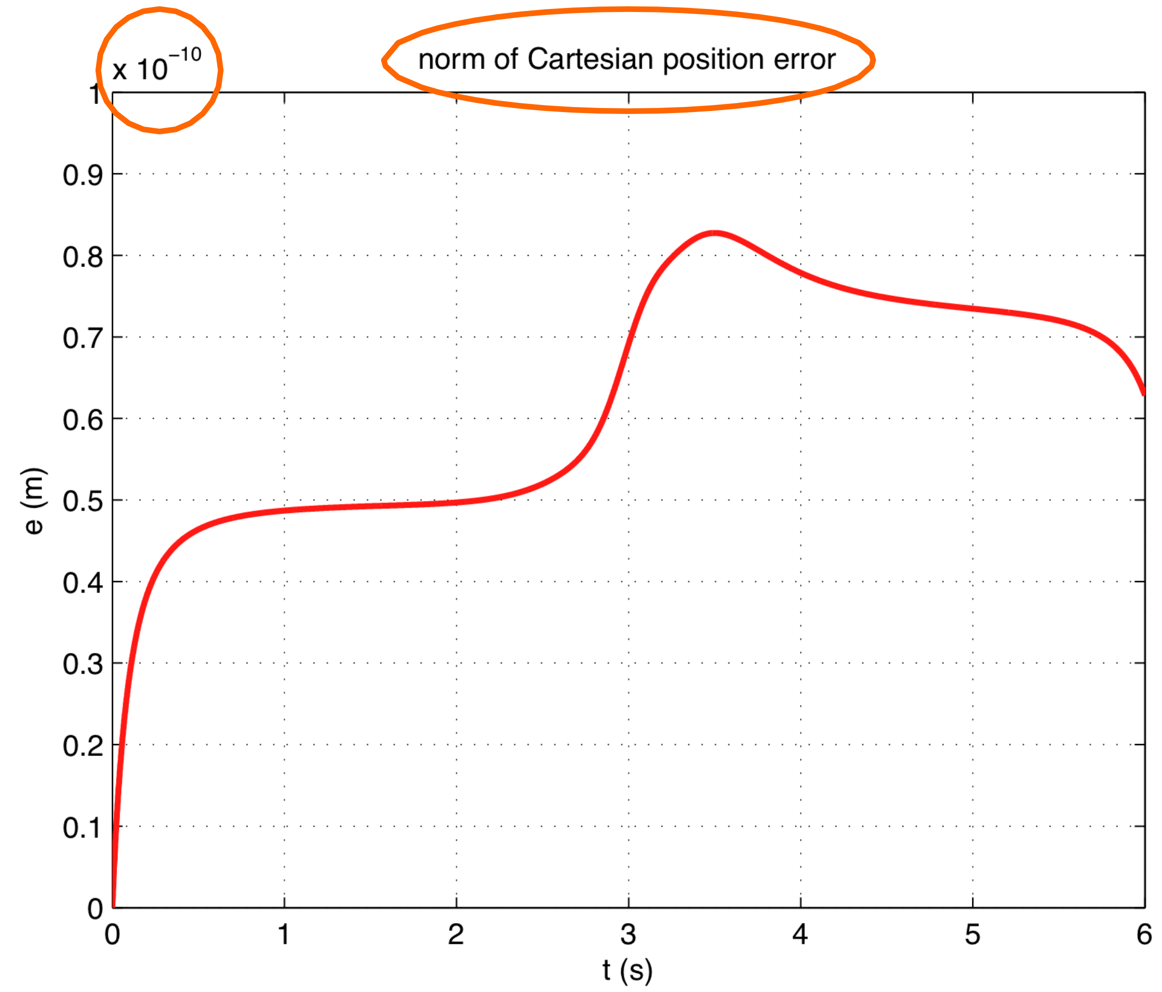
regular case





distance to singularity by the minimum singular value $\sigma_{min}(= \sigma_2) > 0$ of Jacobian J



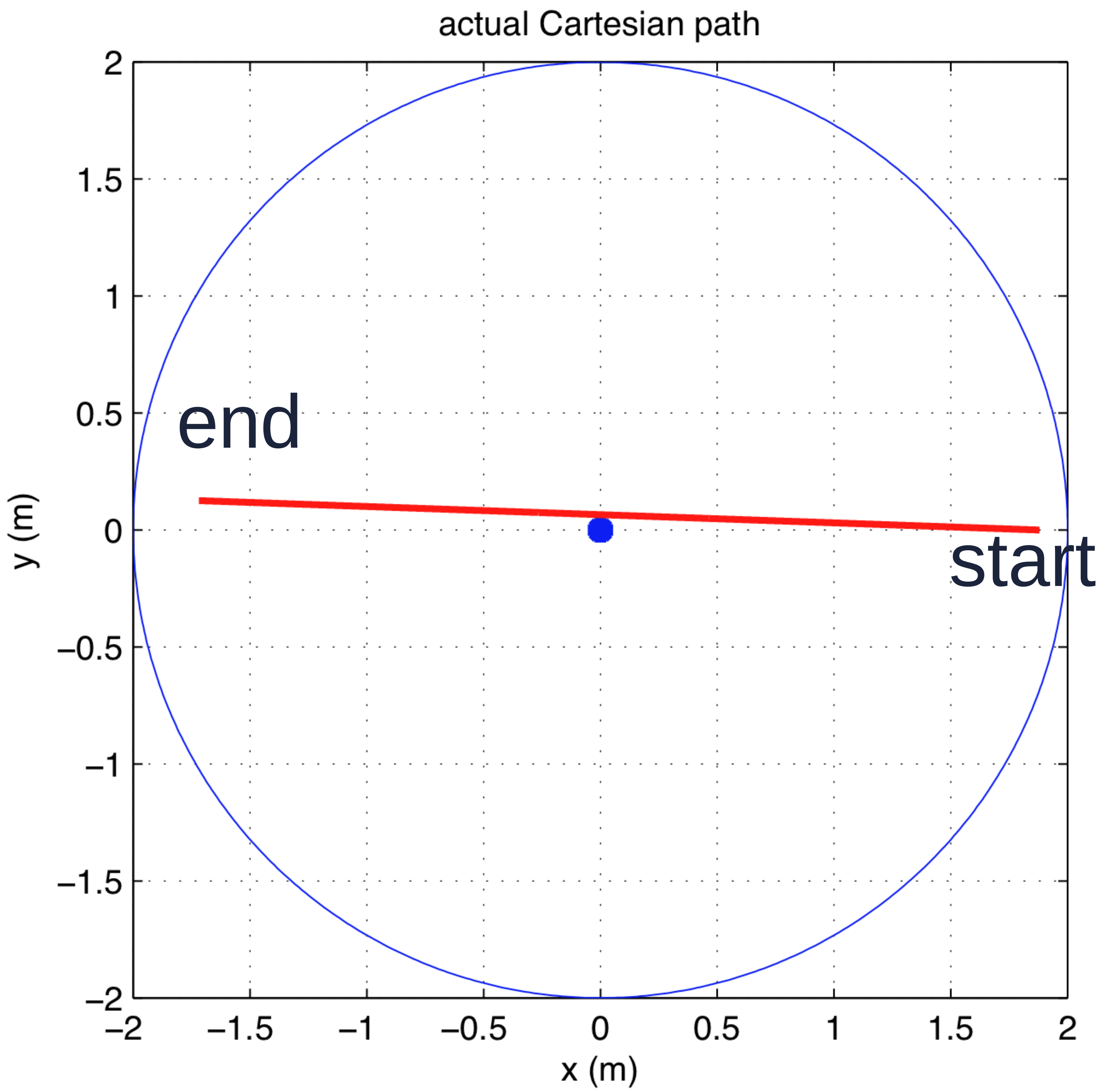


error due only to numerical integration (10^{-10})

Simulation

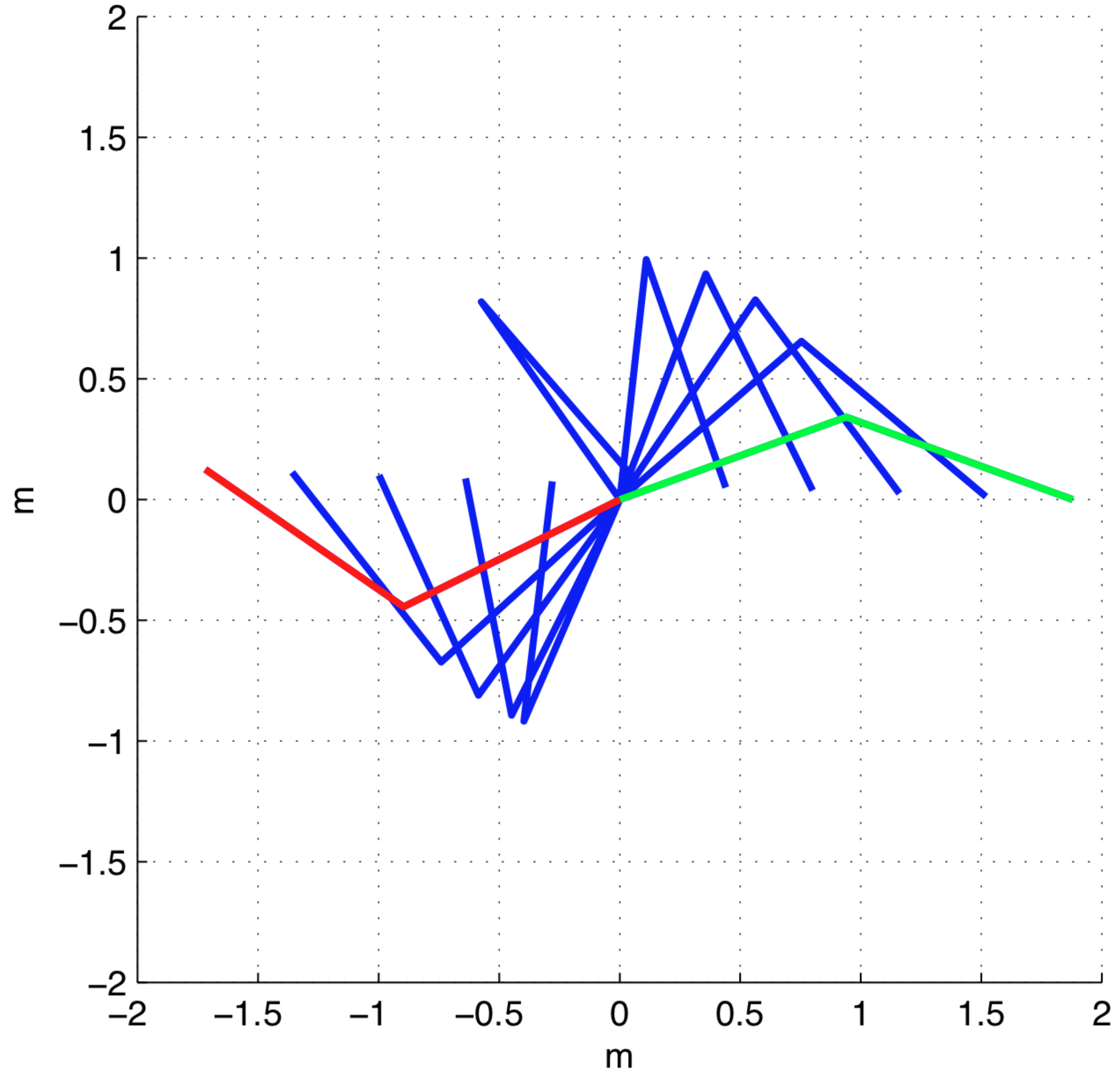
$\dot{q} = J^{-1}(q)v$

Simulation results
(planar 2R robot in straight line Cartesian motion)

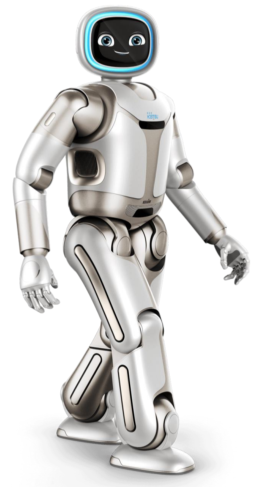


close to singular case

stroboscopic view



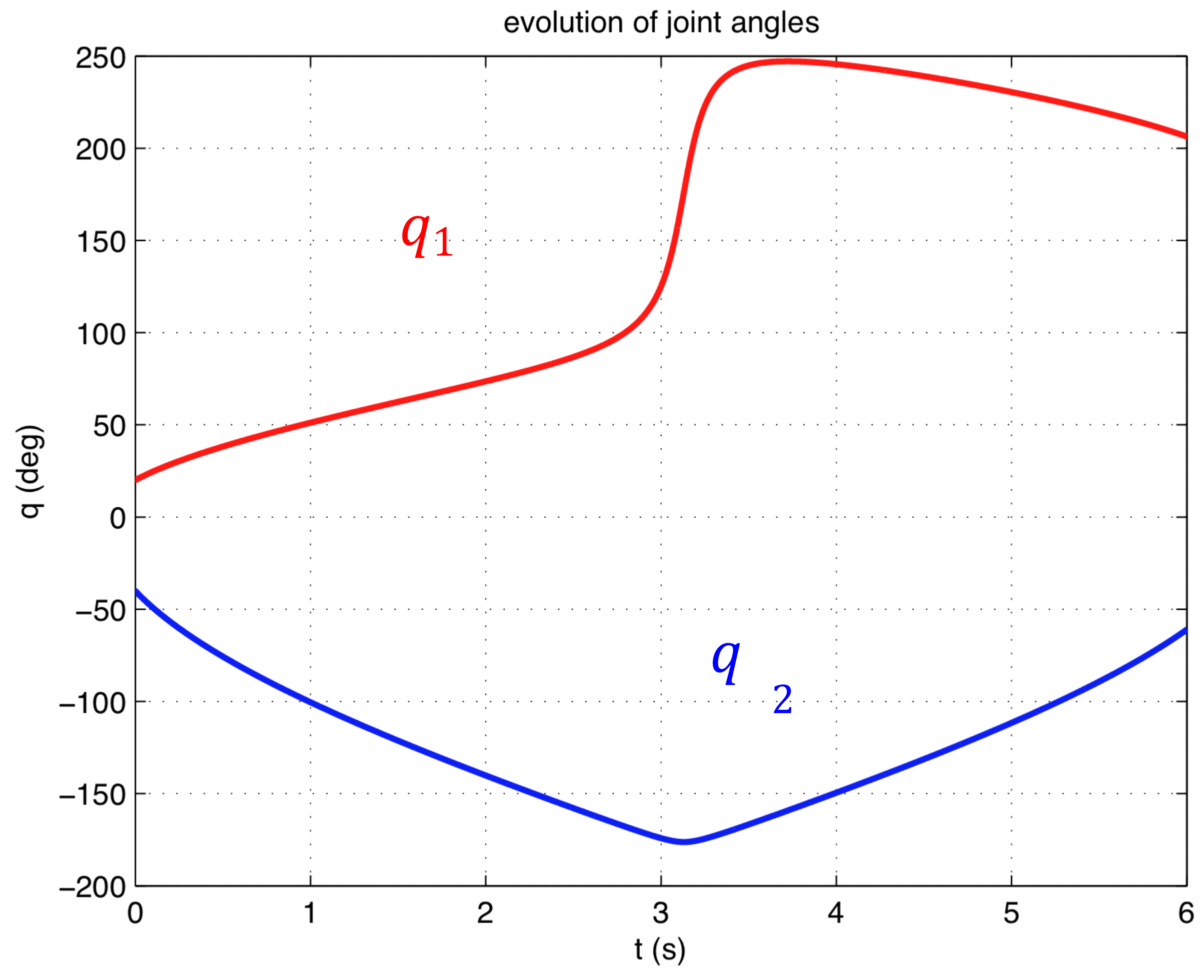
a line from right to left, at $\alpha = 178°$ angle with x-axis,
executed at constant speed $v = 0.6\ m/s$ for $T = 6\ s$

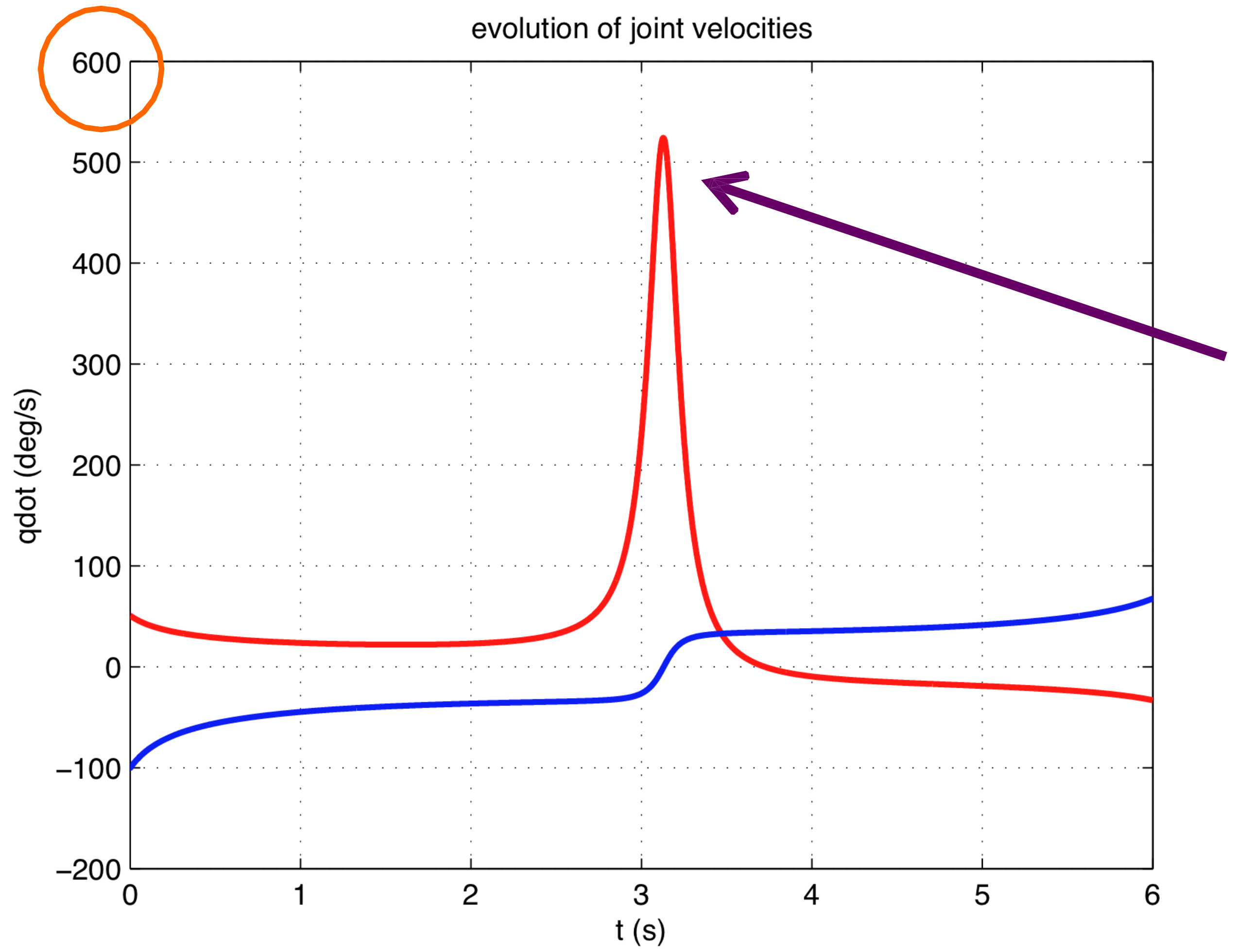


Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

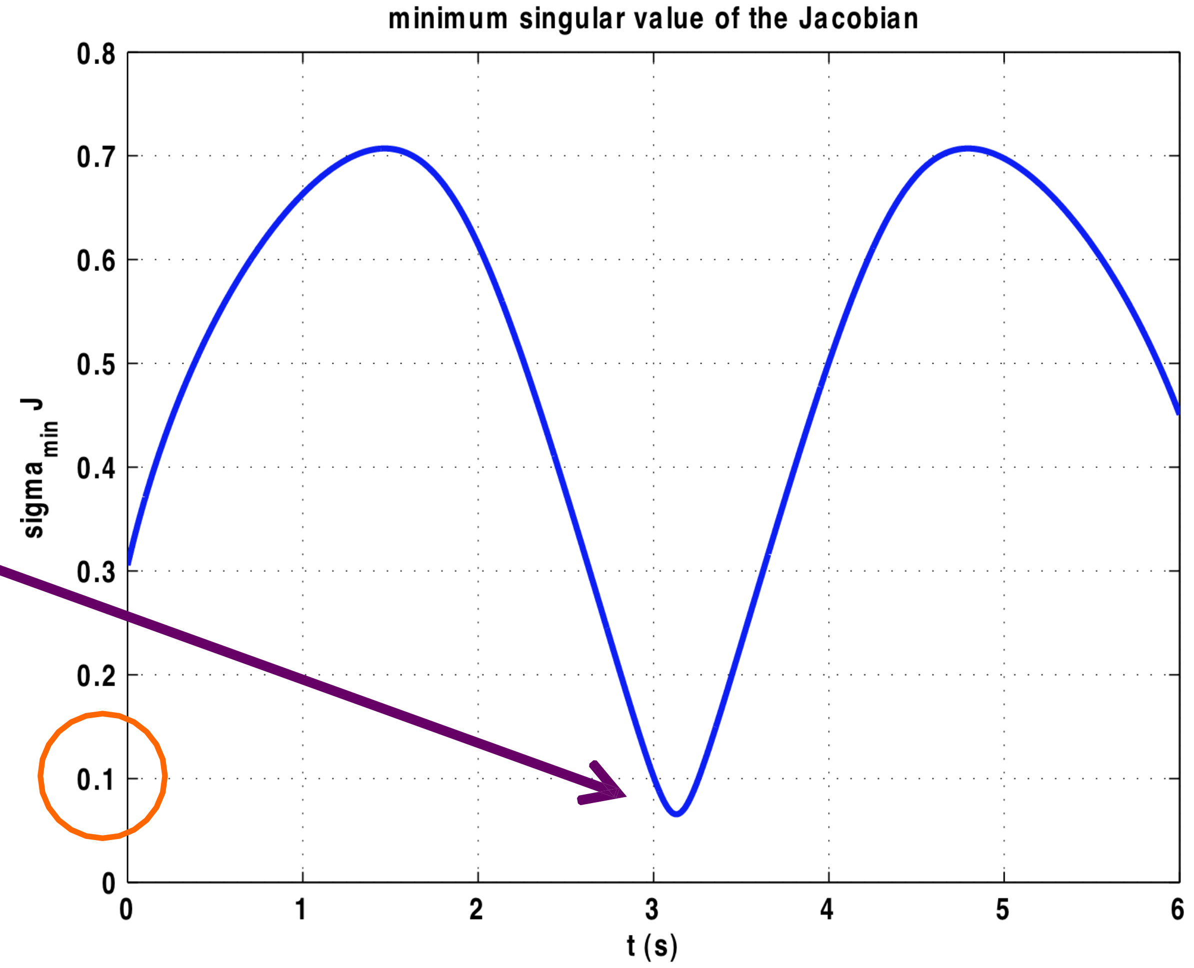
path at
$\alpha = 178°$

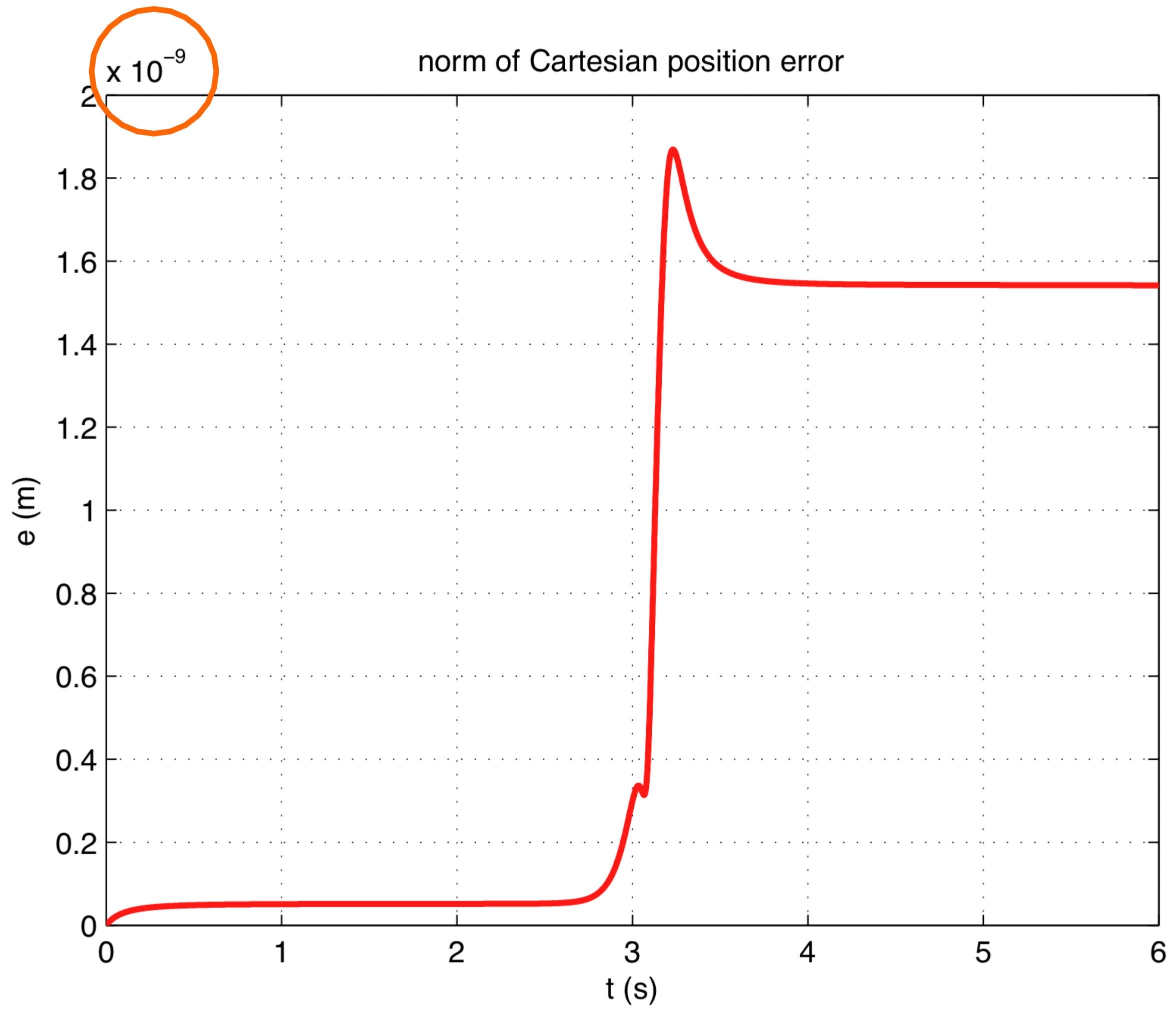




large
peak
of $\dot{q}_1$

close to
singular
case





still very small,
but increased
numerical
integration
error(2×10^{-9})

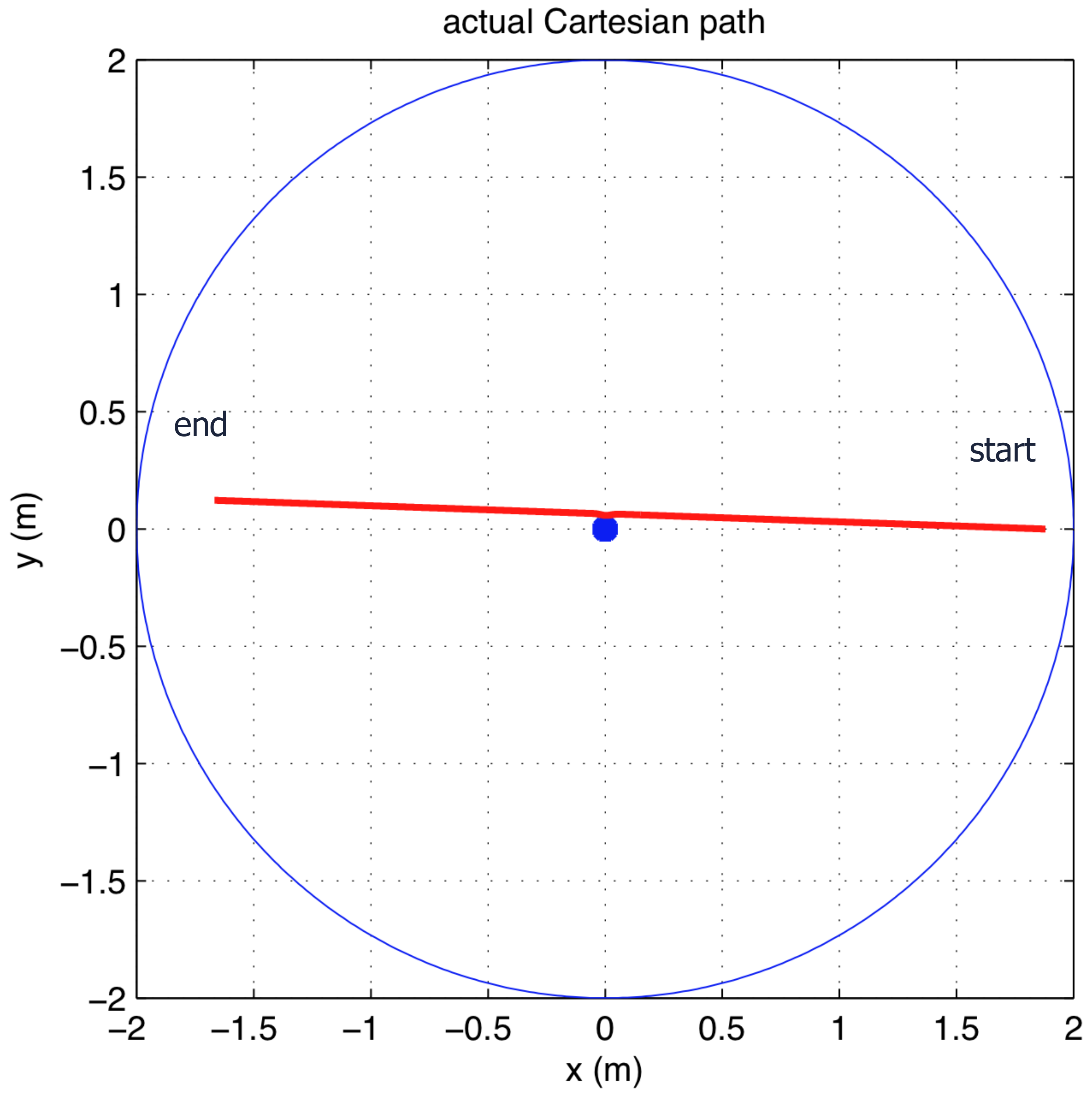
香港中文大學
The Chinese University of Hong Kong

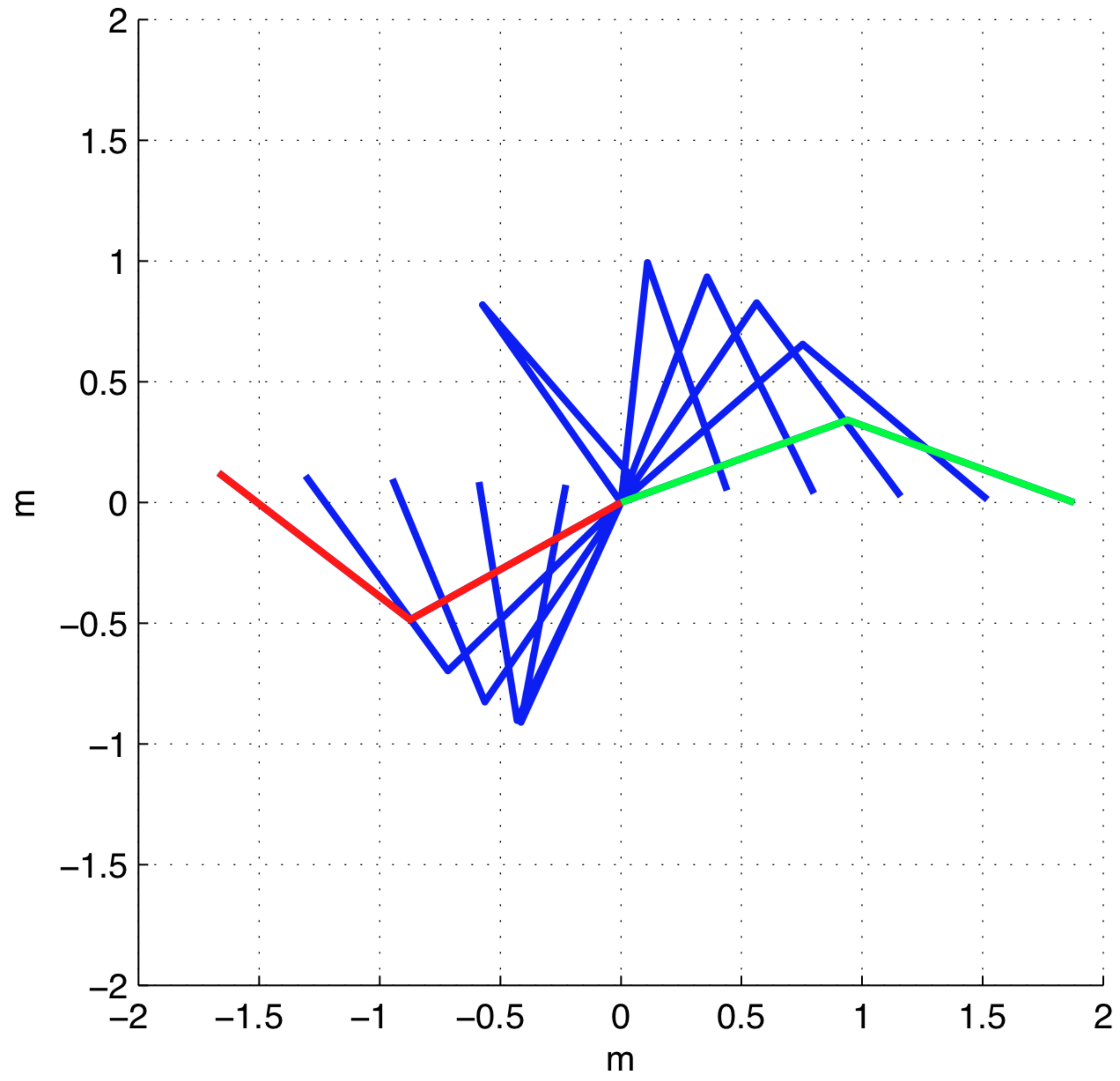
Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

$\dot{q} = J^{-1}(q)v$

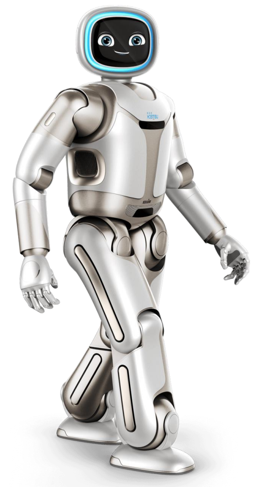
close to singular case with joint velocity saturation at $V_i = 300°/s$





stroboscopic view

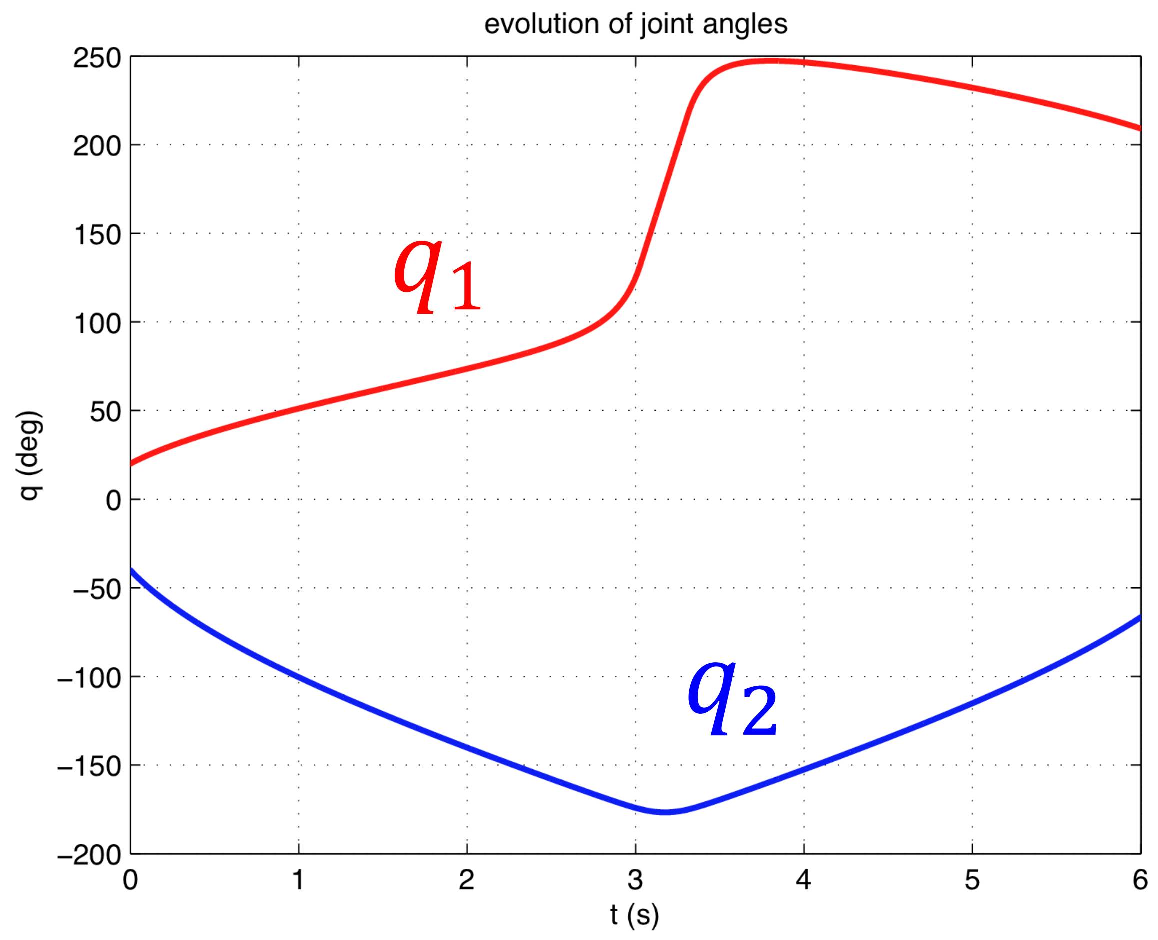
a line from right to left, at $\alpha = 178°$ angle with x-axis, executed at constant speed $v = 0.6\ m/s$ for $T = 6\ s$

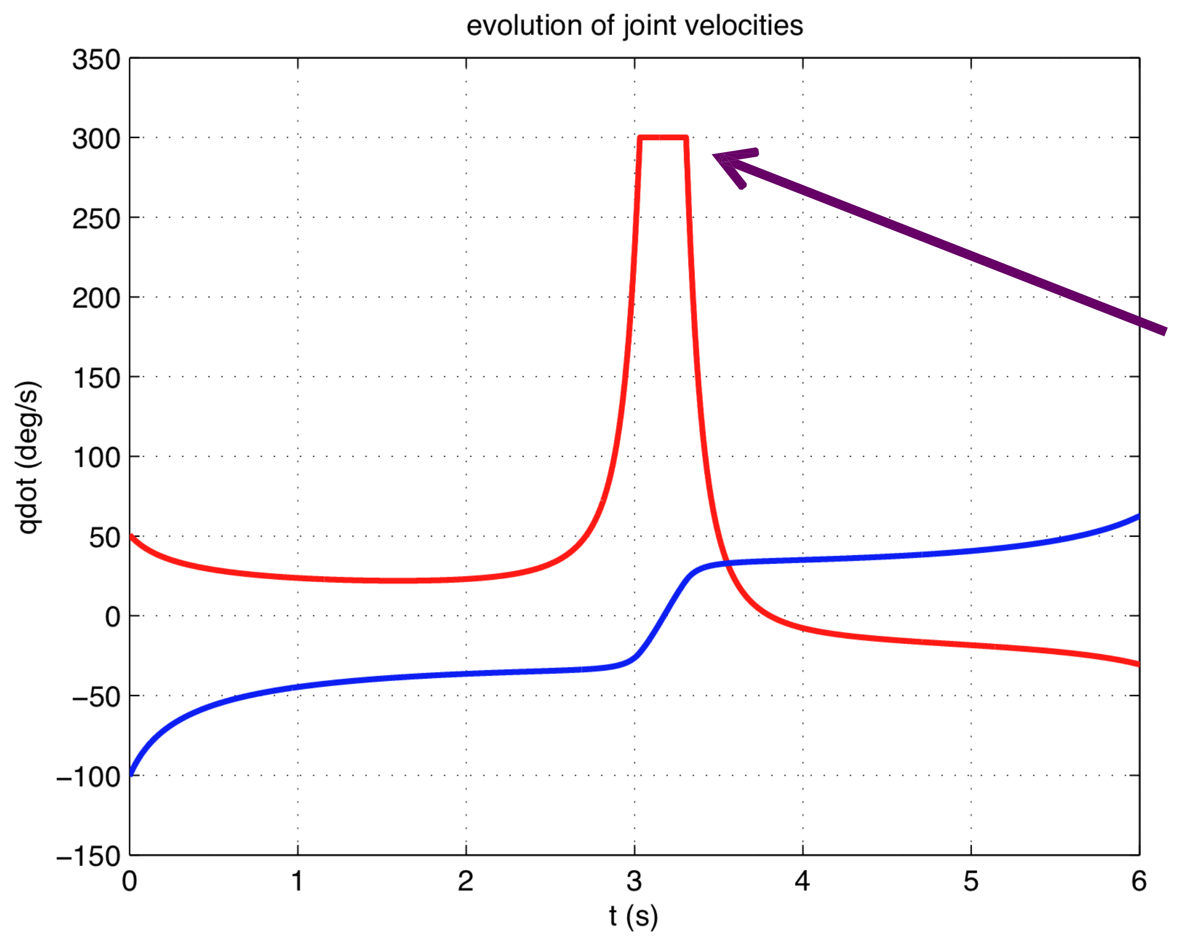


Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

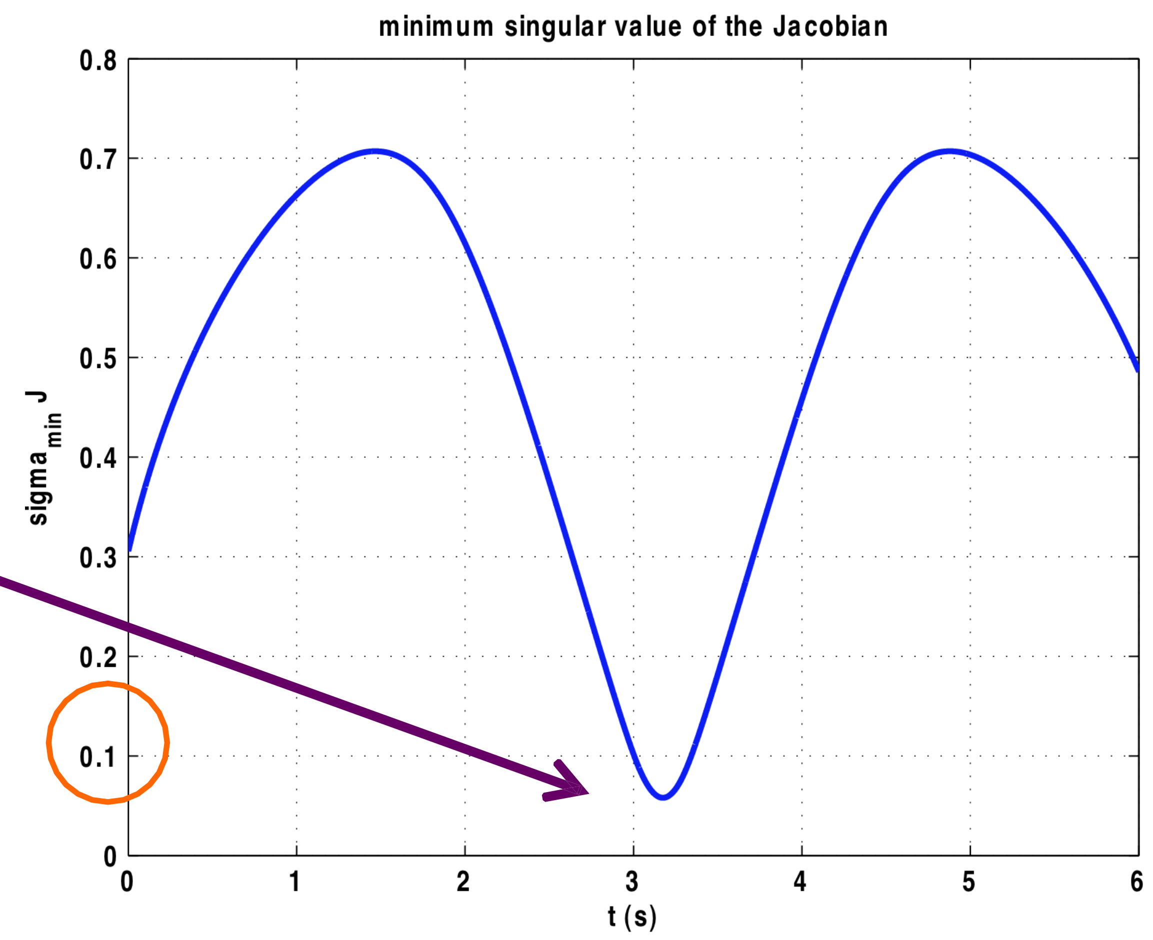
path at
$\alpha = 178°$

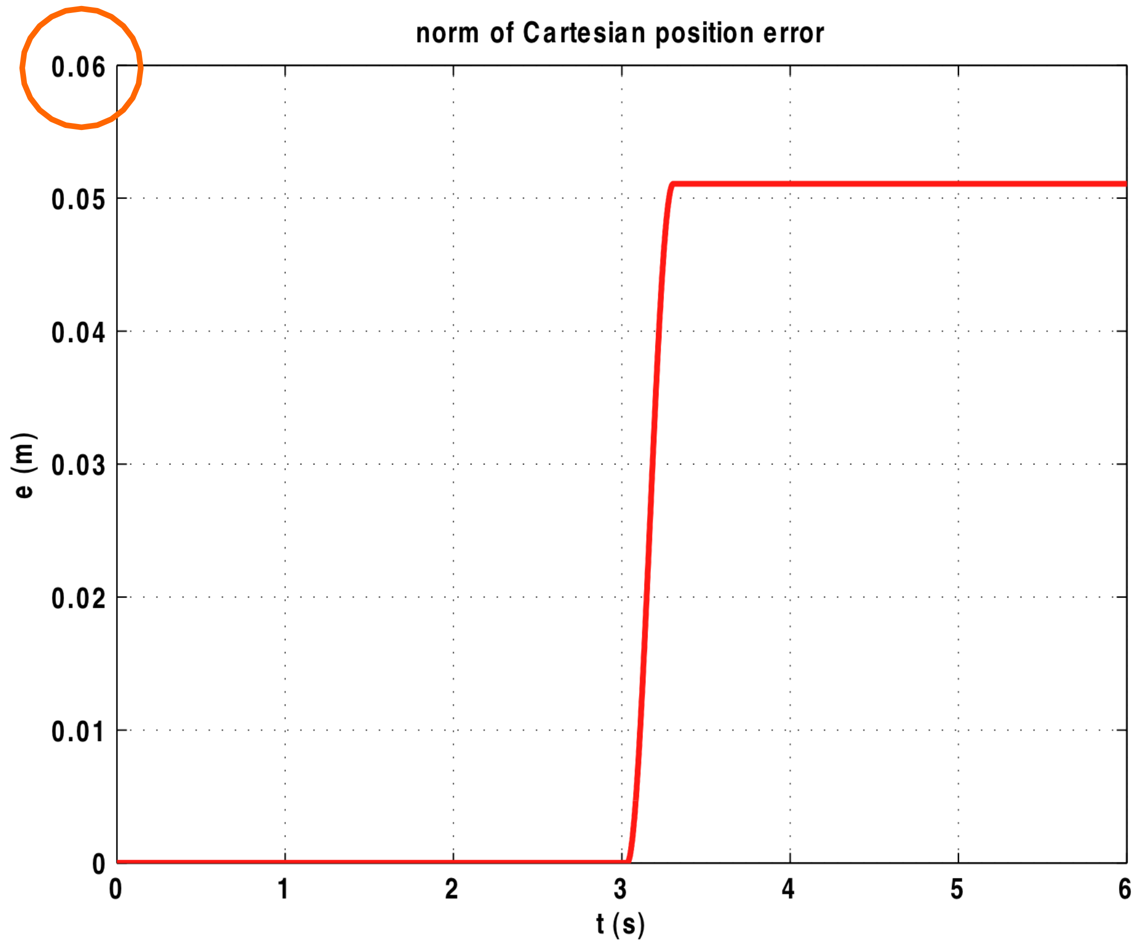




saturated value
of q_1

close to
singular
case





actual position
error!! (6 cm)

Damped Least Squares Method

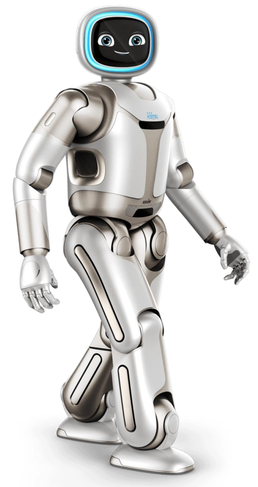
Bruno: 3.5.1

$$\min_{\dot{q}} H = \frac{\lambda}{2}\|\dot{q}\|^2 + \frac{1}{2}\|J\dot{q} - v\|^2, \quad \lambda \geq 0$$

$$\dot{q} = (\lambda I_n + J^T J)^{-1} J^T v = J^T(\lambda I_m + JJ^T)^{-1} v = J_{DLS} v$$

two equivalent expressions, but the second is more convenient in redundant robots!

- inversion of differential kinematics as unconstrained optimization problem
- function $H =$ weighted sum of two objectives (norm of joint velocity and error norm on achieved end-effector velocity) to be minimized
- J_{DLS} can be used for both cases: $m = n$ (square) and $m < n$ (redundant)
- $\lambda = 0$ when "far enough" from singularities: $J_{DLS} = J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1} = J^{-1}$ or $J^{\#}$
- with $\lambda > 0$, there is a (vector) *error* ϵ $(= v - J\dot{q})$ in executing the desired end-effector velocity v (check that $\epsilon = \lambda(\lambda I_m + (JJ^T)^{-1} v)$, but the joint velocities are always reduced ("damped")

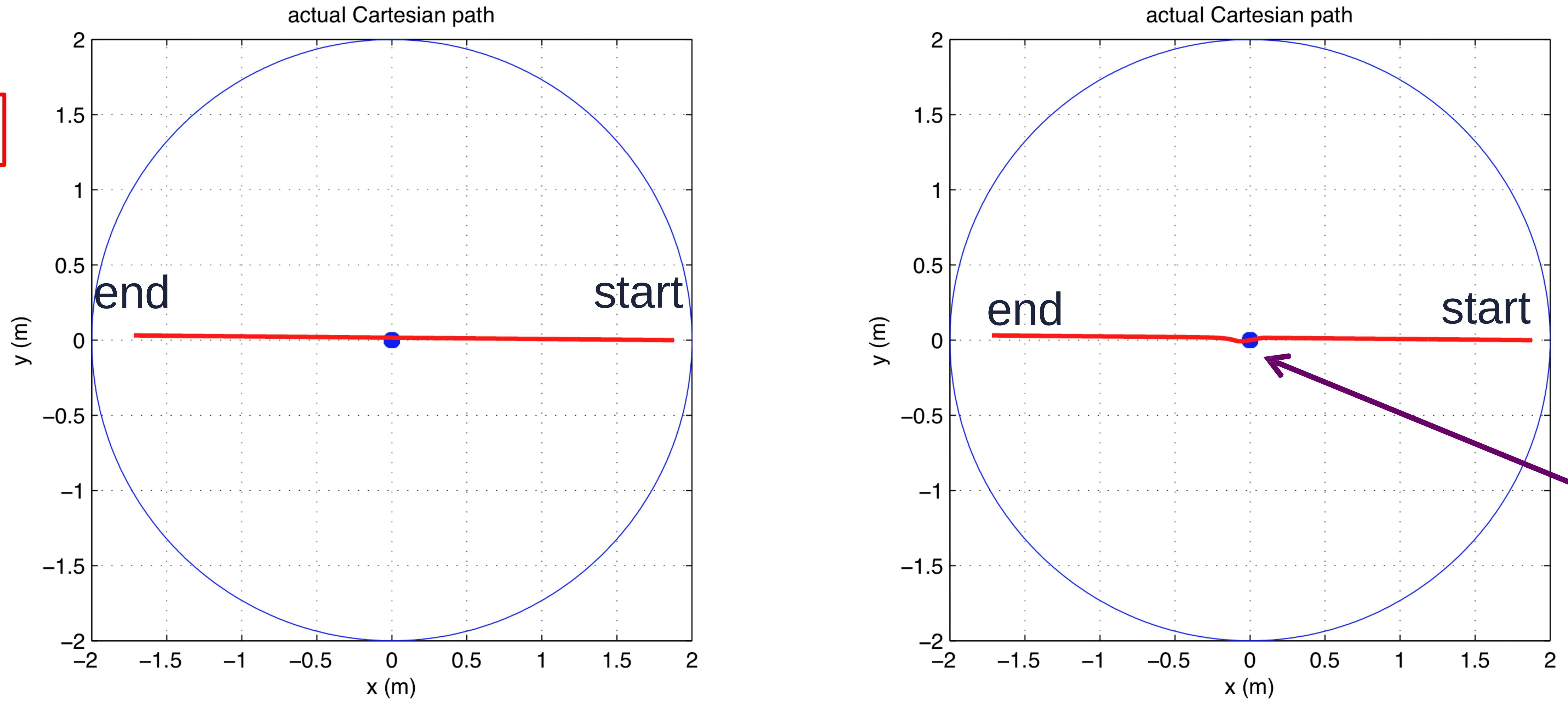


Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

a comparison of inverse and damped inverse Jacobian methods even closer to singular case

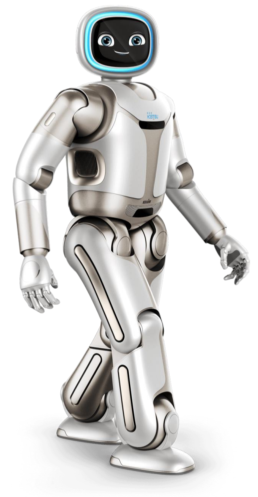
$\dot{q} = J^{-1}(q)v$



$\dot{q} = J_{DLS}(q)v$

some position error imported.

a line from right to left, at $\alpha = 179.5°$ angle with x-axis, executed at constant speed $v = 0.6\ m/s$ for $T = 6\ s$



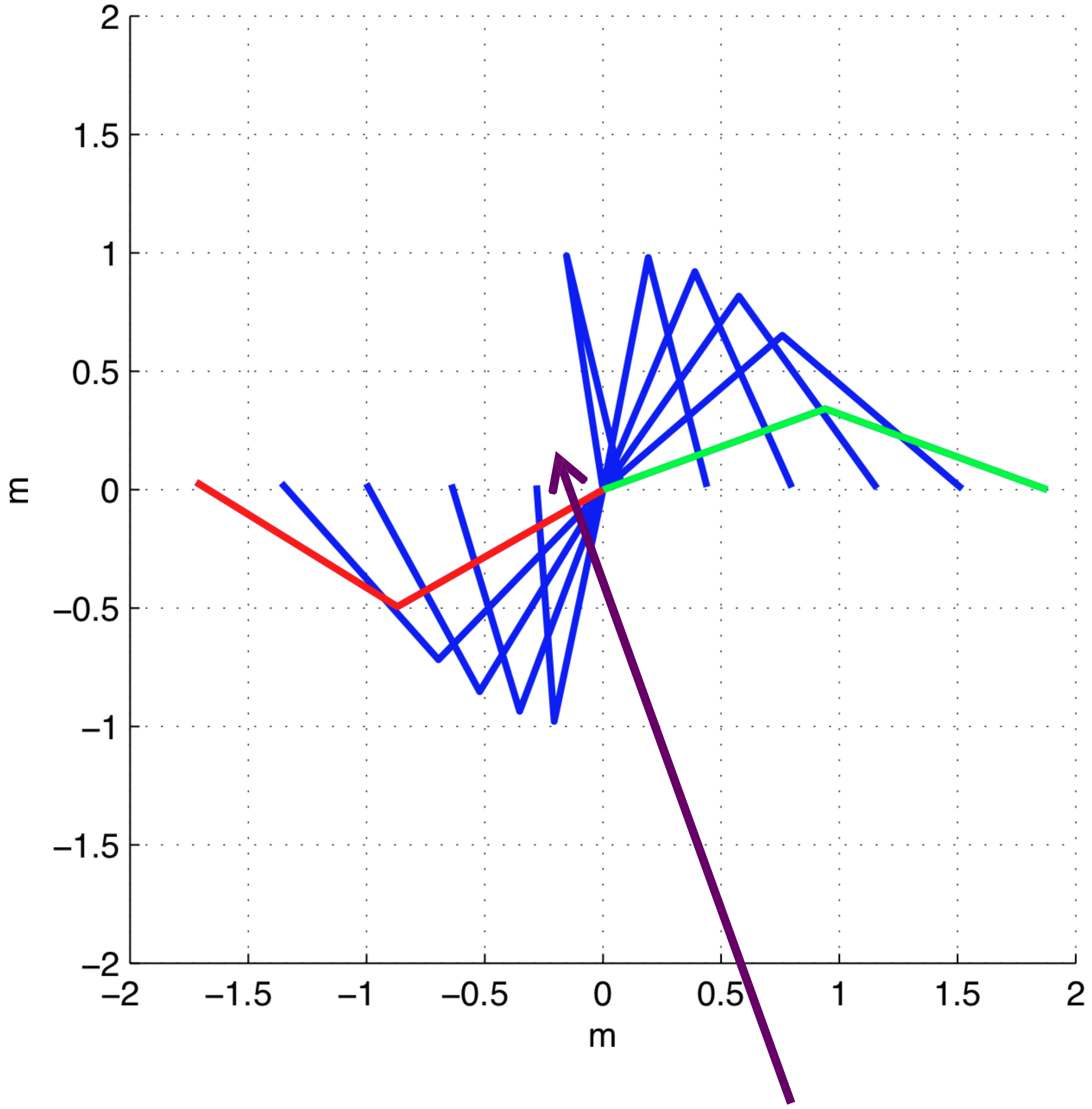
Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

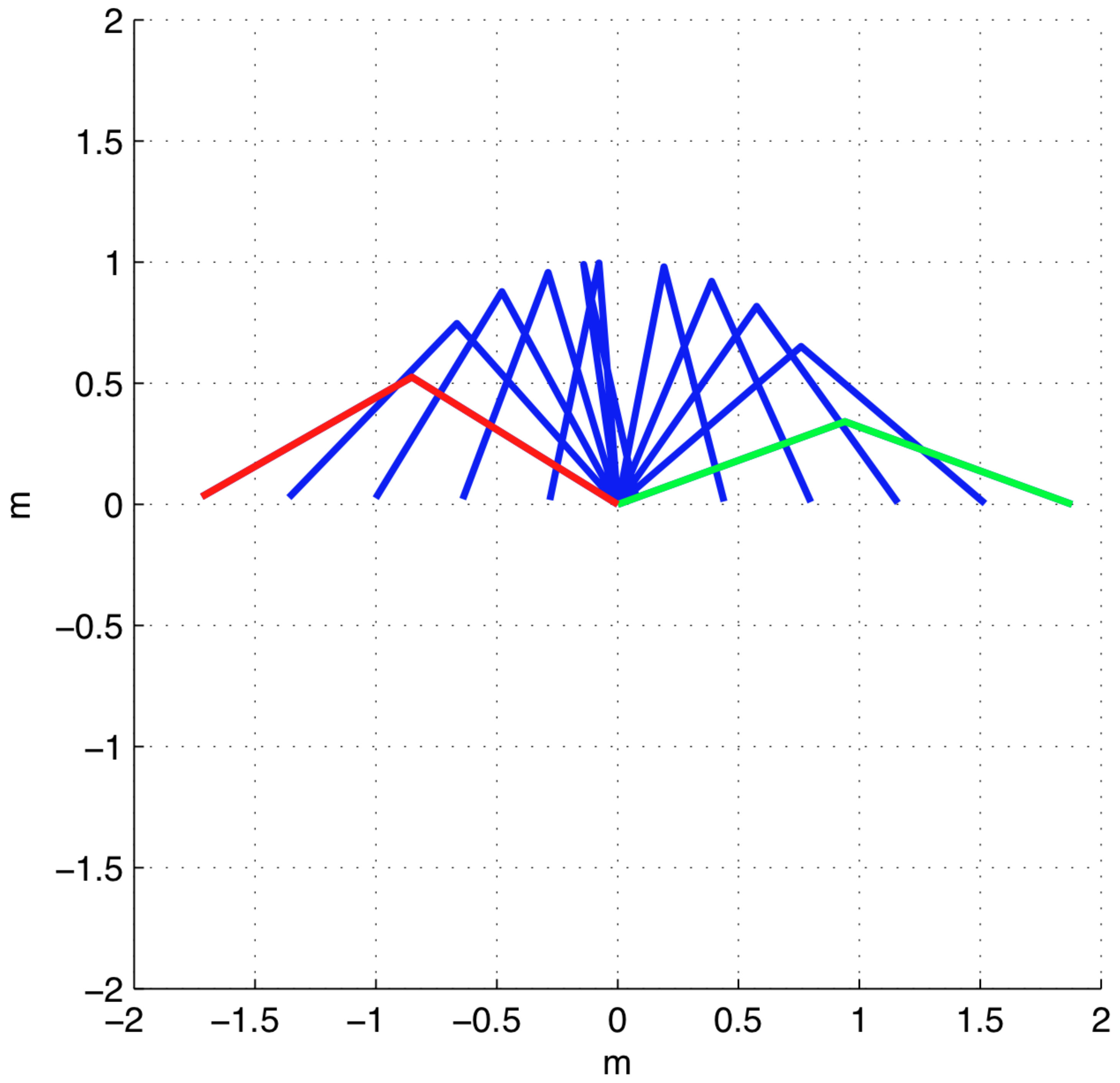
$\dot{q} = J^{-1}(q)v$

path at
$\alpha = 179.5°$

$\dot{q} = J_{DLS}(q)v$

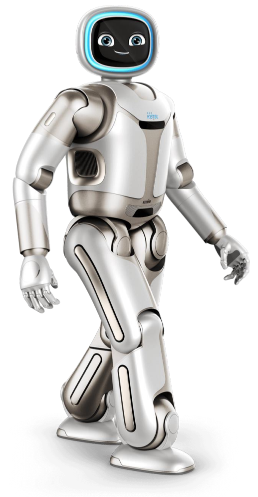


here, a very fast reconfiguration of first joint ...



stroboscopic views

a completely different inverse solution, around/after crossing the region close to the folded singularity



Simulation

Simulation results
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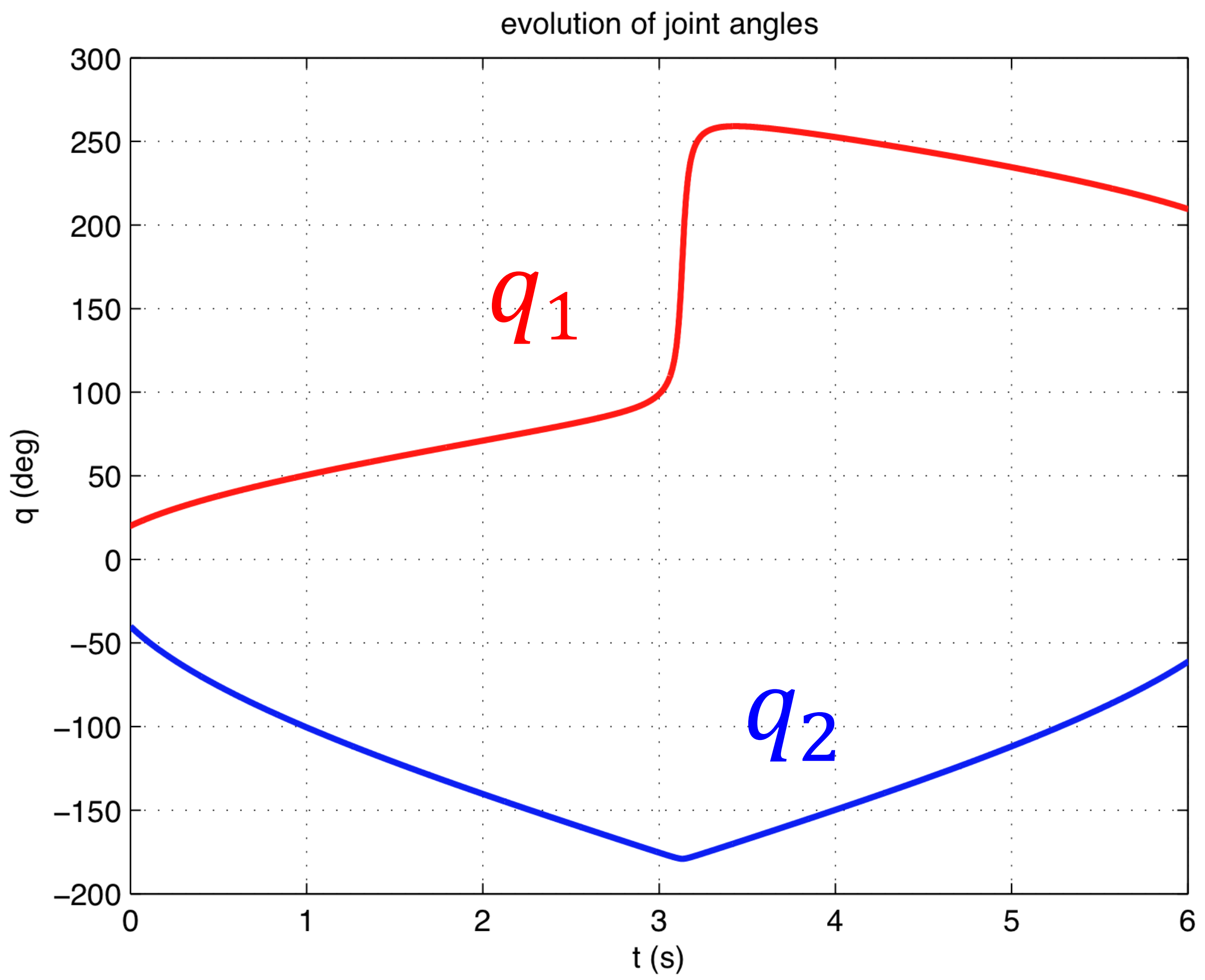
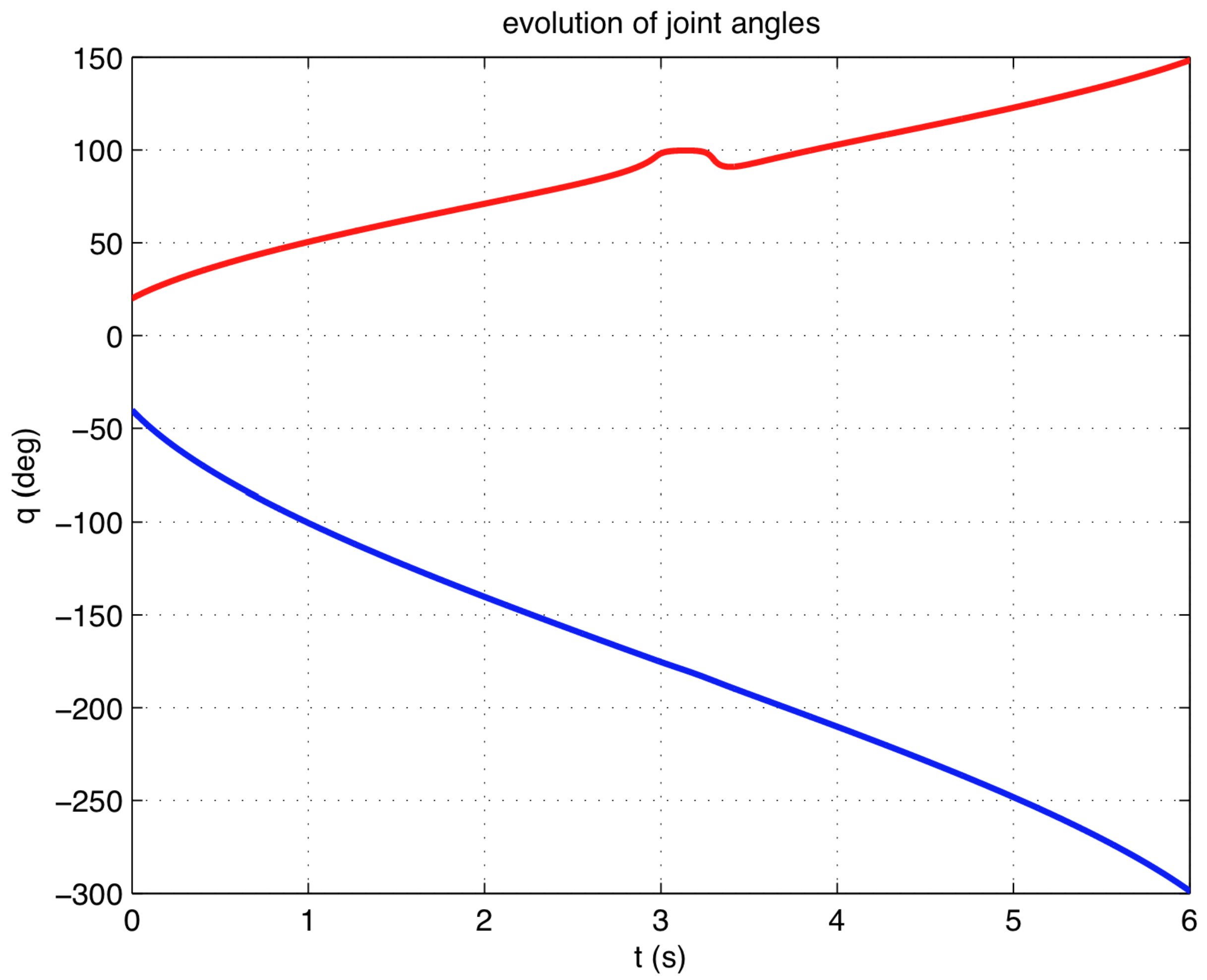
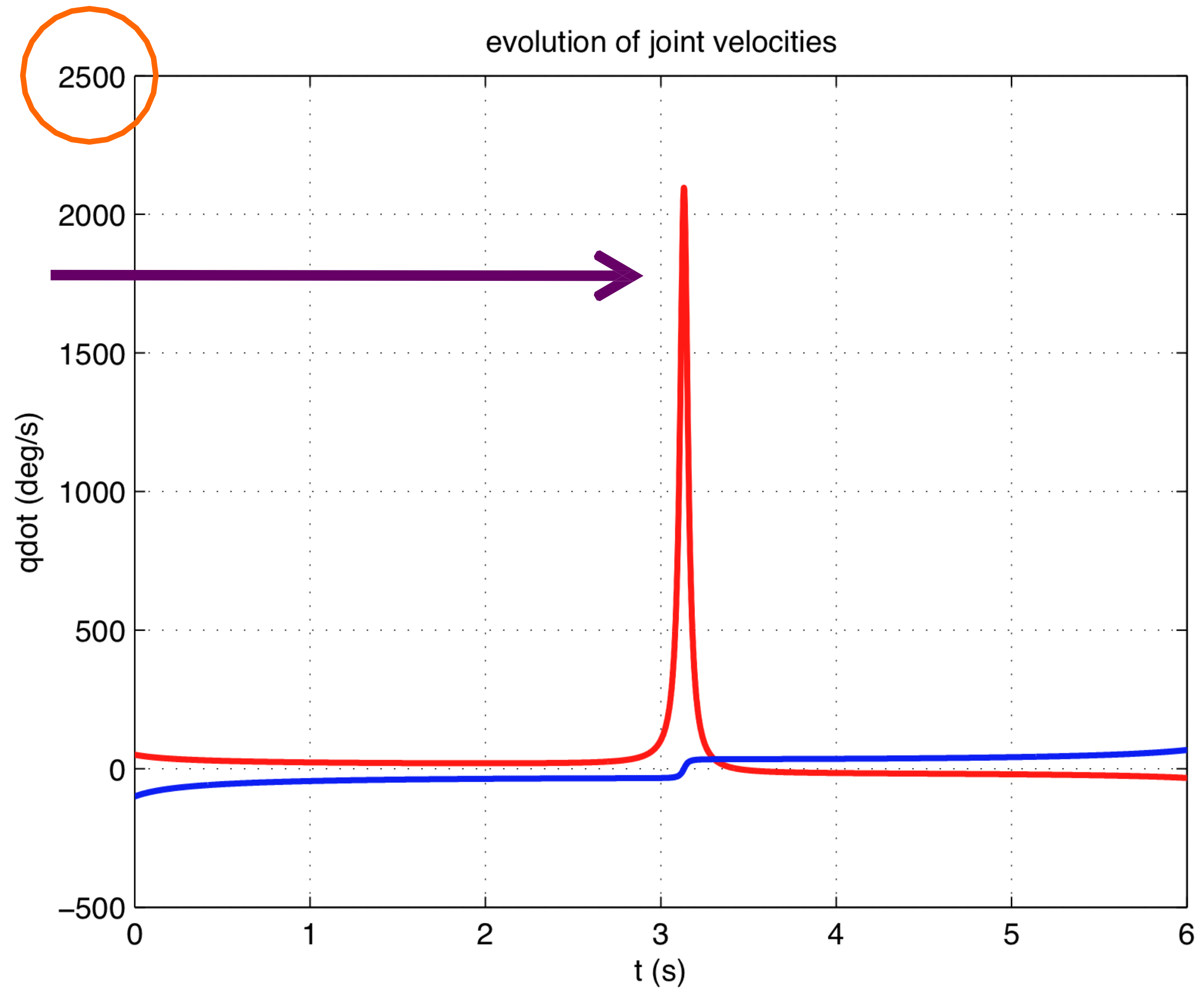
$\dot{q} = J^{-1}(q)v$

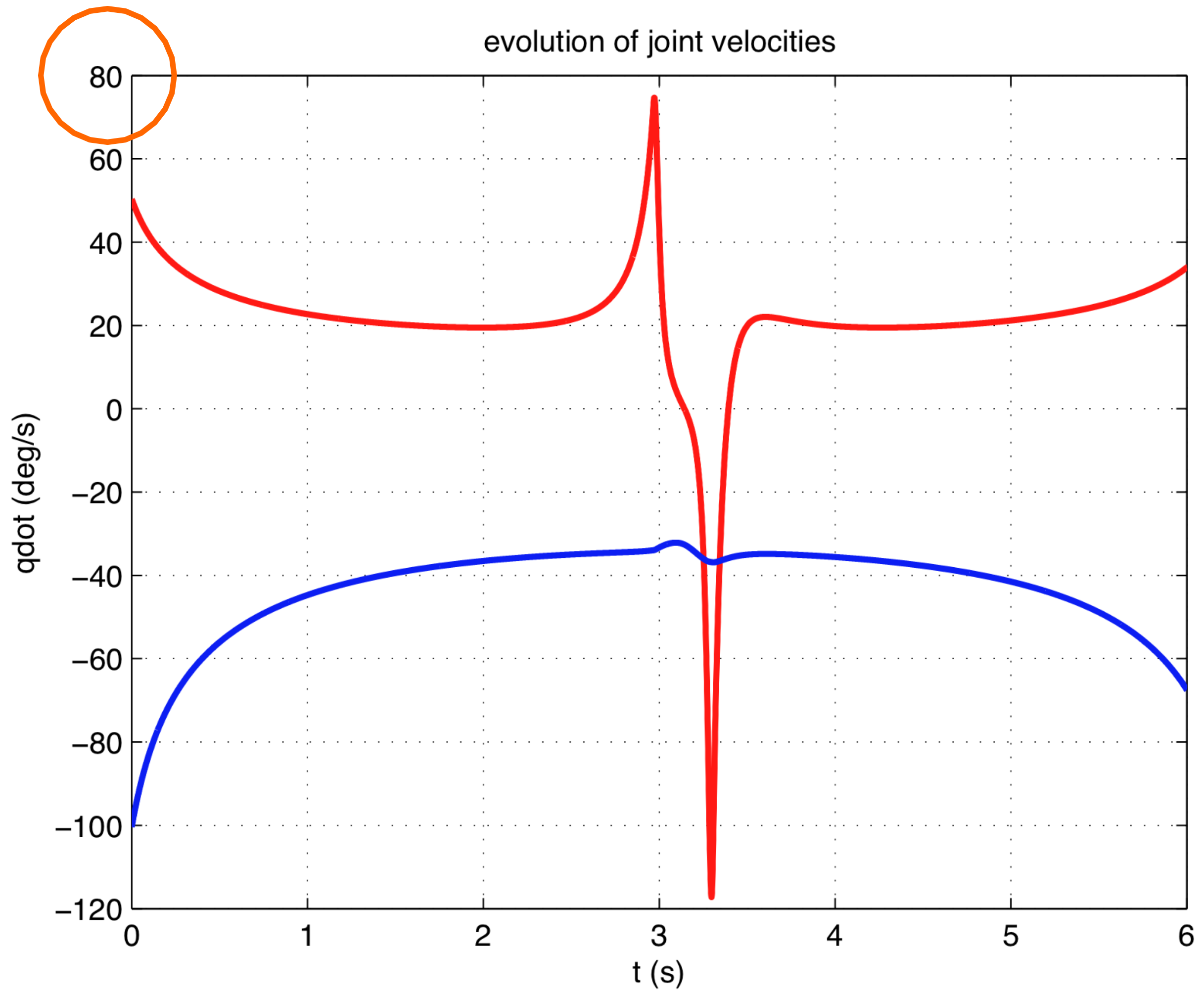




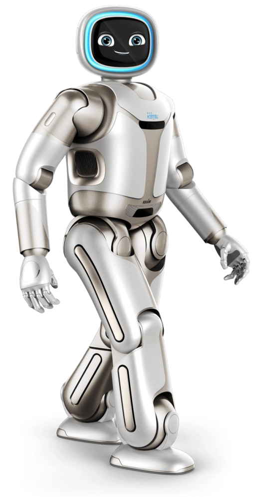
$\dot{q} = J_{DLS}(q)v$

extremely large peak velocity of first joint!!





smoother joint motion with limited joint velocities!

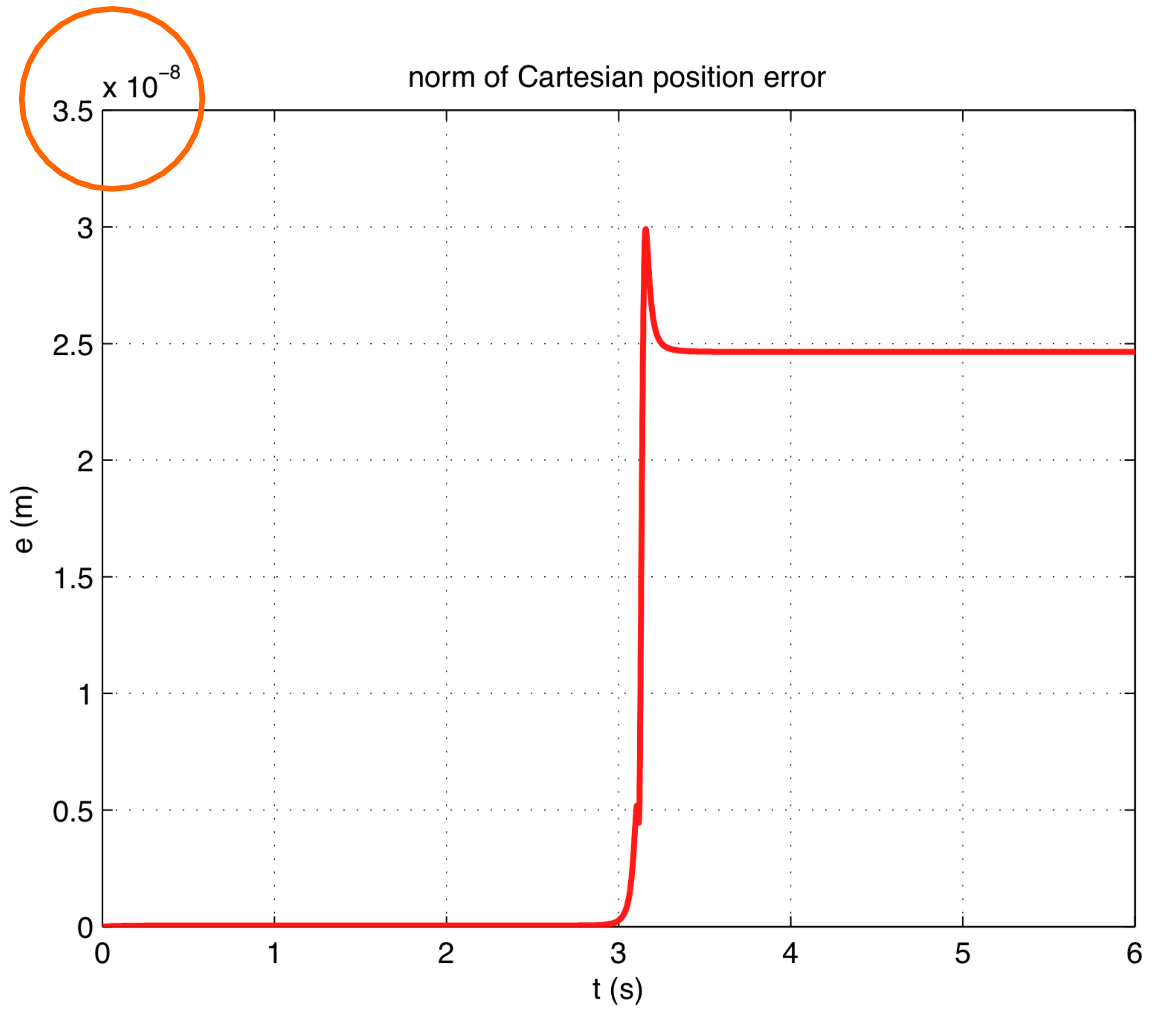


Simulation

Simulation results
(planar 2R robot in straight line Cartesian motion)

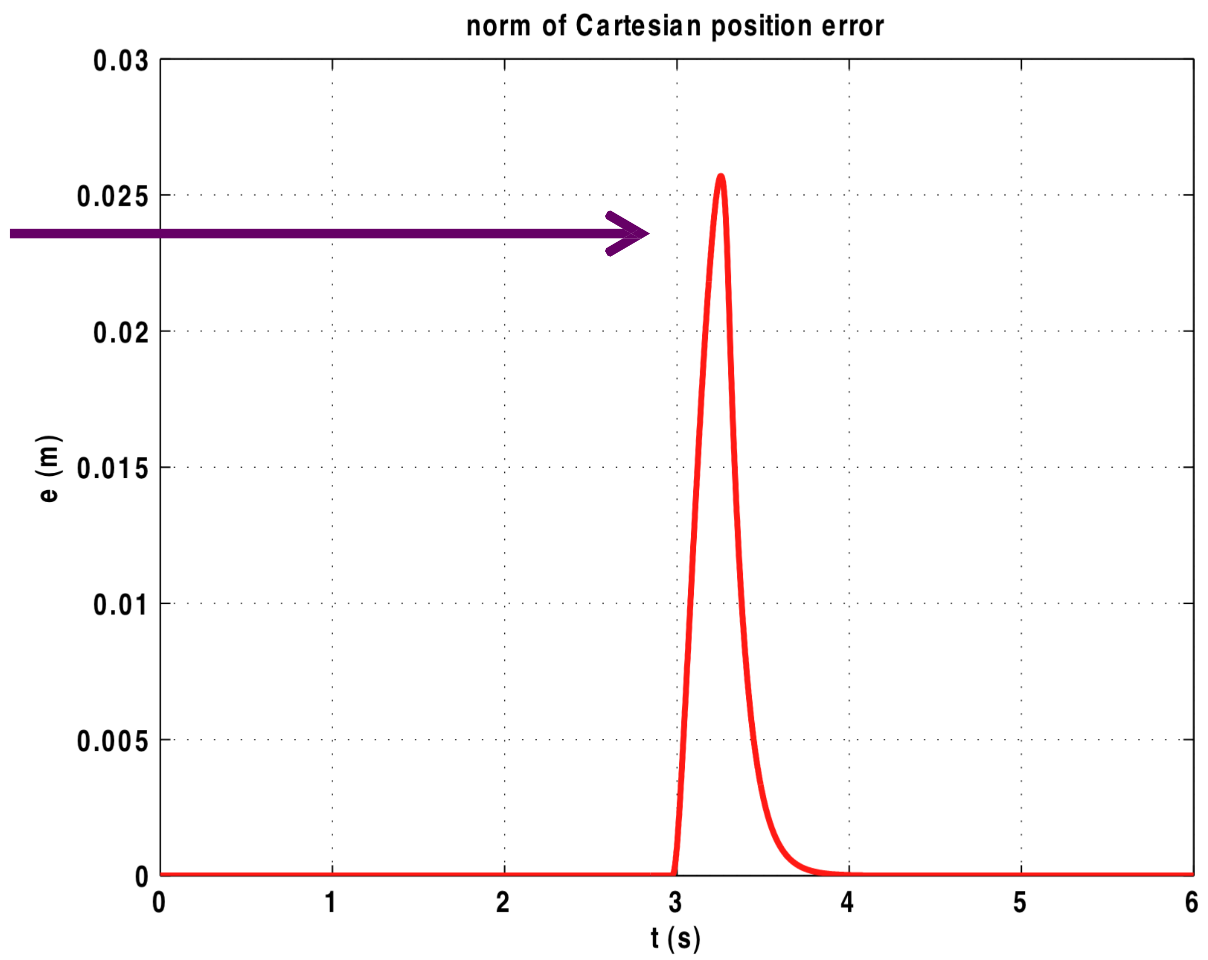
$\dot{q} = J^{-1}(q)v$

$\dot{q} = J_{DLS}(q)v$



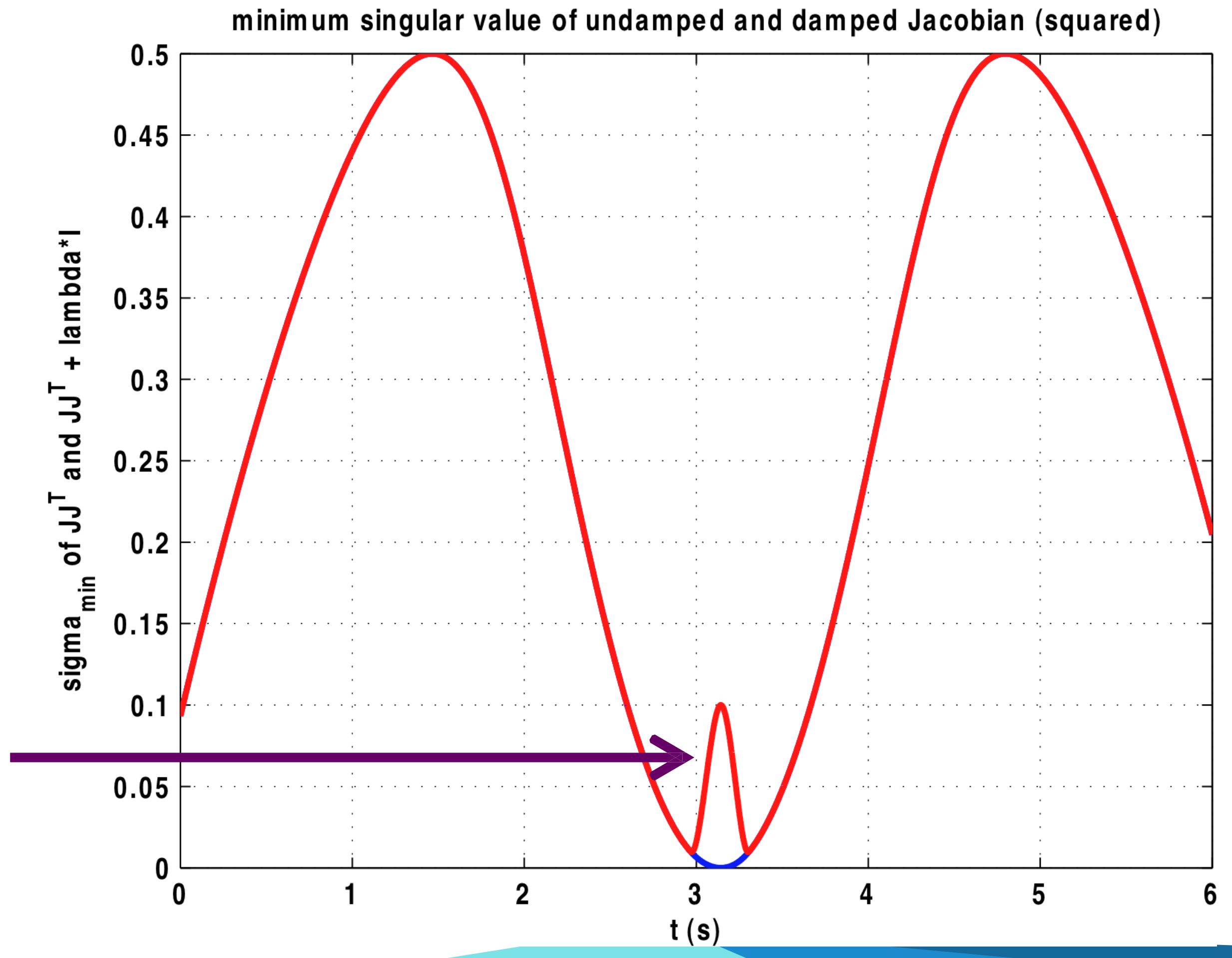
increased numerical integration error (3×10^{-8})

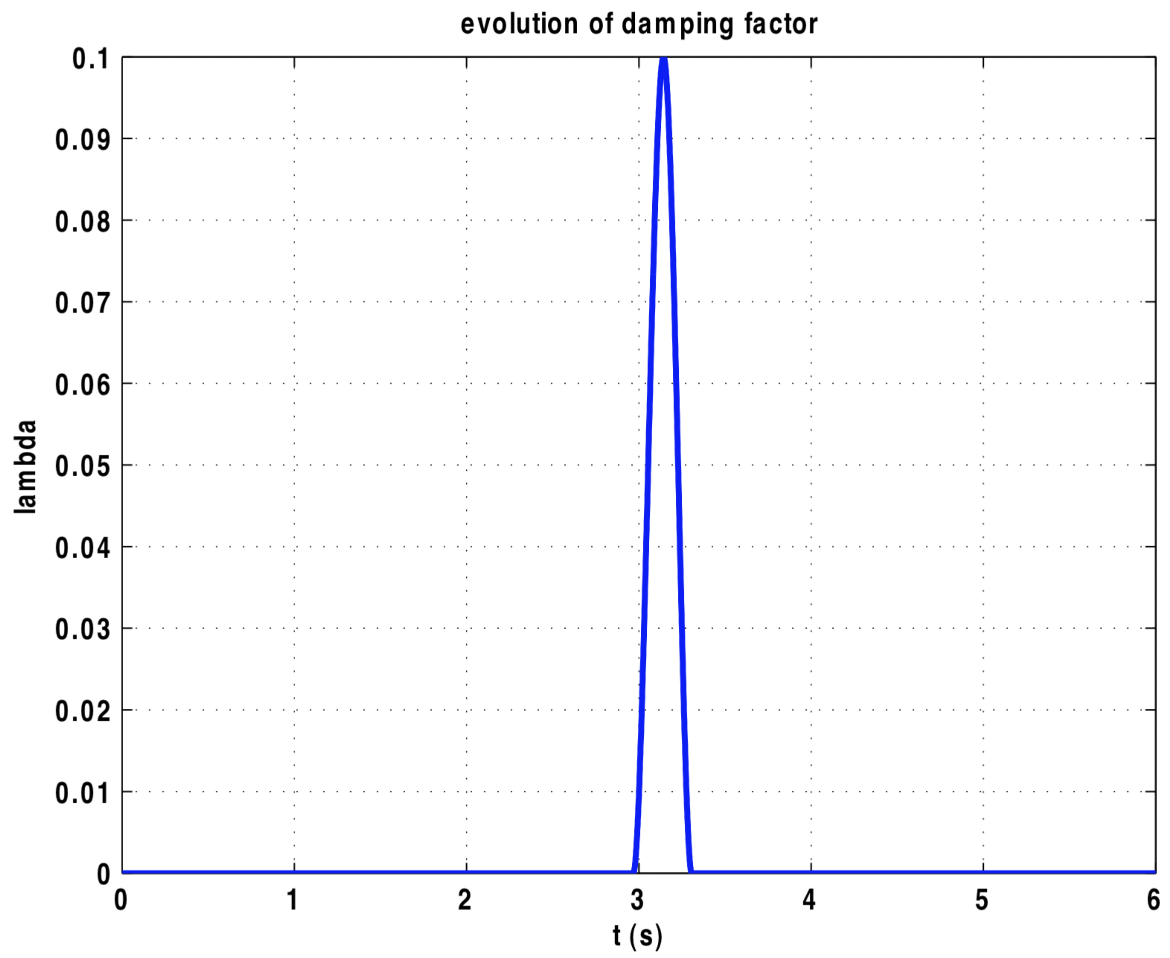
error (25 mm) when crossing the singularity, later recovered by a feedback action ($v \Rightarrow v + K_p e_p$ with $e_p = p_d - p(q)$)



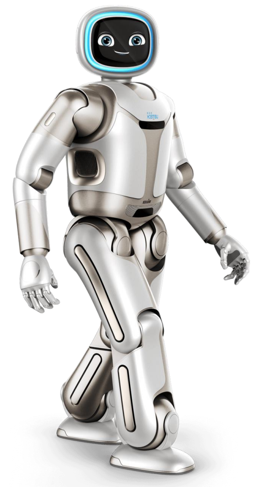
minimum singular value of JJ^T and $\lambda I + JJ^T$

they differ only when damping factor is non-zero





damping factor λ is chosen non-zero only close to singularity!



Pseudoinverse Method

a constrained optimization (minimum norm) problem

$$\min_{\dot{q}} H = \frac{1}{2}\|\dot{q}\|^2$$

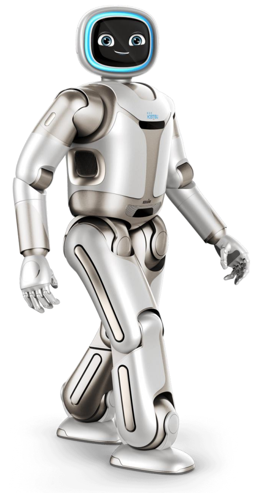
such that $J\dot{q} = v$

solution $\dot{q} = J^{\#}v$

pseudoinverse of J

- if $v \in \mathcal{R}(J)$, the differential constraint is satisfied (v is feasible)
- else, $J\dot{q} = JJ^{\#}v = v^{\perp}$, where $v^{\perp}$ minimizes the error $\|J\dot{q} - v\|$

orthogonal projection of v on $\mathcal{R}(J)$



Pseudoinverse Method

Definition of the pseudoinverse

given J, is the unique matrix $J^{\#}$ satisfying the four relationships

$$JJ^{\#}J = J \qquad J^{\#}JJ^{\#} = J^{\#}$$

$$(JJ^{\#})^T = JJ^{\#} \qquad (J^{\#}J)^T = J^{\#}J$$

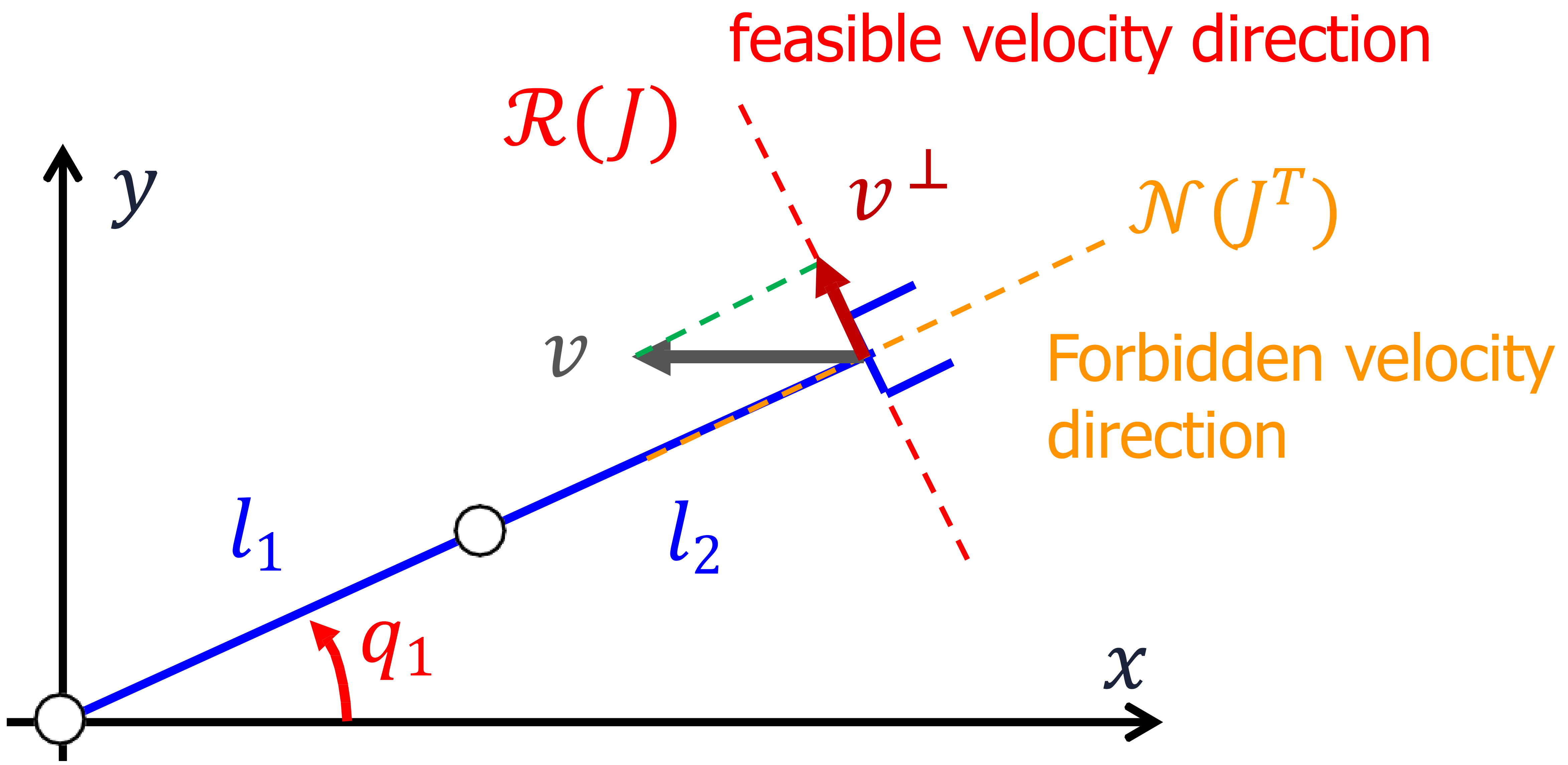
- explicit expressions for full rank cases
  - if $\rho(J) = m = n$: $J^{\#} = J^{-1}$
  - if $\rho(J) = m < n$: $J^{\#} = J^{T}(JJ^{T})^{-1}$
  - if $\rho(J) = n < m$: $J^{\#} = (J^{T}J)^{-1}J^{T}$
- $J^{\#}$ always exists and is computed in general numerically using the SVD = Singular Value Decomposition of J
  - e.g., with the MATLAB function **pinv** (which uses in turn **svd**)

Numerical Example

Jacobian of 2R robot with $l_1 = l_2 = 1$ at $q_2 = 0$ (rank $\rho(J) = 1$)

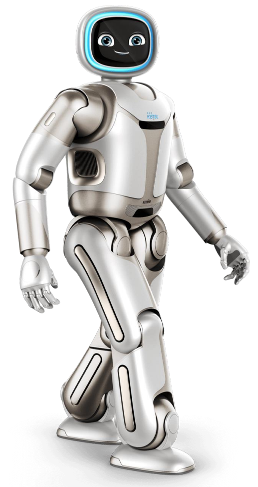
$$J = \begin{pmatrix} -2s_1 & -s_1 \\ 2c_1 & c_1 \end{pmatrix} \qquad J^{\#} = \frac{1}{5}\begin{pmatrix} -2s_1 & 2c_1 \\ -s_1 & c_1 \end{pmatrix}$$

$$JJ^{\#} = \begin{pmatrix} s_1^2 & -s_1 c_1 \\ -s_1 c_1 & c_1^2 \end{pmatrix}$$



$\dot{q} = J^{\#} v$ is the minimum norm joint velocity vector that realizes exactly $v^{\perp}$

- at $q_1 = \pi/6$: for $v = (-0.5 \quad 0)^T \left[\frac{m}{s}\right]$, $\dot{q} = J^{\#} v = (0.1 \quad 0.05)^T \left[\frac{rad}{s}\right] \Rightarrow v^{\perp} = JJ^{\#} v = \left(-1/8 \quad \sqrt{3}/8\right)^T \left[\frac{m}{s}\right]$
- at $q_1 = \pi/2$: $J = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -0.4 & 0 \\ -0.2 & 0 \end{pmatrix}$; now the same $v \in \mathcal{R}(J)$, $\dot{q} = (0.2 \quad 0.1)^T \Rightarrow v^{\perp} = v$ (no error!)



General Solution for $m<n$

Slightly modified constrained optimization problem (“biased” toward the joint velocity ξ, exploring redundancy, ie, chosen to avoid obstacles, joint limits, etc.)

$$\min_{\dot{q}} H = \frac{1}{2}\|\dot{q} - \xi\|^2 \text{ such that } J\dot{q} = v$$

ALL solutions of the inverse differential kinematics problem can be written as

$$\dot{q} = J^{\#}v + (I - J^{\#}J)\xi$$

any joint velocity...

projection matrix of ξ to $\mathcal{N}(J)$ – aka, “null space method”

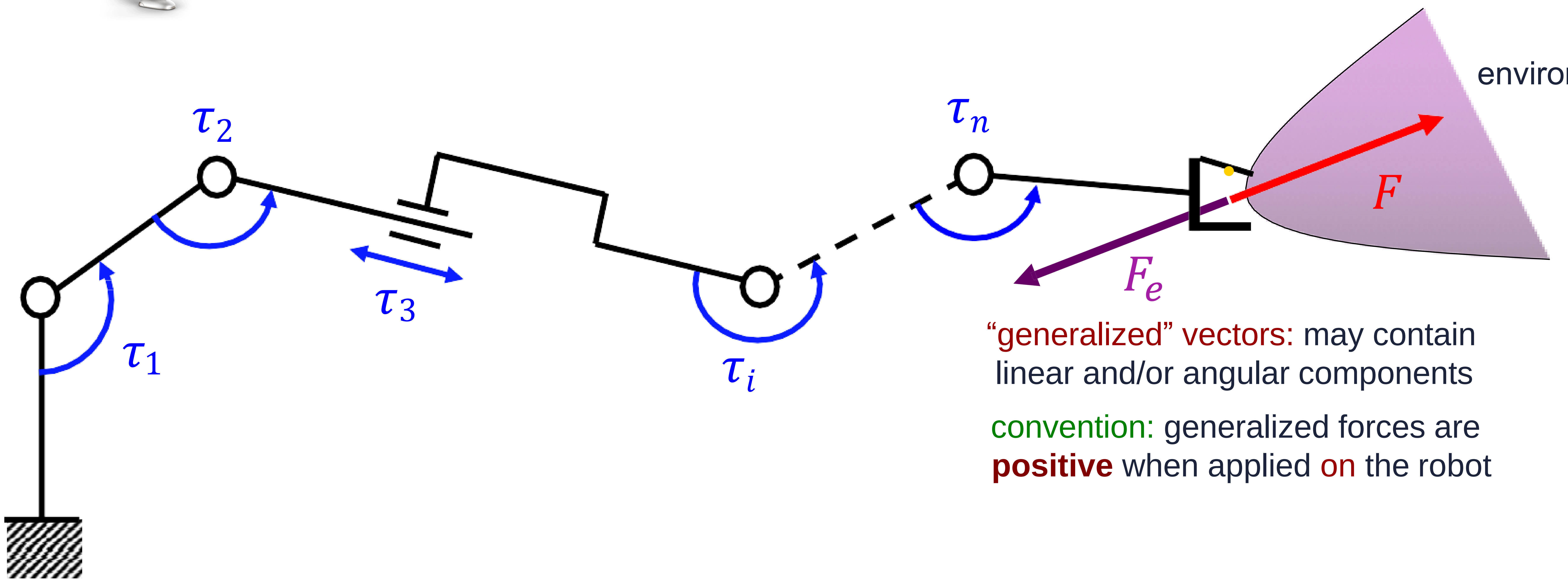
Bruno: Eqt. 3.54

verification of the actual task velocity that is being obtained

$$v_{\text{actual}} = J\dot{q} = J\left(J^{\#}v + (I - J^{\#}J)\xi\right) = JJ^{\#}v + \cancel{J(I - J^{\#}J)\xi} = JJ^{\#}(Jw) = Jw = v$$

if $v \in \mathcal{R}(J) \Rightarrow v = Jw$ for some $w \in \mathbb{R}^n$

Generalized Forces and Torques



"generalized" vectors: may contain linear and/or angular components

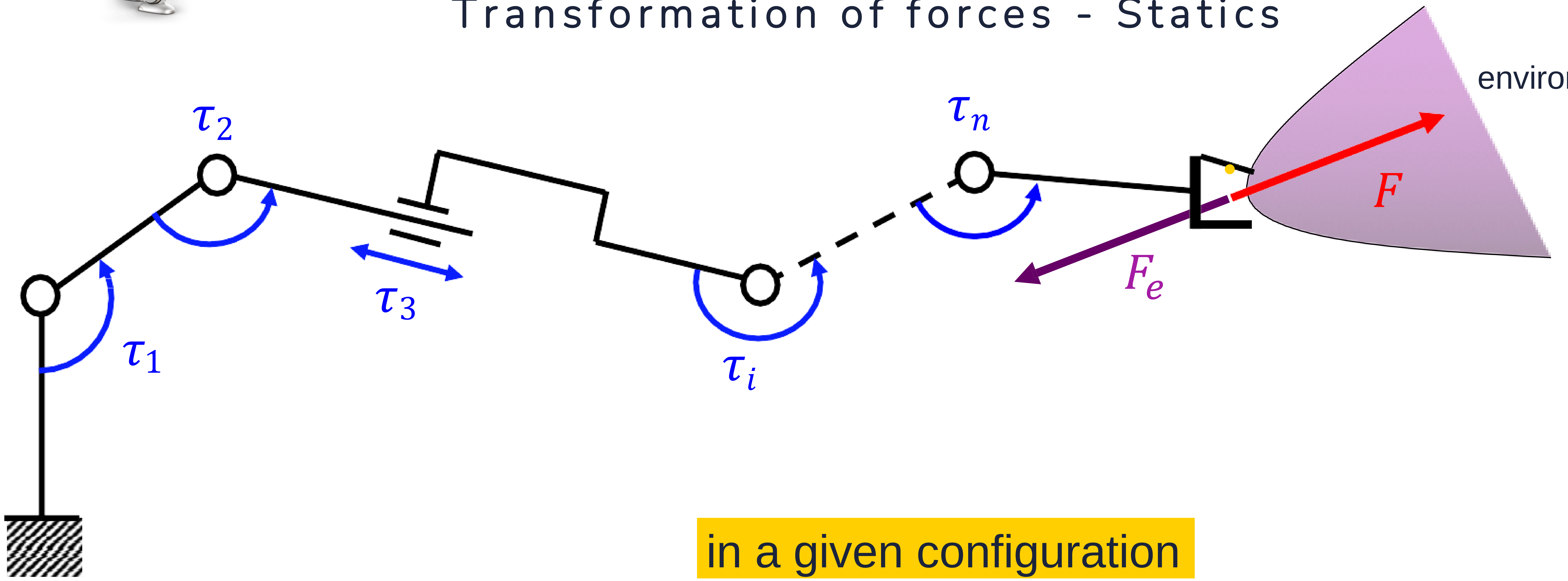
convention: generalized forces are **positive** when applied on the robot

- τ = forces/torques exerted by the motors at the robot joints
- F = equivalent forces/torques exerted by the robot end-effector
- F_e = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: $F_e = -F$

reaction from environment is equal and opposite to the robot action on it

Transformation of Forces

Transformation of forces - Statics



in a given configuration

- what is the transformation between F at robot end-effector and τ at joints?

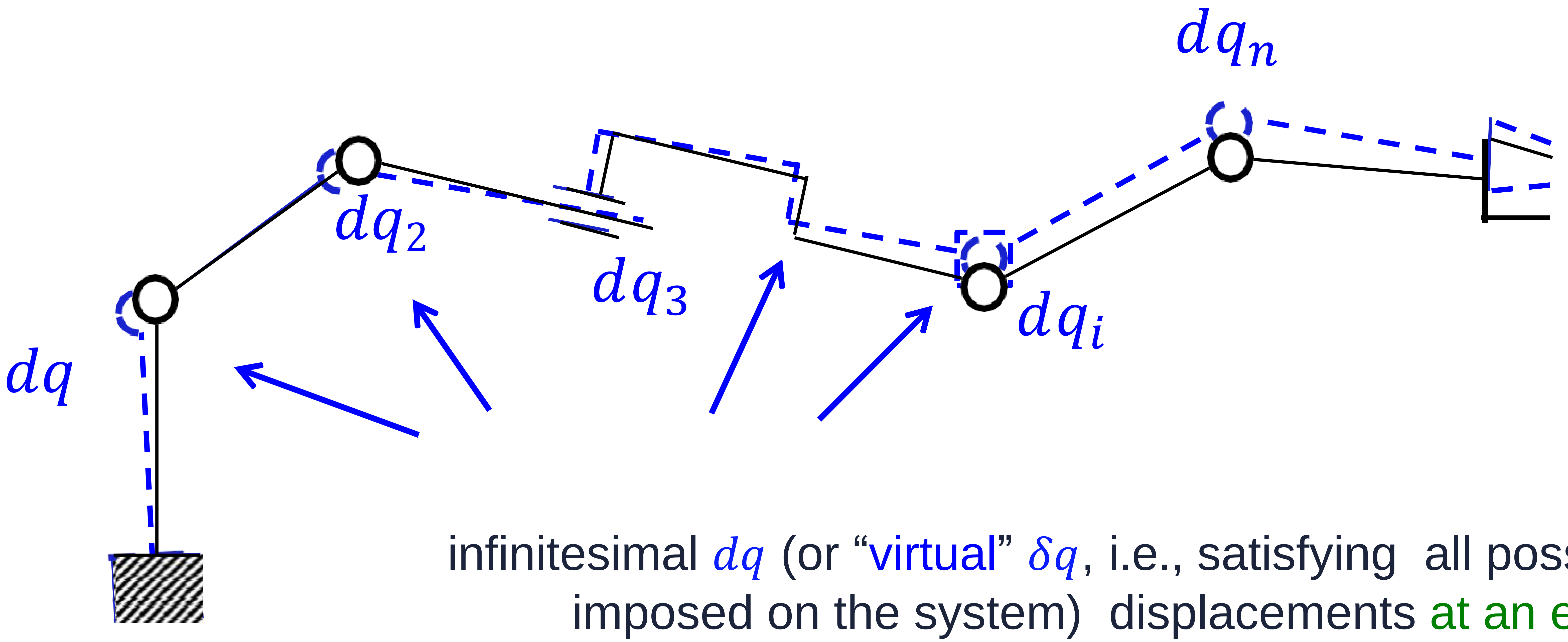
in **static equilibrium** conditions (i.e., **no motion**):

- what F will be exerted on environment by a τ applied at the robot joints?
- what τ at the joints will balance a $F_e(= -F)$ exerted by the environment?

all equivalent formulations

Virtual Displacements and Works

Virtual displacements and works

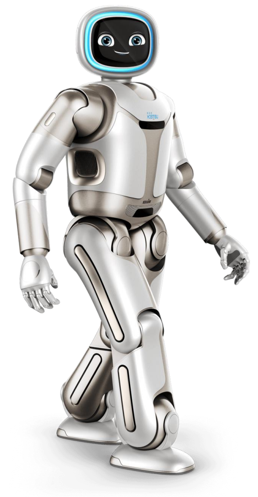


$$\begin{pmatrix} dp \\ \omega dt \end{pmatrix} = J dq$$

infinitesimal dq (or "virtual" δq, i.e., satisfying all possible constraints imposed on the system) displacements at an equilibrium

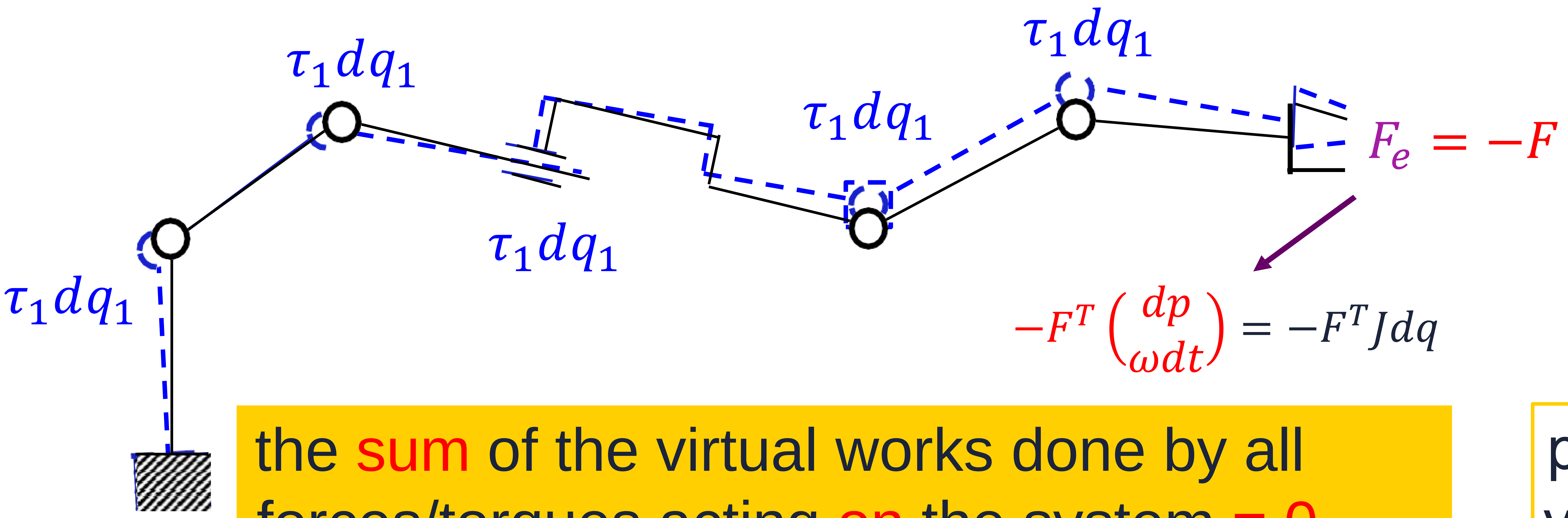
- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the virtual work is the work done by all forces/torques acting on the system for a given virtual displacement



Principle of Virtual Work

Principle of virtual work

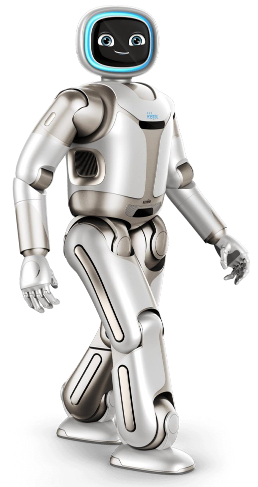


the sum of the virtual works done by all forces/torques acting on the system = 0

principle of virtual work

$$\tau^T dq - F^T\left(\frac{dp}{\omega dt}\right) = \tau^T dq - F^T J dq = 0 \qquad \forall dq$$

$$\Rightarrow \quad \tau = J^{\mathrm{T}}(q)F$$



Duality

Duality between velocity and force

$J(q)$

velocity q
(or displacement dq)
in the joint space

generalized velocity v(or
e-e displacement $\begin{pmatrix} dp \\ \omega dt \end{pmatrix}$in
the Cartesian space

forces/torques τ
at the joints

generalized forces F
at the Cartesian e-e

$J^{\mathrm{T}}(q)$

the singular configurations for the velocity map are the same as those for the force map ← $\rho\, J \;=\; \rho(J\ )$

Dual Subspaces of Velocity and Force

Dual subspaces of velocity and force (summary of definitions)

$$\mathcal{R}(J) = \{v \in \mathbb{R}^m : \exists \dot{q} \in \mathbb{R}^n, J\dot{q} = v\}$$

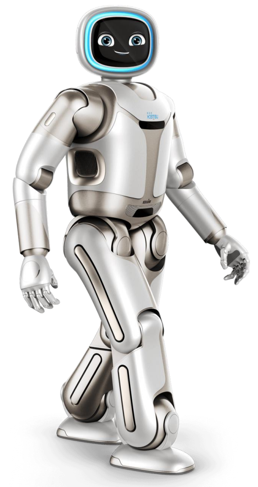
$$\mathcal{N}(J^T) = \{F \in \mathbb{R}^m : J^T F = 0\}$$

$$\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^m$$

$$\mathcal{R}(J^T) = \{\tau \in \mathbb{R}^n : \exists F \in \mathbb{R}^m, J^T F = \tau\}$$

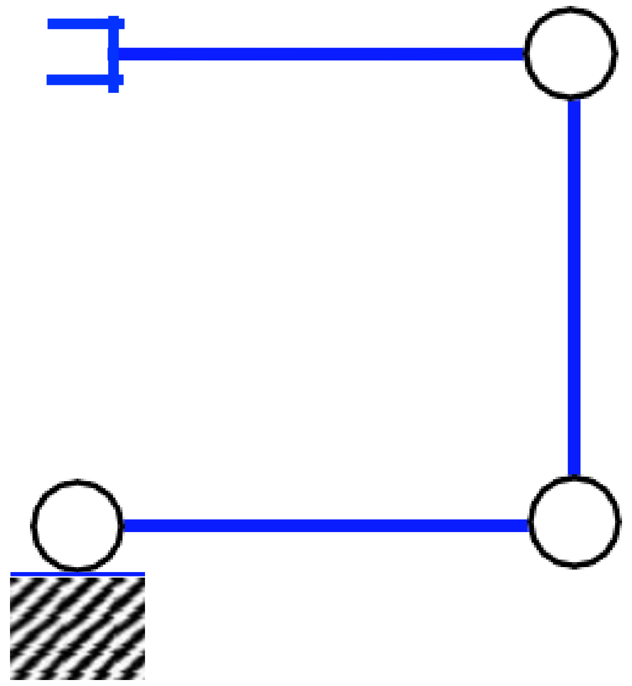
$$\mathcal{N}(J) = \{\dot{q} \in \mathbb{R}^n : J\dot{q} = 0\}$$

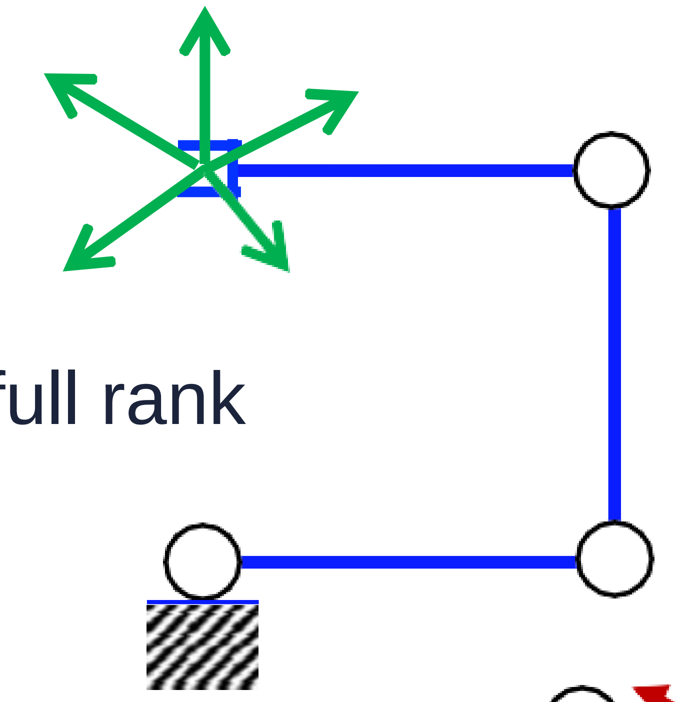
$$\mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^n$$



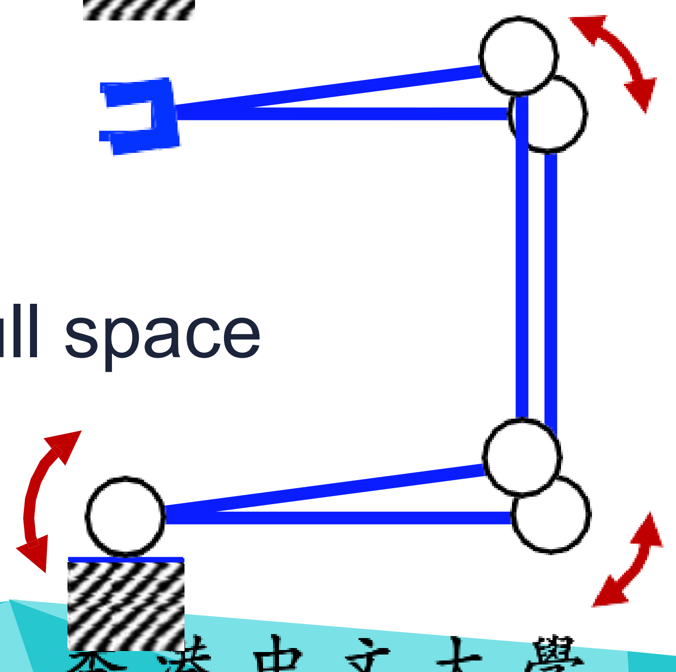
Mobility Analysis

Mobility analysis for a planar 3R robot





If full rank



If null space

case 1)

$q = (0, \pi/2, \pi/2)$

$$J = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad J^T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}$$

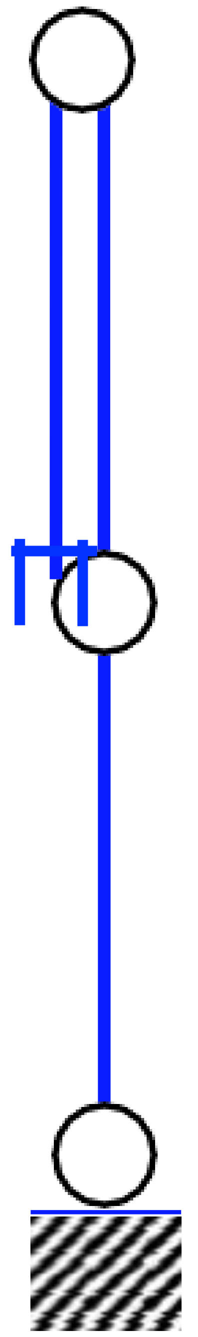
$$\rho(J) = 2 = m \qquad \rho(J^T) = \rho(J) = 2$$

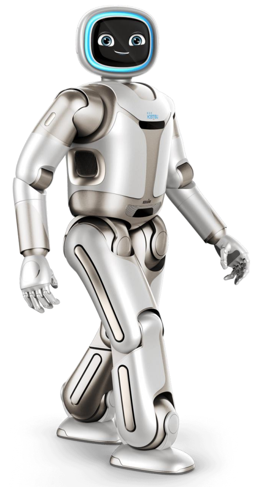
full rank, non-singular case

$$\mathcal{R}(J) = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2 \qquad \mathcal{N}(J) = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} \qquad \dim\ \mathcal{N}(J) = 1 = n - \rho(J) = n - m$$

$$\mathcal{R}(J^T) = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{N}(J^T) = 0 \qquad \dim\ \mathcal{R}(J^T) = 2 = m$$

$$\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^2$$
$$\mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^3$$





Mobility Analysis

Mobility analysis for a planar 3R robot

case 2)

$q = (\pi/2, 0, \pi)$

$$J = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rho(J) = 1 < m$$

$$J^T = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\rho(J^T) = \rho(J) = 1$$

$$\mathcal{R}(J) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \qquad \mathcal{N}(J) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

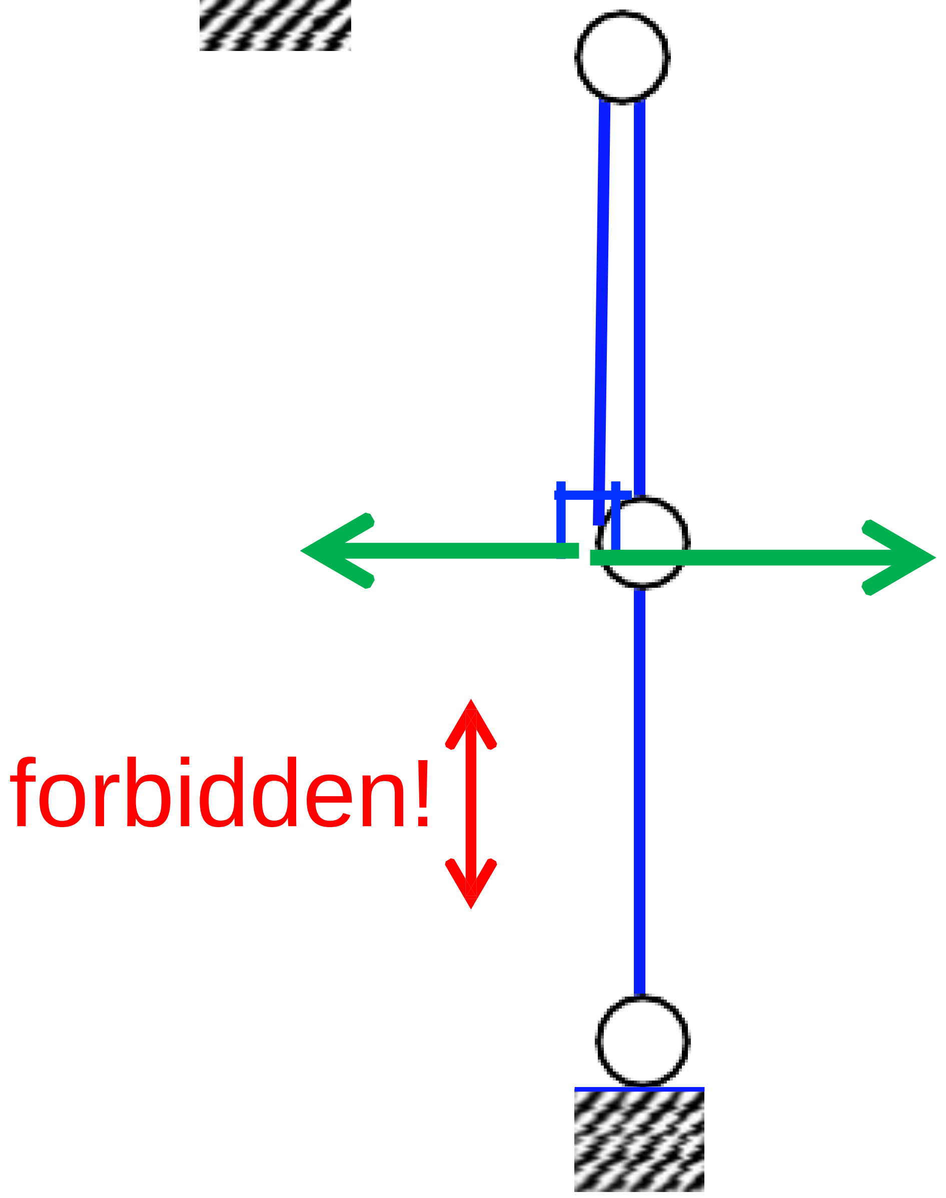
$$\dim \mathcal{R}(J) = 1 = \rho(J)$$

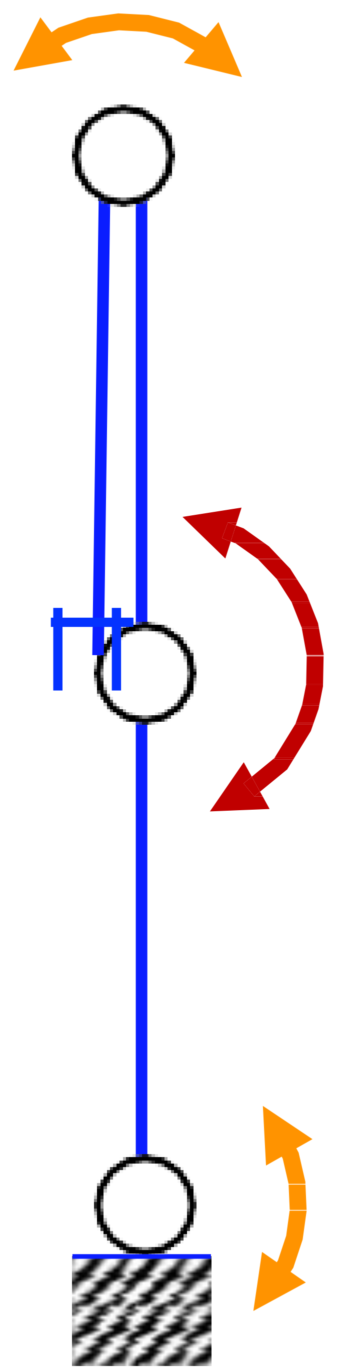
$$\dim \mathcal{N}(J) = 2 = n - \rho(J)$$

$$\mathcal{R}(J^T) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{N}(J^T) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim \mathcal{R}(J^T) = 1 = m - \rho(J)$$

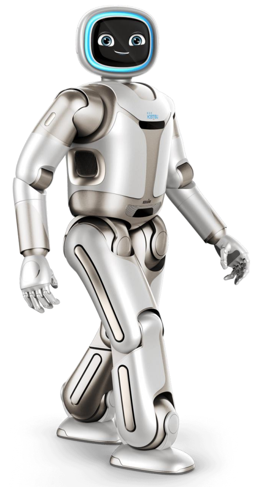
$$\dim \mathcal{N}(J^T) = 1 = n - \rho(J)$$





$$\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^2$$

$$\mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^3$$



Velocity Manipulability

Velocity manipulability

- in a given configuration, evaluate how effective is the transformation between joint and end-effector velocities
  - "how easily" can the end-effector be moved in various directions of the task space
  - equivalently, "how far" is the robot from a singular condition
- we consider all end-effector velocities that can be obtained by choosing joint velocity vectors of unit norm

$$\dot{q}^T\dot{q} = 1 \quad \Rightarrow \quad v^{\mathrm{T}} J^{\#\mathrm{T}} J^{\#} v = 1$$

task velocity manipulability ellipsoid

if $\rho(J) = m$, full rank

$J^{\#} = J^{\mathrm{T}}(JJ^T)^{-1}$

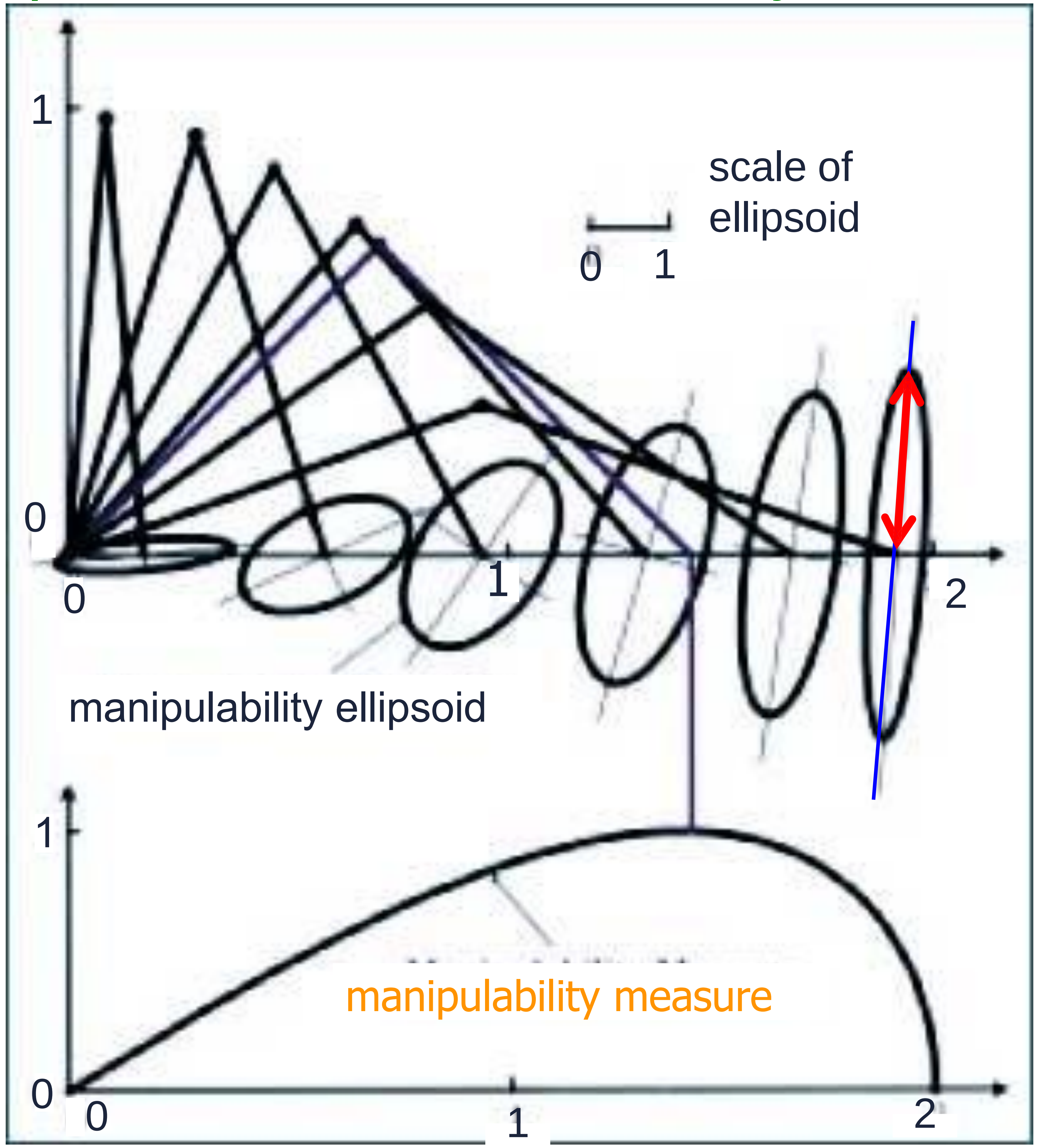
$(JJ^T)^{-1}$

note: the "core" matrix of the ellipsoid equation $v^{\mathrm{T}}A^{-1}v = I$ is the matrix A!

Velocity Manipulability

Manipulability ellipsoid (in velocity)

planar 2R arm with unitary links



length of principal (semi-) axes

singular values σ_i of J (in its SVD)

$$\sigma_i(J) = \sqrt{\lambda_i(JJ^T)}$$

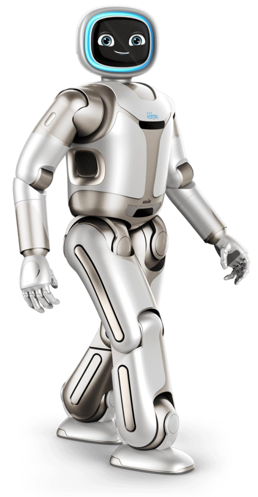
in a singularity, the ellipsoid loses a dimension
(for m = 2, it becomes a segment)

direction of principal axes

eigenvectors associated to λ_i

$$w = \sqrt{\det(JJ^T)} = \prod_{i=1}^{m} \sigma_i \geq 0$$

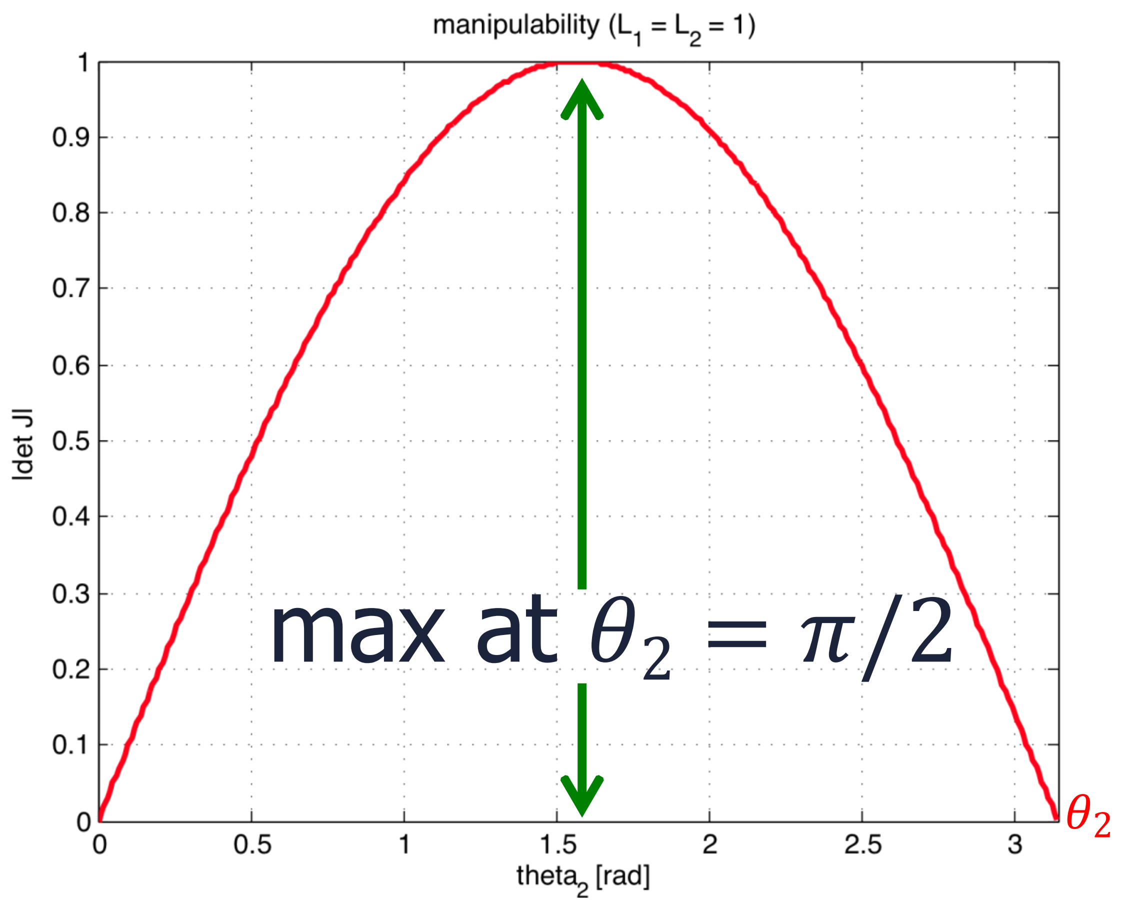
proportional to the volume of the ellipsoid (for m = 2, to its area)

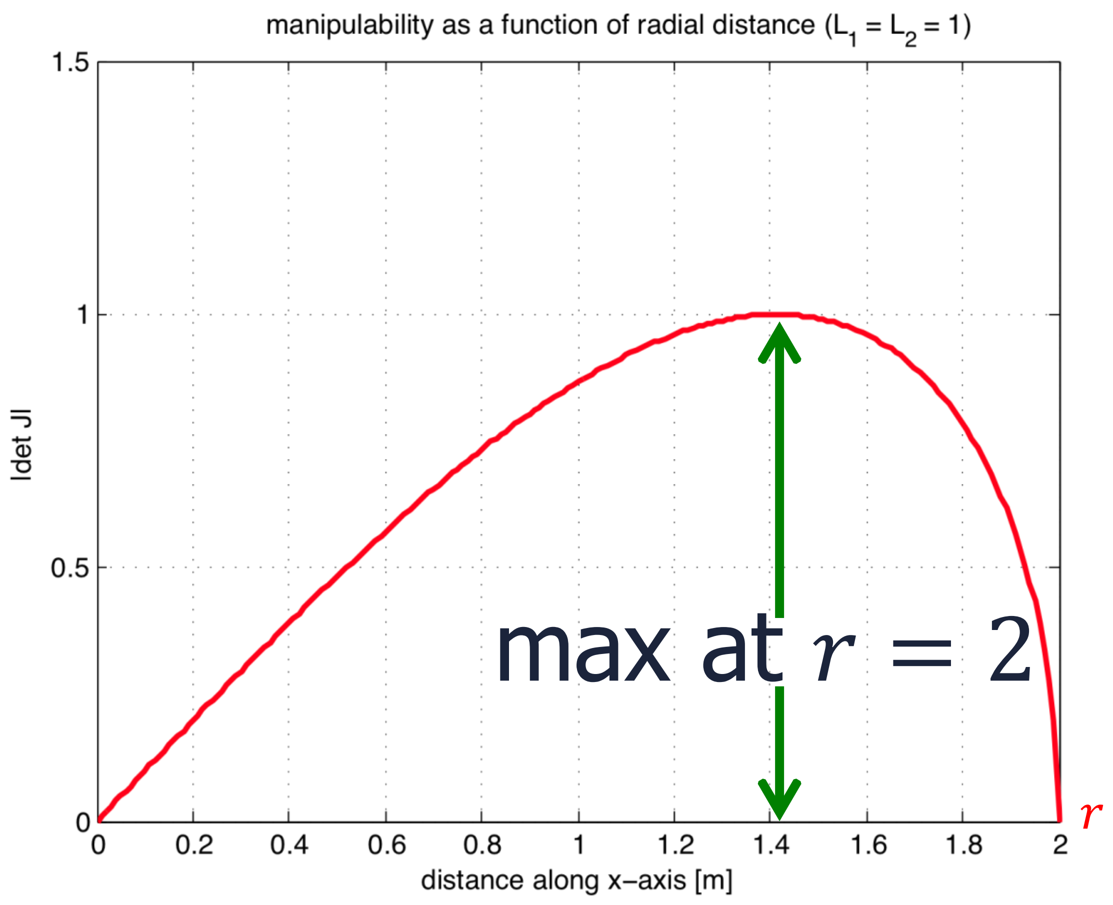


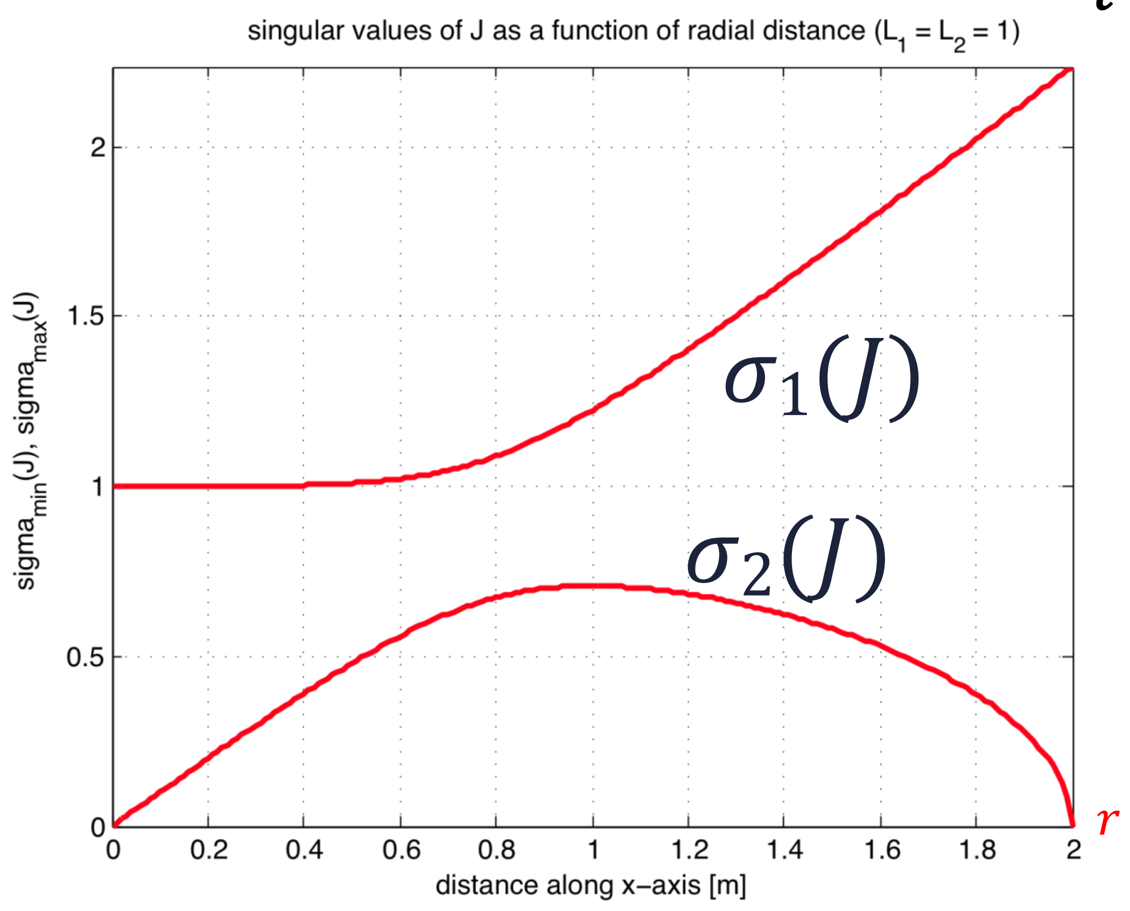
Manipulability Measure

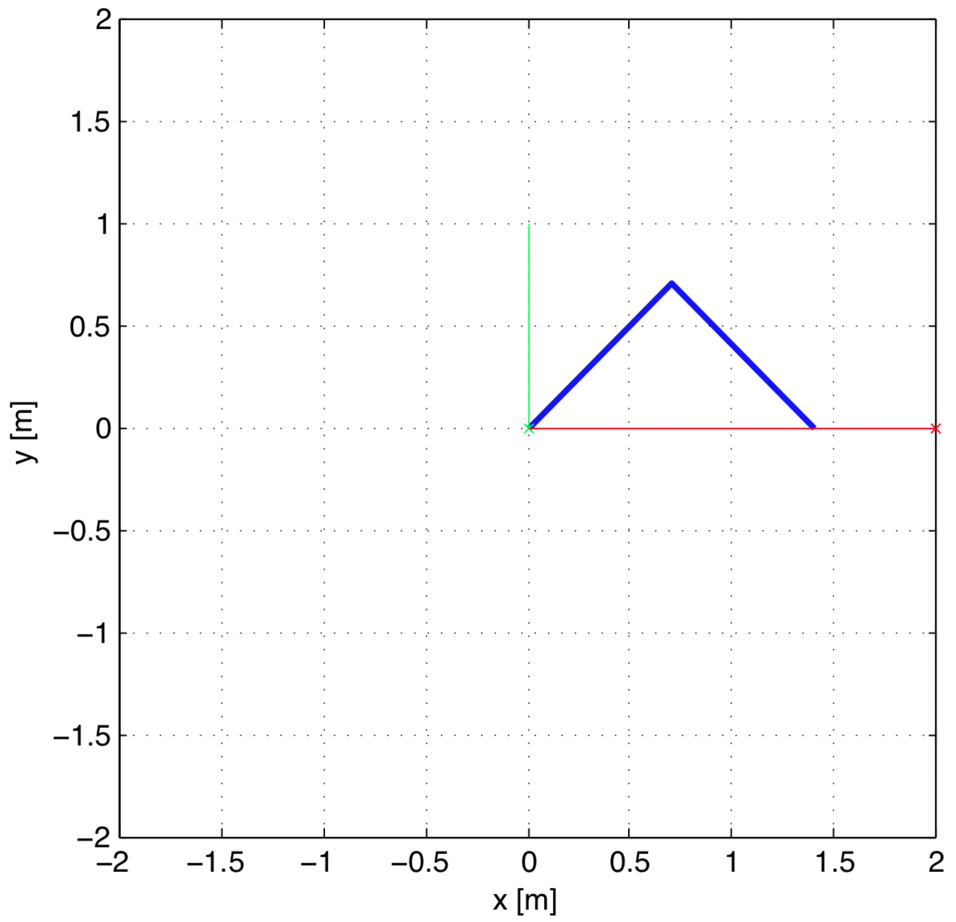
Manipulability measure

planar 2R arm (with $l_1 = l_2 = 1$): $\sqrt{\det(JJ^T)} = \sqrt{\det(J) \cdot \det(J^T)} = |\det J| = \prod_{i=1}^{2} \sigma_i$



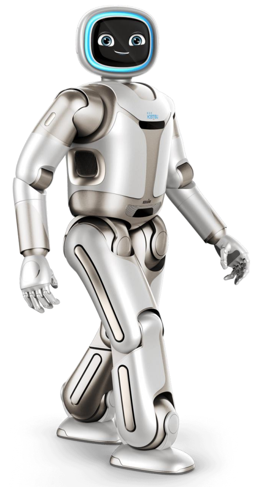






best posture for manipulation (similar to a human arm!)

no full isotropy here, since it is always $\sigma_1 \neq \sigma_2$



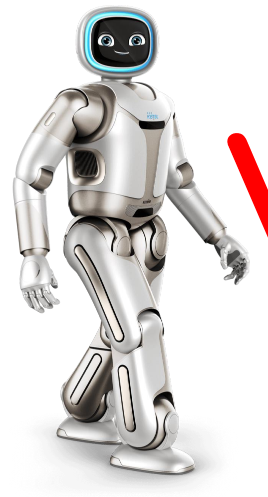
Force Manipulability

- in a given configuration, evaluate how effective is the transformation between joint torques and end-effector forces
  - "how easily" can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space
  - in singular configurations, there are directions in the task space where external forces are balanced without the need of any joint torque
- we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of unit norm

$$\tau^T \tau = 1 \quad \Rightarrow \quad F^T J J^T F = 1$$

task force manipulability ellipsoid

same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths

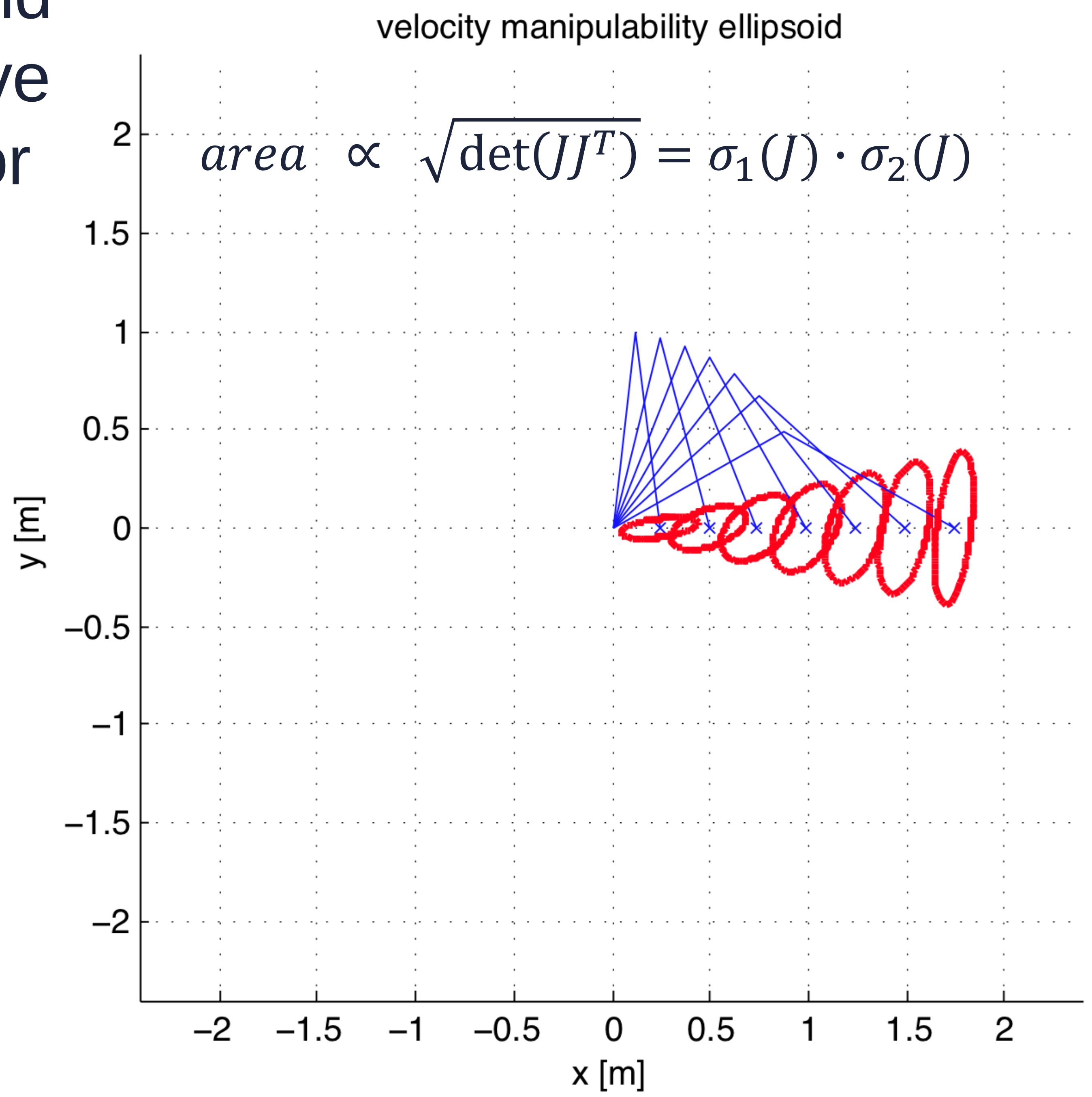


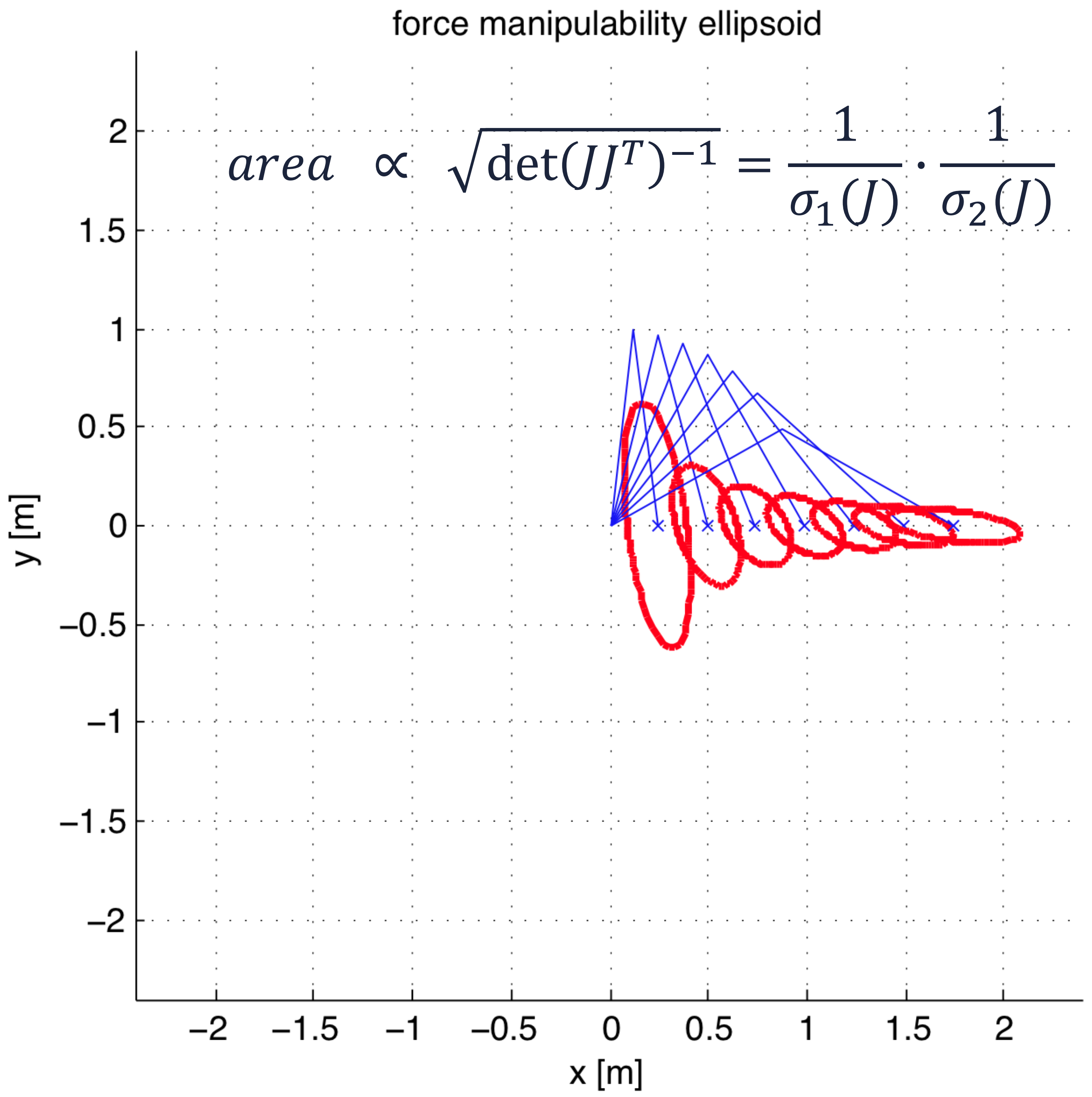
Velocity and Force Manipulability

Velocity and force manipulability (dual comparison of actuation vs. control)

planar 2R arm with unitary links

note: velocity and force ellipsoids have a different scale for a better view

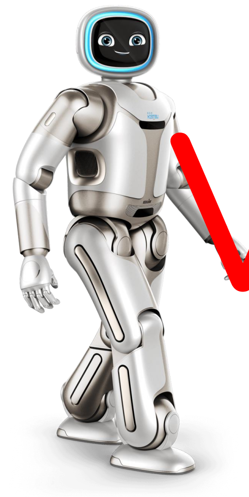




Cartesian **actuation** task (joint-to-task high transformation ratio): preferred velocity (or force) directions are those where the ellipsoid stretches

⇕

Cartesian **control** task (low transformation ratio = high resolution): preferred velocity (or force) directions are those where the ellipsoid shrinks



Velocity and Force Transformations

Velocity and force transformations

- same reasoning made for relating end-effector to joint forces/torques (virtual work principle + static equilibrium) used also transforming forces and torques applied at different places of a rigid body and/or expressed in different reference frames

transformation among generalized velocities

$$\begin{bmatrix} {}^A v_A \\ {}^A\omega \end{bmatrix} = \begin{bmatrix} {}^A R_B & -{}^A R_B S({}^B r_{BA}) \\ 0 & {}^A R_B \end{bmatrix} \begin{bmatrix} {}^B v_B \\ {}^B\omega \end{bmatrix} = J_{BA} \begin{bmatrix} {}^B v_B \\ {}^B\omega \end{bmatrix}$$

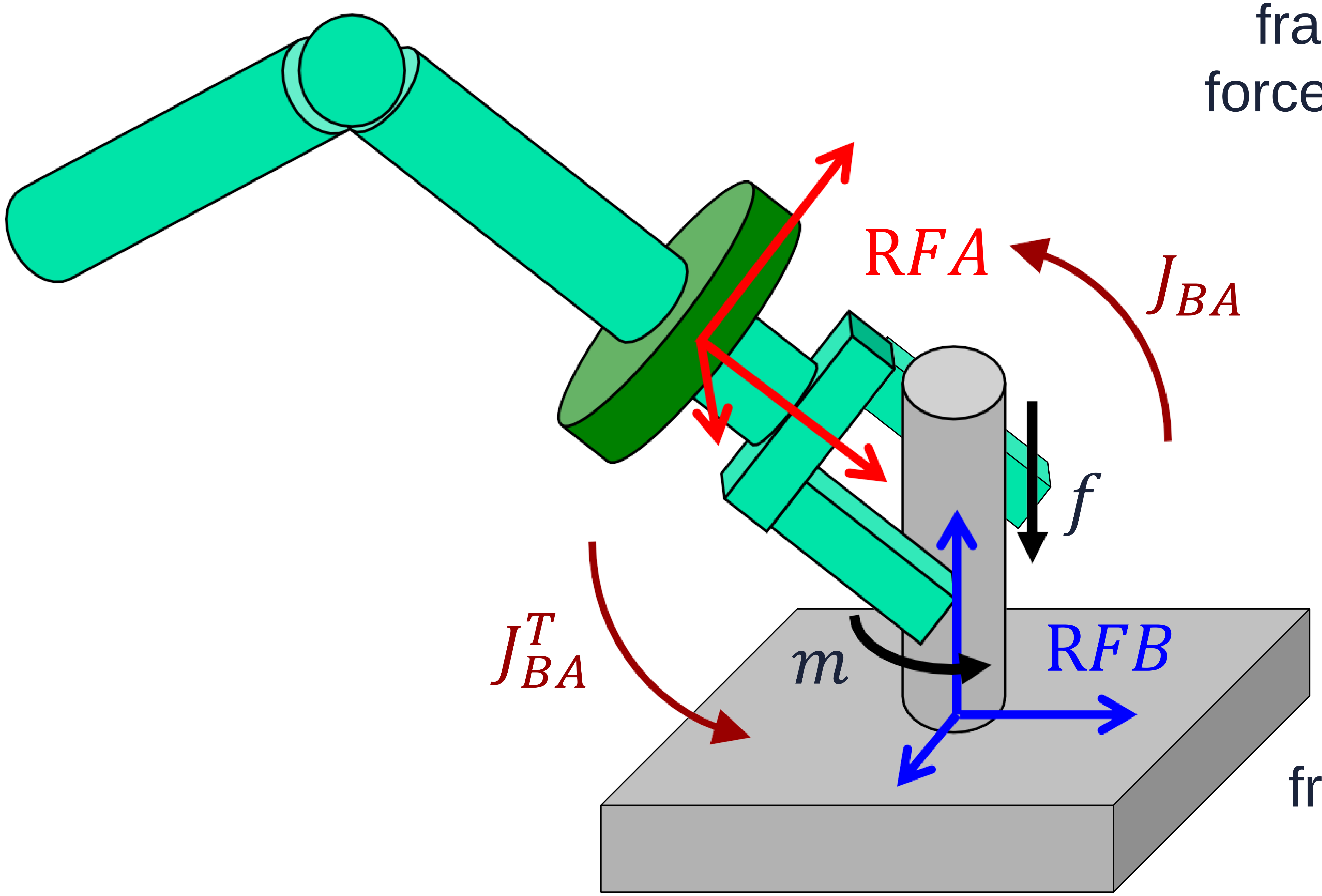
$$\begin{bmatrix} {}^B f_B \\ {}^B m \end{bmatrix} = J_{BA}^T \begin{bmatrix} {}^A f_A \\ {}^A m \end{bmatrix} = \begin{bmatrix} {}^B R_A & 0 \\ -S^T({}^B r_{BA}) {}^B R_A & {}^B R_A \end{bmatrix} \begin{bmatrix} {}^A f_A \\ {}^A m \end{bmatrix}$$

transformation among generalized forces

for skew-symmetric matrices, it is: $-S^{\mathrm{T}} r = S(r)$

Example 1

Example: 6D force/torque sensor

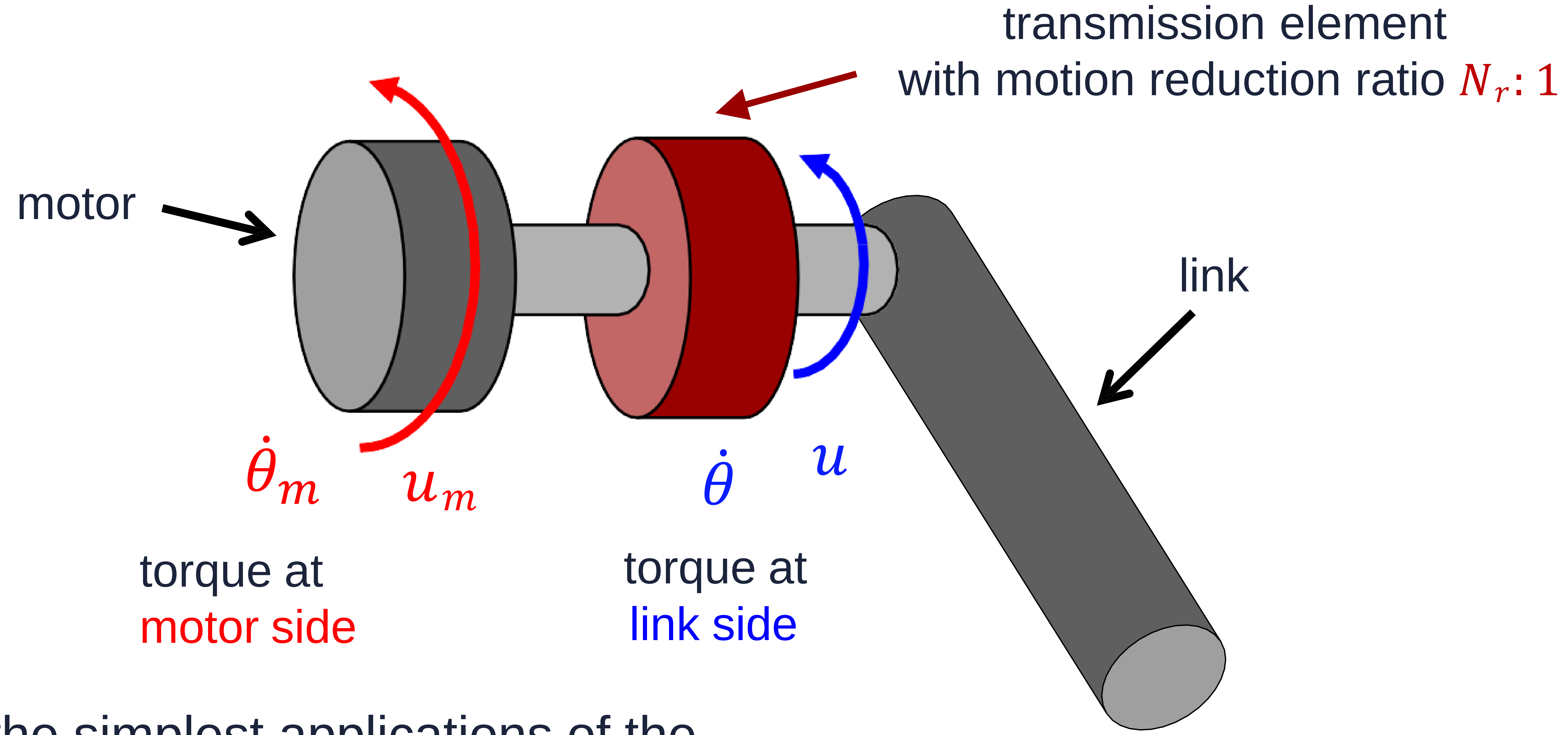


frame of measure for the forces/torques (attached to the wrist sensor)

frame of interest for evaluating forces/torques in a task with environment contact

Example 2

Example: Gear reduction at joints

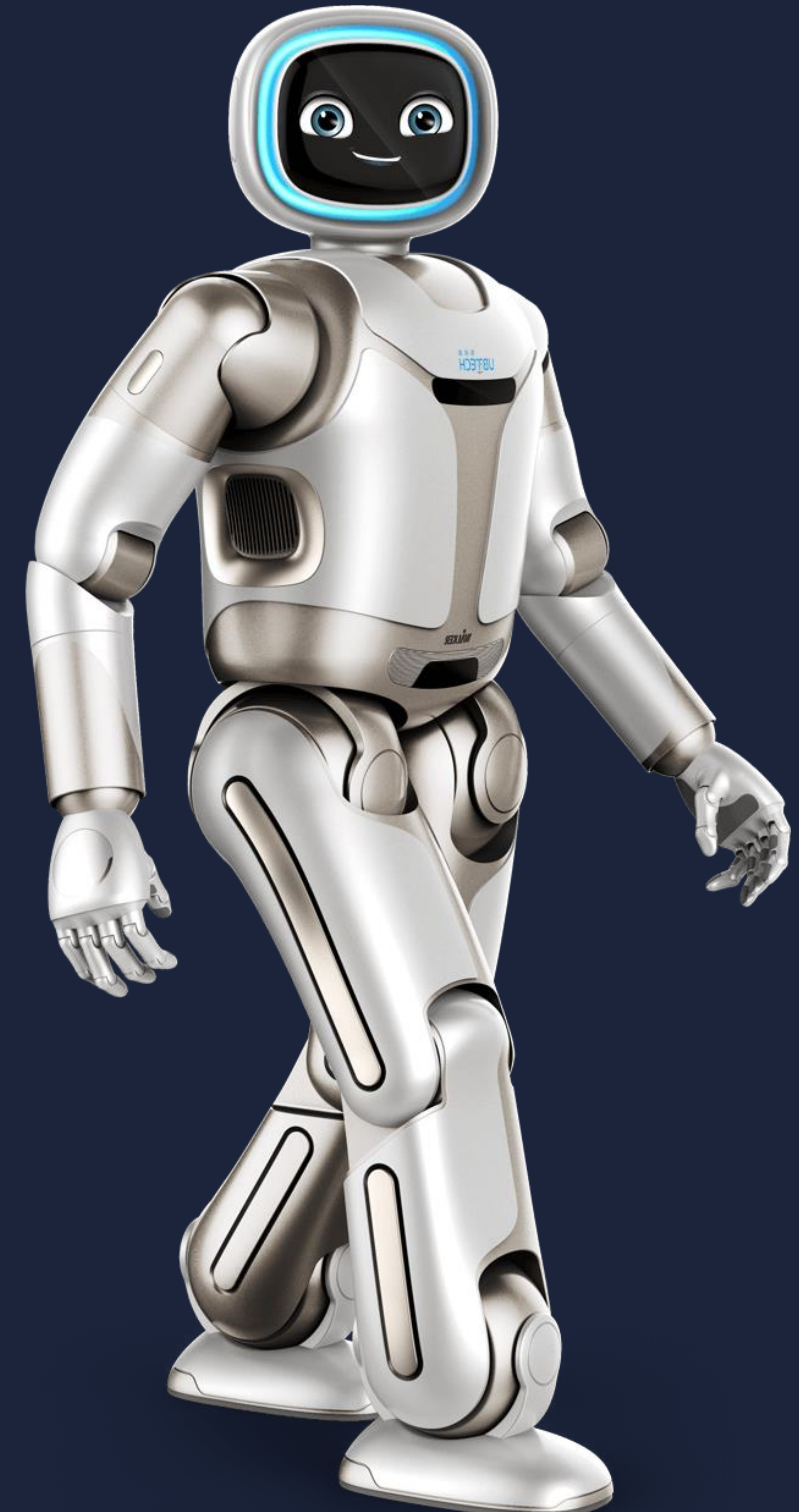


one of the simplest applications of the principle of virtual work:

$$P_m = u_m \dot{\theta}_m = u\dot{\theta} = P$$

$$\dot{\theta}_m = N_r \dot{\theta}$$
$$u = N_r u_m$$

here, $J = J^{\mathrm{T}} = N_r$ (a scalar!)



Q&A