Advanced Robotics

ENGG5402 Spring 2023

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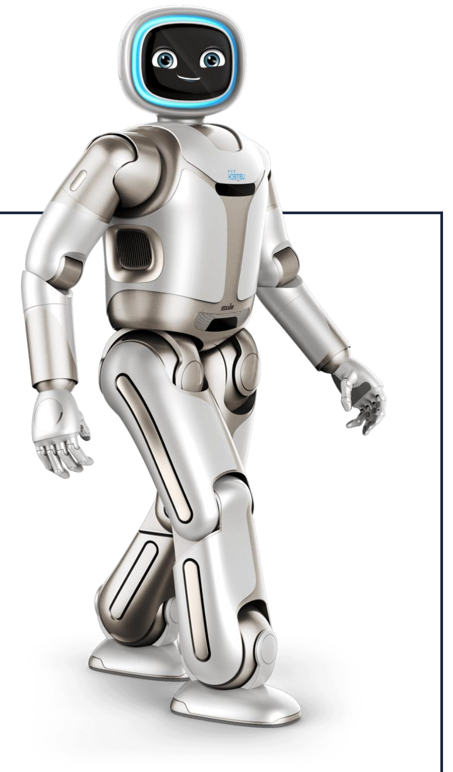
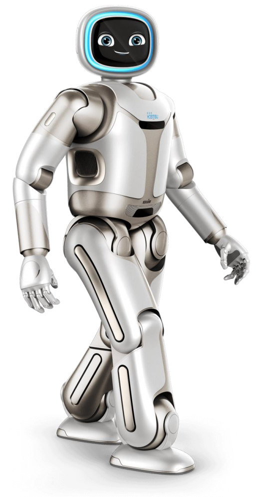


Topics:

- Dynamic model of robots: Lagrangian approach

Readings:

- Siciliano: Sec. 7



Introduction

- Dynamic modeling of manipulators
  - Direct and inverse dynamics
  - Euler-Lagrange formulation
  - Newton-Euler formulation
  - properties of the dynamic model
  - identification of dynamic parameters
  - inclusion of flexibility at the joints
  - inclusion of geometric constraints

all on fixed-base
robot manipulators

Dynamic model

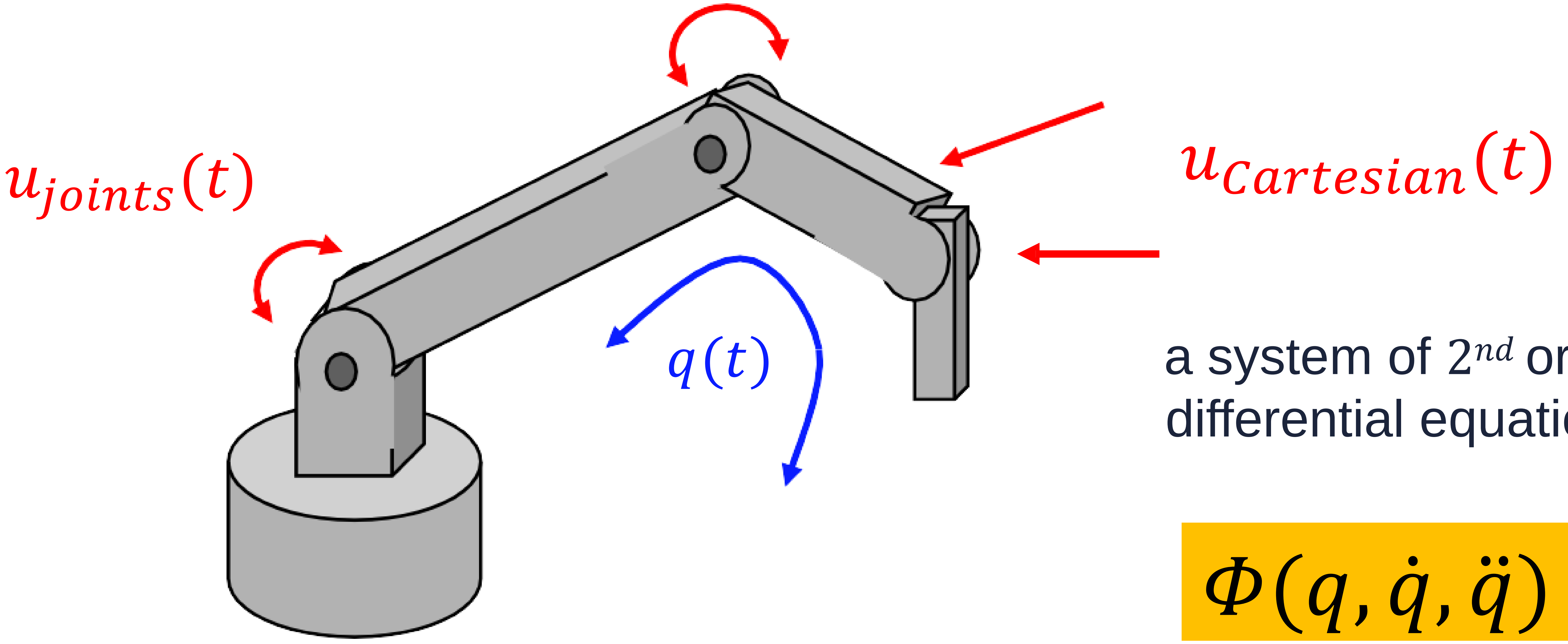
Dynamic model

- provides the relation between

  generalized forces $u(t)$ acting on the robot

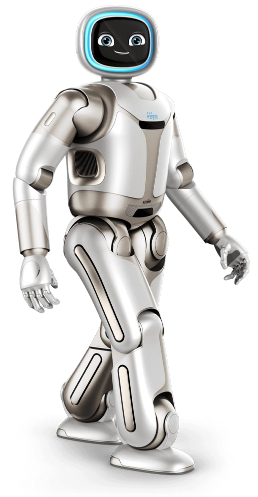
  ⬍

  robot motion, i.e.,
  assumed configurations $q(t)$ over time



a system of 2^{nd} order differential equations

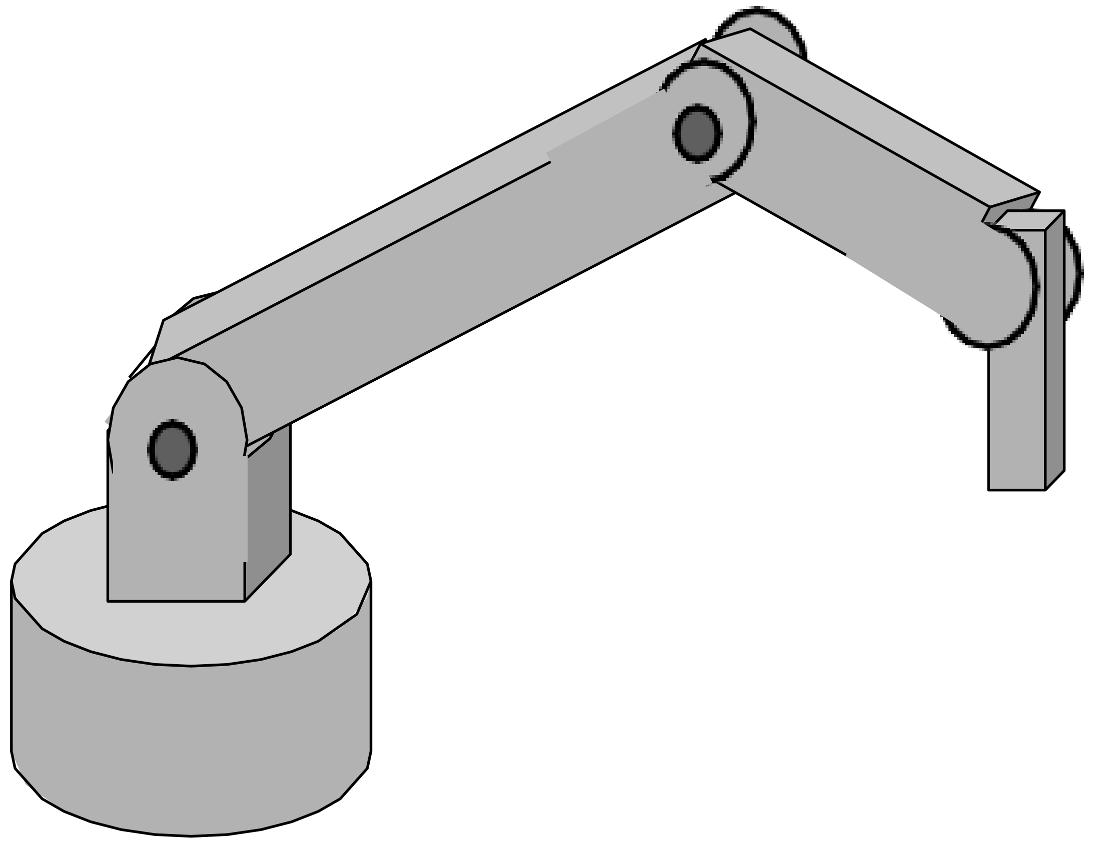
$$\Phi(q, \dot{q}, \ddot{q}) = u$$



Direct Dynamics

- direct relation

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$



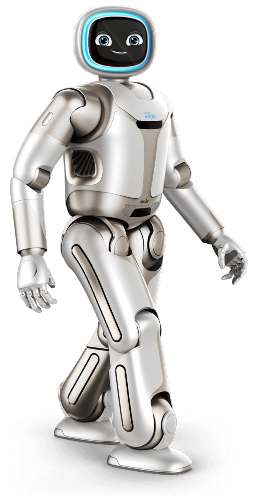
$$q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

input for $t \in [0, T]$ + $q(0), \dot{q}(0)$

initial state at $t = 0$

resulting motion

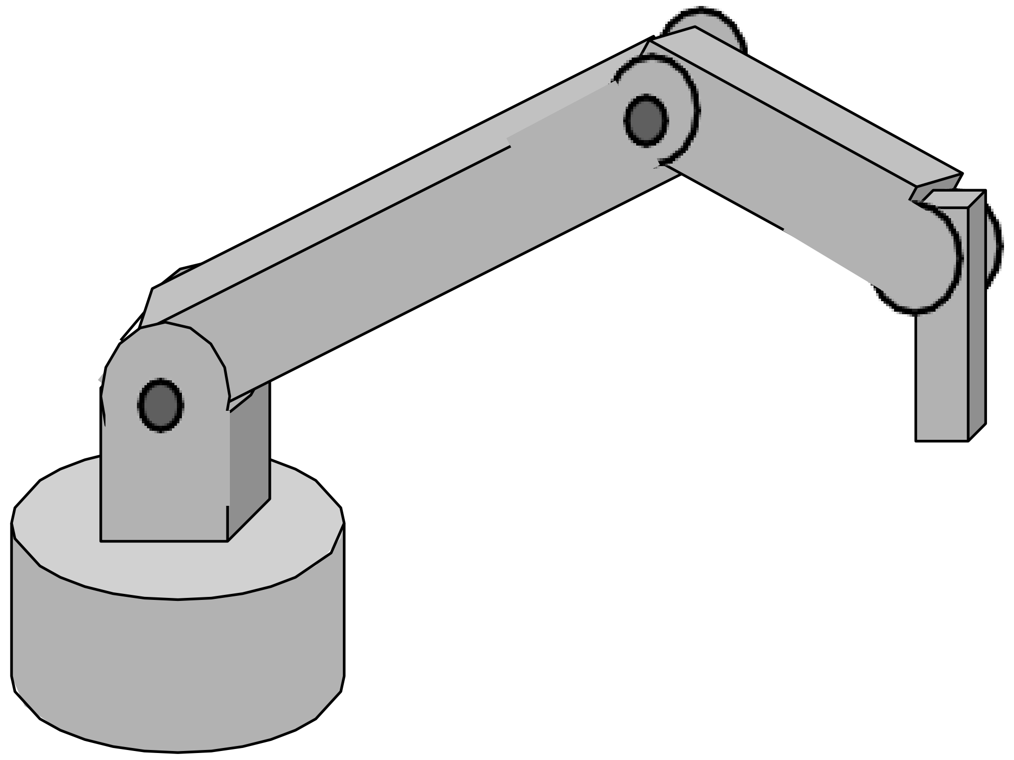
- experimental solution
  - apply torques/forces with motors and measure joint variables with encoders (with control sampling step T_c)
- solution by simulation ⟷ given $\Phi(q, \dot{q}, \ddot{q}) = u$
  - use dynamic model and integrate numerically the differential equations (with simulation sampling step $T_s \leq T_c$)



Inverse Dynamics

- inverse relation

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$



u_d(t)

desired motion
for $t \in [0, T]$

required input
for $t \in [0, T]$

- experimental solution

repeated motion trials of direct dynamics using $u_k(t)$, with iterative learning of nominal torques updated on trial $k + 1$ based on the error in $[0, T]$ measured in trial k: $lim_{k\to\infty}\, u_k(t) \;\Rightarrow\; u_d(t)$

- analytic solution ⟷ given $\Phi(q, \dot{q}, \ddot{q}) = u$

use dynamic model and compute algebraically the values $u_d(t)$ at every time instant t

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Approaches to dynamic modeling

Euler-Lagrange method (energy-based approach)

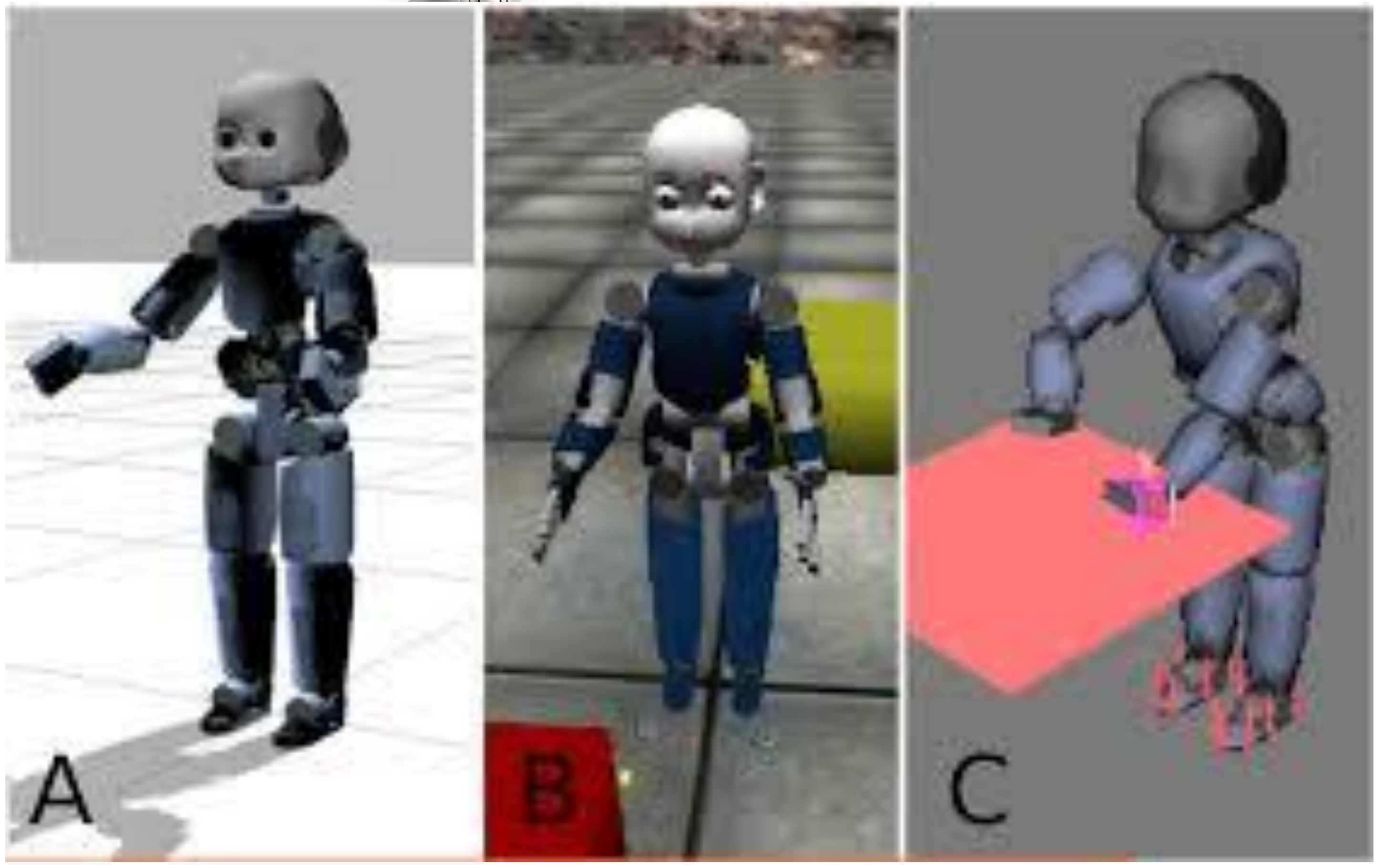
- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

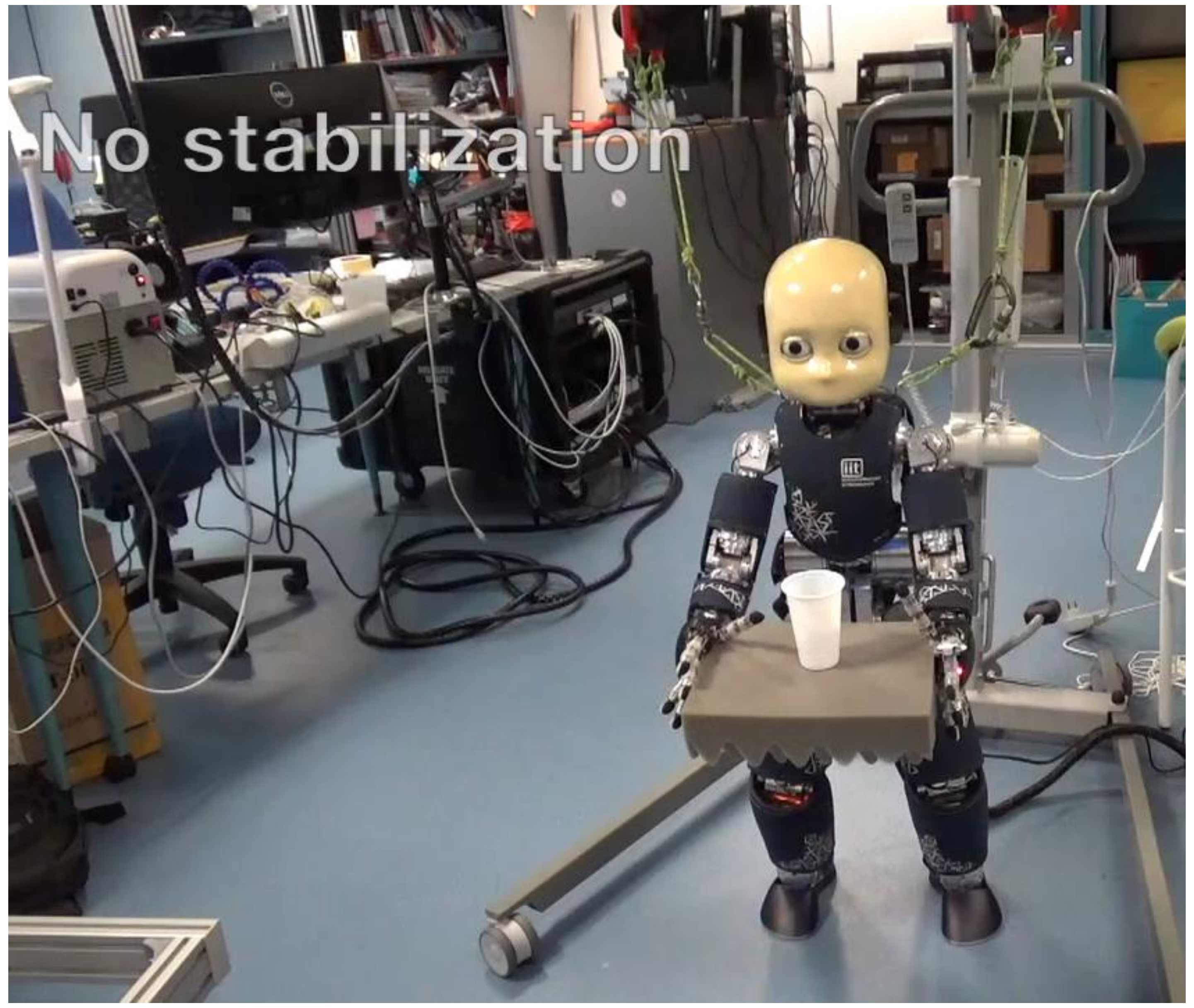
Newton-Euler method (balance of forces/torques)

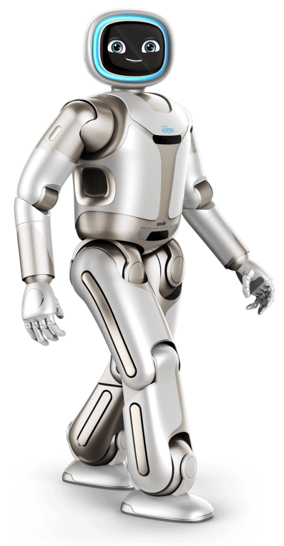
- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
- principle of d’Alembert, of Hamilton, of virtual works, …

Approaches to Dynamic Modeling







Euler-Lagrange Method

energy-based approach

basic assumption: the N links in motion are considered as **rigid bodies**, e.g., typical industrial arm
*(+ nowadays, include also **concentrated elasticity** at the joints, e.g., KUKA iiwa collaborative arm)*

generalized coordinates (e.g., joint variables, but not only!)

Lagrangian

$$\boxed{L(q,\dot{q}) = T(q,\dot{q}) - U(q)}$$

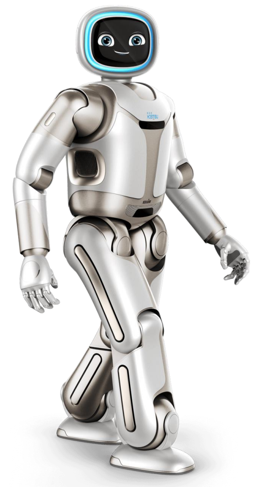
kinetic energy – potential energy

- *principle of least action of Hamilton*
- *principle of virtual works*

Euler-Lagrange equations

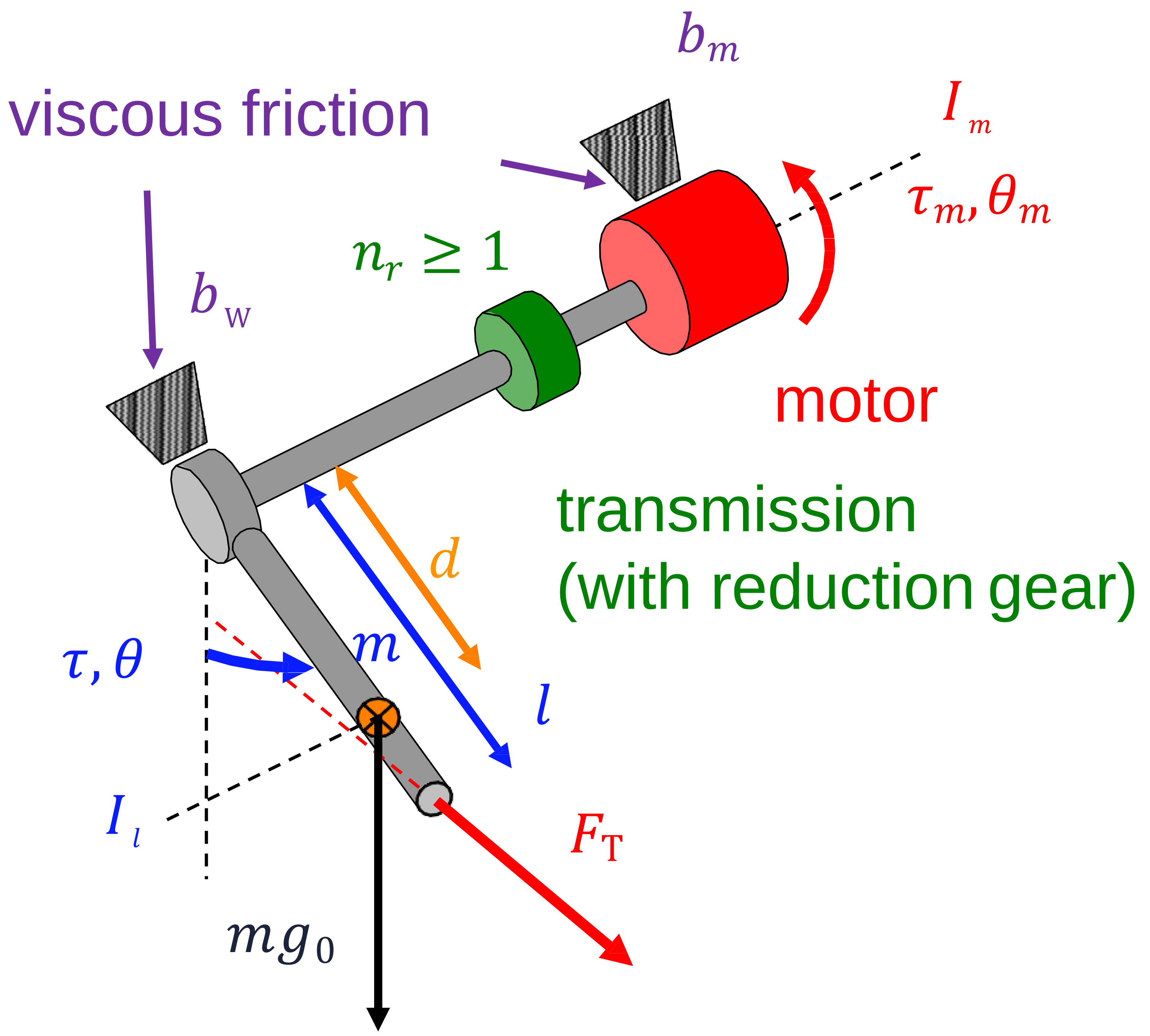
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \qquad i = 1, \ldots, N$$

non-conservative (external or dissipative) generalized forces performing work on q_i



Euler-Lagrange Method

Dynamic of an actuated pendulum
(a first example)



kinetic energy

$$\dot{\theta}_m = n_r \dot{\theta} \quad \Rightarrow \quad \theta_m = n_r \theta + \cancel{\theta_{m0}}$$

$$\tau = n_r \tau_m \qquad\qquad = 0$$

$\boxed{q = \theta}$ (or $q = \theta_m$)

$$\boxed{T = T_m + T_l}$$

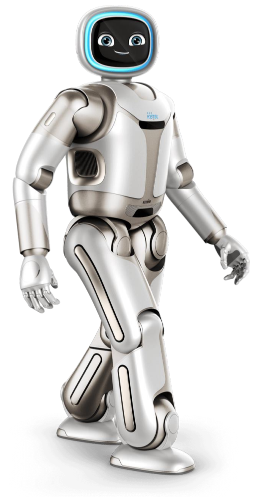
$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \qquad T_l = \frac{1}{2}(I_l + md^2)\dot{\theta}^2$$

motor inertia (around its spinning axis)

link inertia (around the z-axis through its center of mass…)

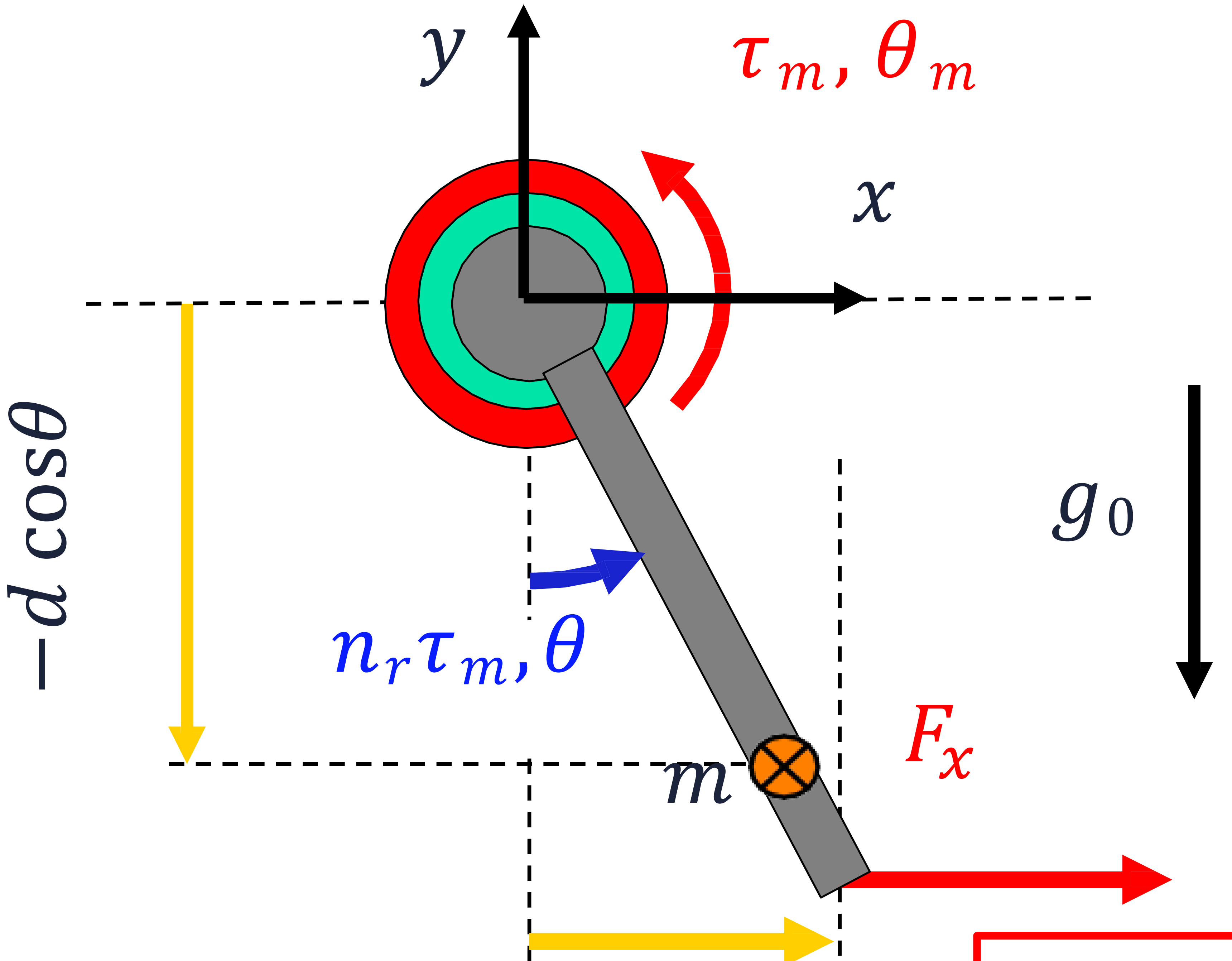
(… around the ॥ axis through its base)

$$\boxed{T = \frac{1}{2}(I_l + md^2 + I_m n_r^2)\dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2}$$



Euler-Lagrange Method

Dynamics of an actuated pendulum (cont)



$$U = U_0 - mg_0 d\cos\theta$$

potential energy

$$L = T - U = \frac{1}{2}I\dot{\theta}^2 + mg_0 d\cos\theta - U_0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = I\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mg_0 d\sin\theta$$

$$p_x = l\sin\theta$$

$$\boxed{\dot{p}_x = l\cos\theta \cdot \dot{\theta} = J_x\dot{\theta}}$$

$$u = n_r\tau_m - b_l\dot{\theta} - n_r b_m\dot{\theta}_m + J_x^T F_x = n_r\tau_m - (b_l + b_m n_r^2)\dot{\theta} + l\cos\theta\, F_x$$

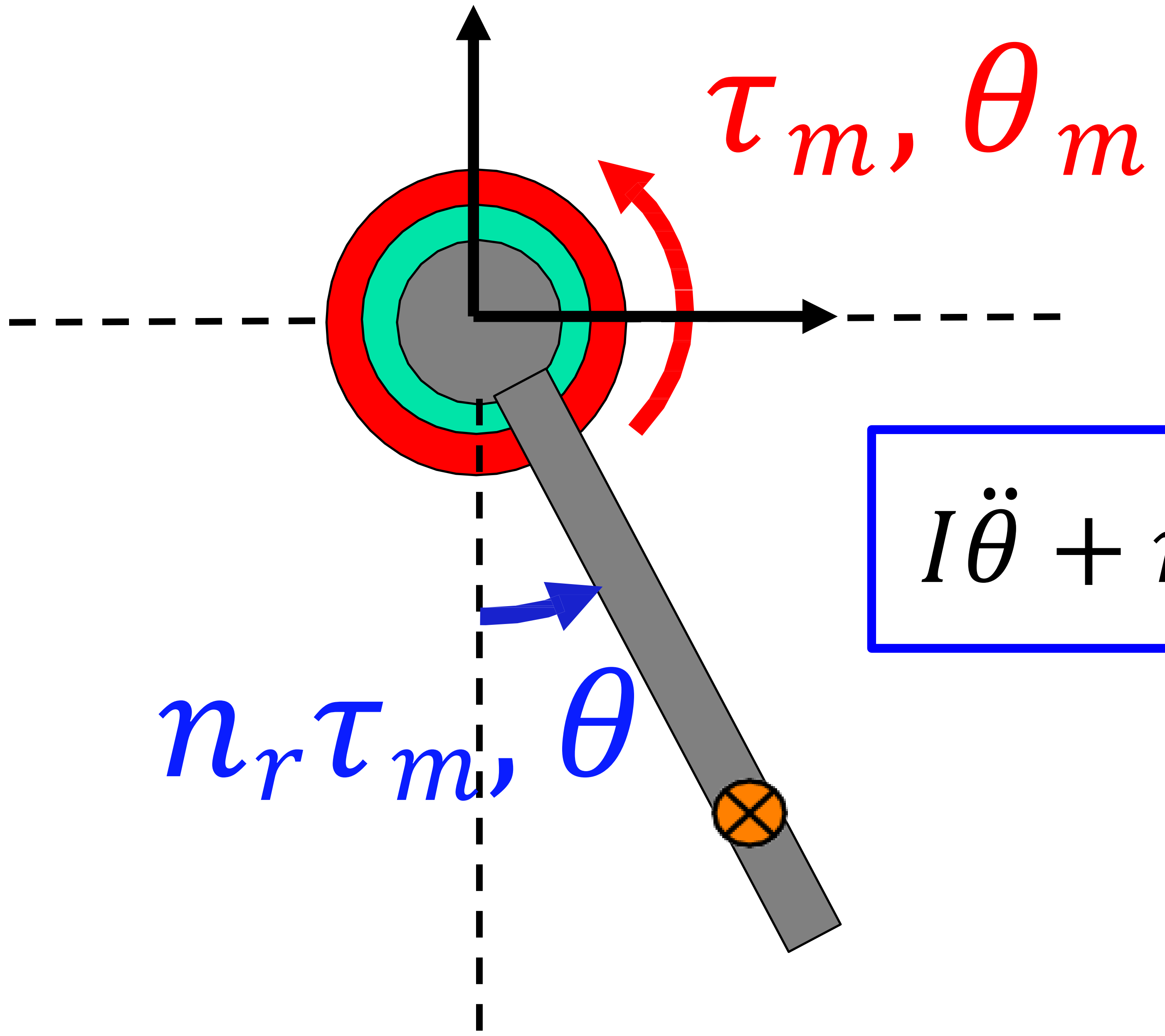
applied or dissipated torques on motor side are multiplied by n_r when moved to the link side

equivalent joint torque due to force F_x applied to the tip at point p_x

"sum" of non-conservative torques

Euler-Lagrange Method

Dynamics of an actuated pendulum (cont)



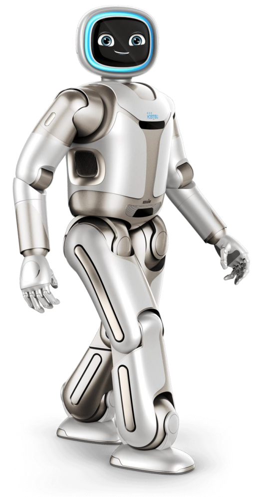
dynamic model in $q = \theta$

$$I\ddot{\theta} + mg_0 d \sin\theta = n_r \tau_m - (b_l + b_m n_r^2)\dot{\theta} + l\cos\theta \cdot F_x$$

dividing by n_r and substituting $\theta = \theta_m / n_r$

$$\frac{I}{n_r^2}\ddot{\theta}_m + \frac{m}{n_r} g_0 d \sin\frac{\theta_m}{n_r} = \tau_m - \left(\frac{b_l}{n_r^2} + b_m\right)\dot{\theta}_m + \frac{l}{n_r}\cos\frac{\theta_m}{n_r} \cdot F_x$$

dynamic model in $q = \theta_m$

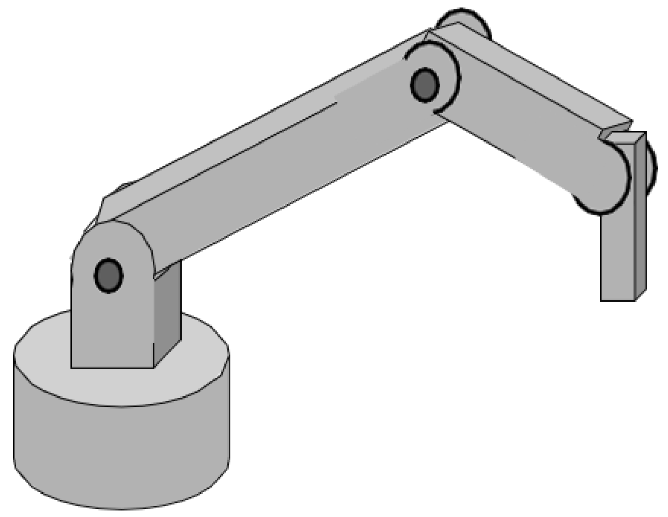


Direct & Inverse Dynamics

$$\boxed{I\ddot{\theta} + m g_0 d \sin\theta = n_r \tau_m - (b_l + b_m n_r^2)\dot{\theta} + l\cos\theta \cdot F_x}$$

- direct relation

$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ $\rightarrow$



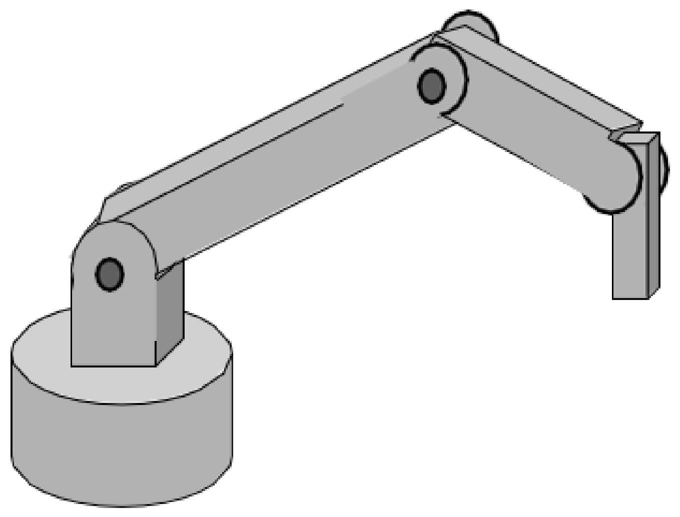
$\rightarrow$ $q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$

input for $t \in [0, T]$ + $q(0), \dot{q}(0)$
initial state at $t = 0$

resulting motion

- inverse relation

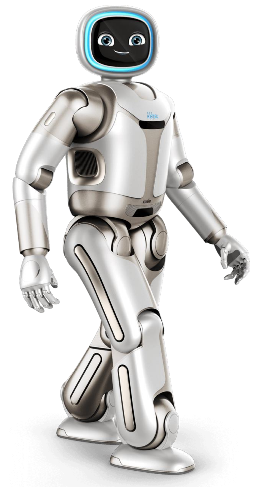
$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ $\rightarrow$



$\rightarrow$ $u_d(t)$

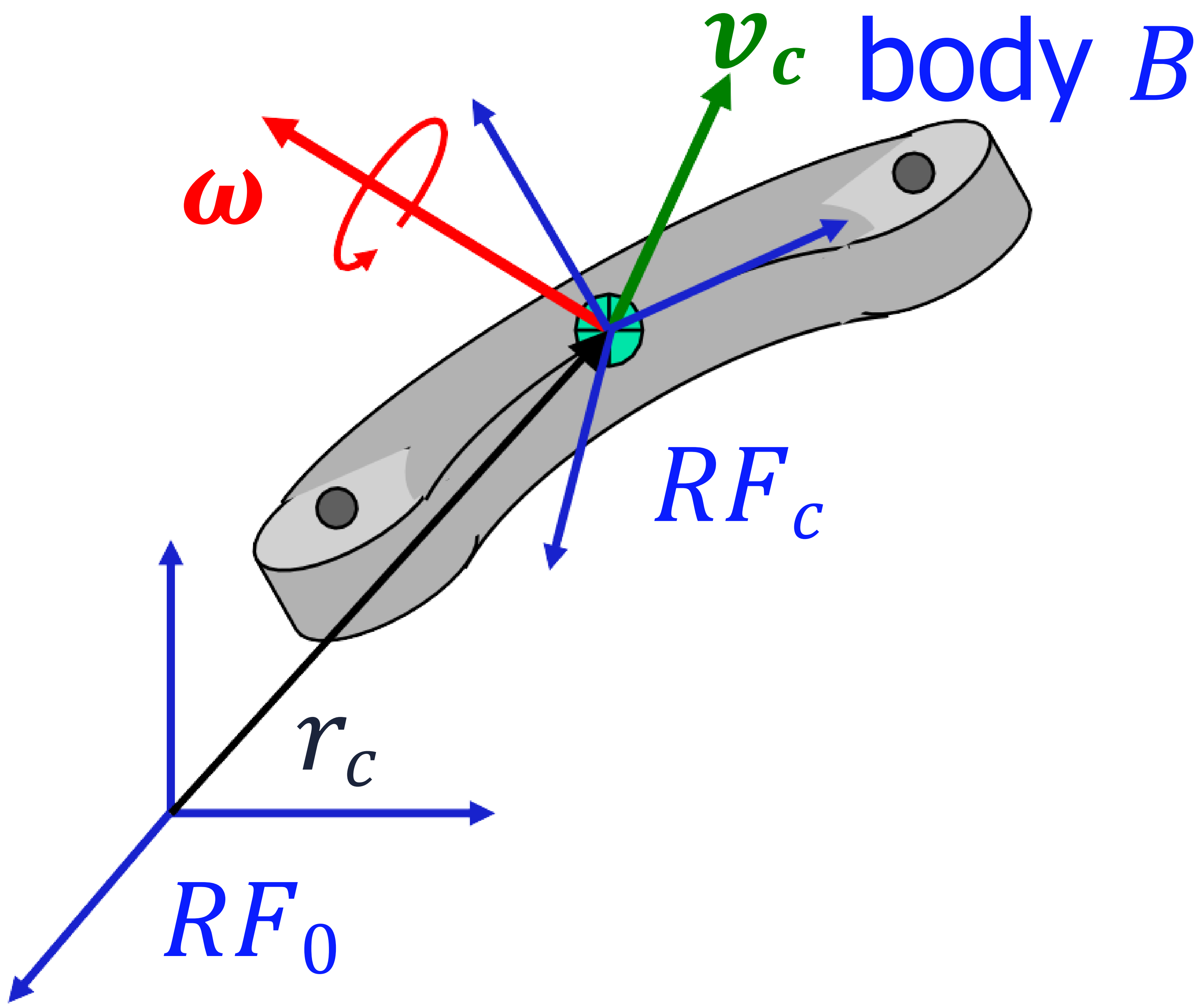
desired motion
for $t \in [0, T]$

required input
for $t \in [0, T]$



Kinetic Energy

Kinetic energy of a rigid body



mass density

mass $$m = \int_B \rho\,(x,y,z)dxdydz = \int_B d\,m$$

position of center of mass (CoM) $$r_c = \frac{1}{m}\int_B r\,dm$$

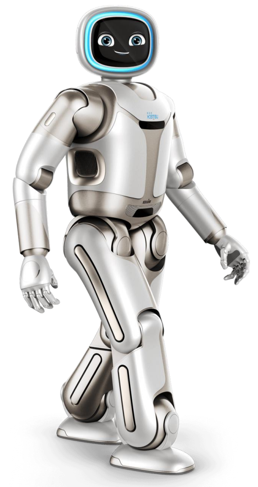
when all vectors are referred to a body frame RF_c attached to the CoM, then

$$r_c = 0 \Rightarrow \int_B r\,dm = 0$$

kinetic energy $$T = \frac{1}{2}\int_B v^T\,(x,y,z)v(x,y,z)dm$$

(fundamental) kinematic relation for a rigid body $$v = v_c + \omega \times r = v_c + S(\omega)r$$

skew-symmetric matrix



Kinetic Energy

Kinetic energy of a rigid body (cont)

$$T = \frac{1}{2}\int_B (v_c + S(\omega)r)^T (v_c + S(\omega)r)dm$$

$$= \frac{1}{2}\int_B v_c^T v_c dm + \int_B v_c^T S(\omega)rdm + \frac{1}{2}\int_B r^T S^T(\omega)S(\omega)rdm$$

$$\boxed{= \frac{1}{2}mv_c^T v_c}$$

$$= v_c^T S(\omega)\int_B r\, dm = 0$$

translational kinetic energy (point mass at CoM)

sum of elements on the diagonal of a matrix

$$a^T b = trace\{ab^T\}$$

$$= \frac{1}{2}\int_B trace\{S(\omega)rr^T S^T(\omega)\}dm$$

$$= \frac{1}{2}\mathrm{trace}\left\{S(\omega)\left(\int_B r\, r^T dm\right)S^T(\omega)\right\}$$

$$= \frac{1}{2}\mathrm{trace}\{S(\omega)J_c S^T(\omega)\}$$

Euler matrix

$$\vdots$$

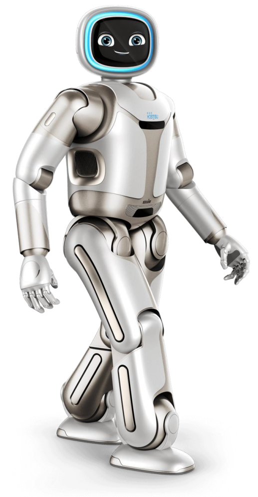
$$\boxed{= \frac{1}{2}\omega^T I_c \omega}$$

body inertia matrix (around the CoM)

+

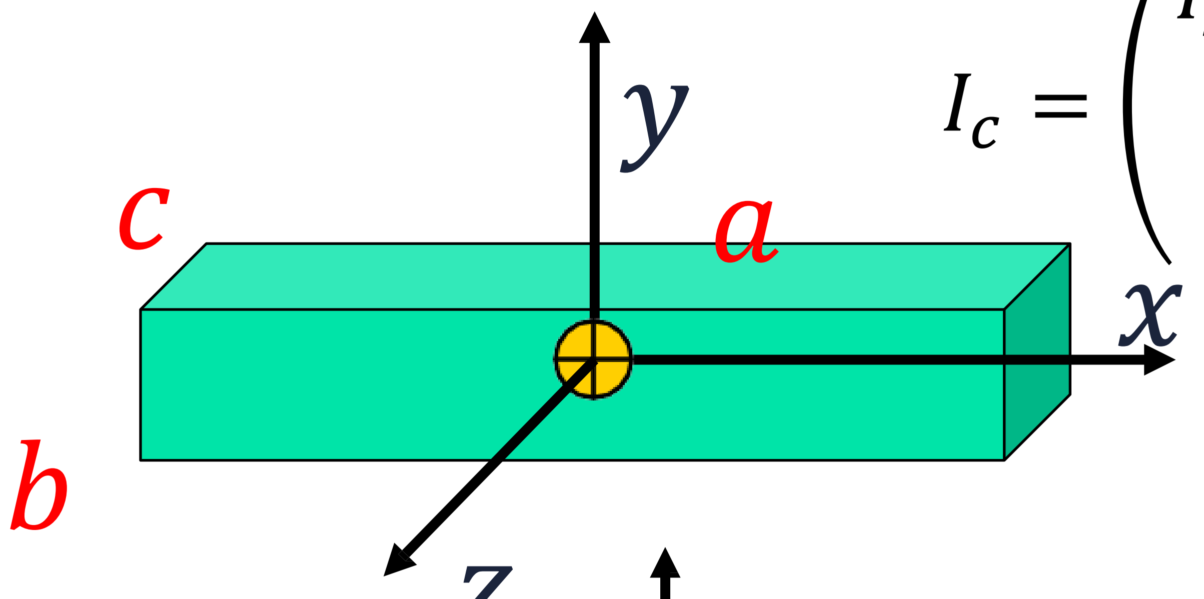
König theorem

rotational kinetic energy (of the whole body)



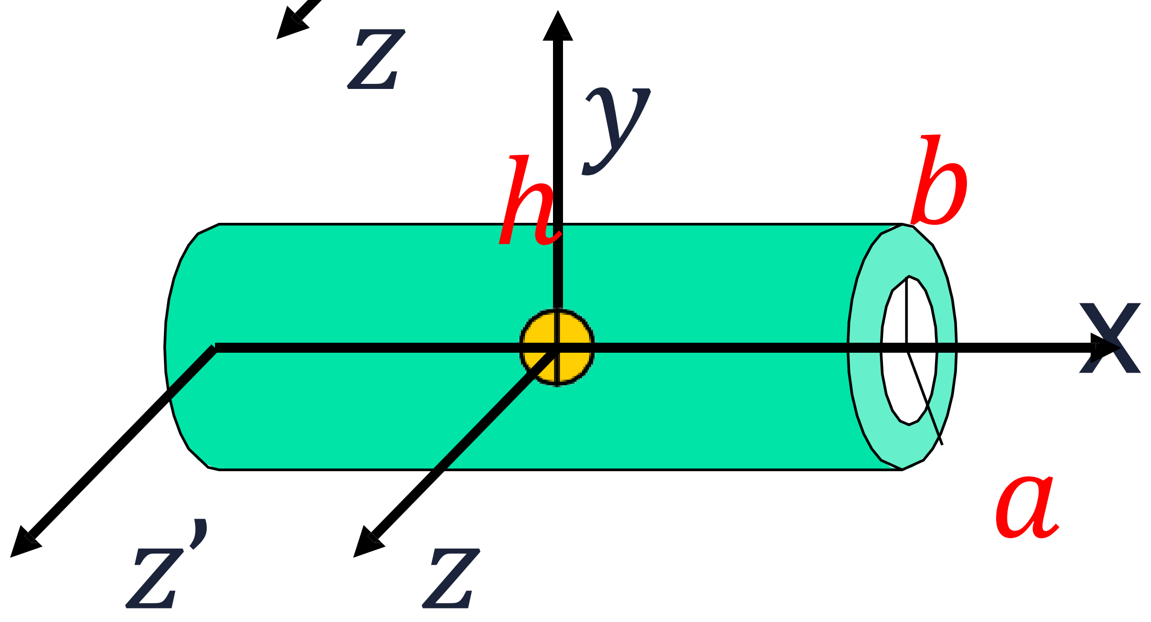
Inertia Matrices

Examples of body inertia matrices
(homogeneous bodies of mass $\boldsymbol{m}$, with axes of symmetry)



$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{pmatrix}$$

parallelepiped with sides a (length/height), b and c (base)



$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m(3(a^2+b^2)+h^2) & \\ & & I_{zz} \end{pmatrix} \qquad I_{zz} = I_{yy}$$

empty cylinder with length h, and external/internal radius a and b

$$I'_{zz} = I_{zz} + m\left(\frac{h}{2}\right)^2$$

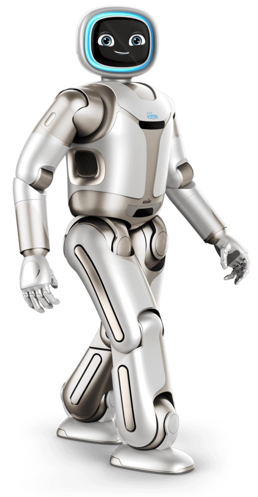
(parallel) axis translation theorem

… its generalization: changes on body inertia matrix due to a pure translation r of the reference frame

Steiner theorem
a.k.a. **parallel axis theorem**

$$\boxed{I = I_c + m(r^T r \cdot E_{3\times3} - rr^T) = I_c + mS^T(r)S(r)}$$

body inertia matrix relative to the CoM identity matrix

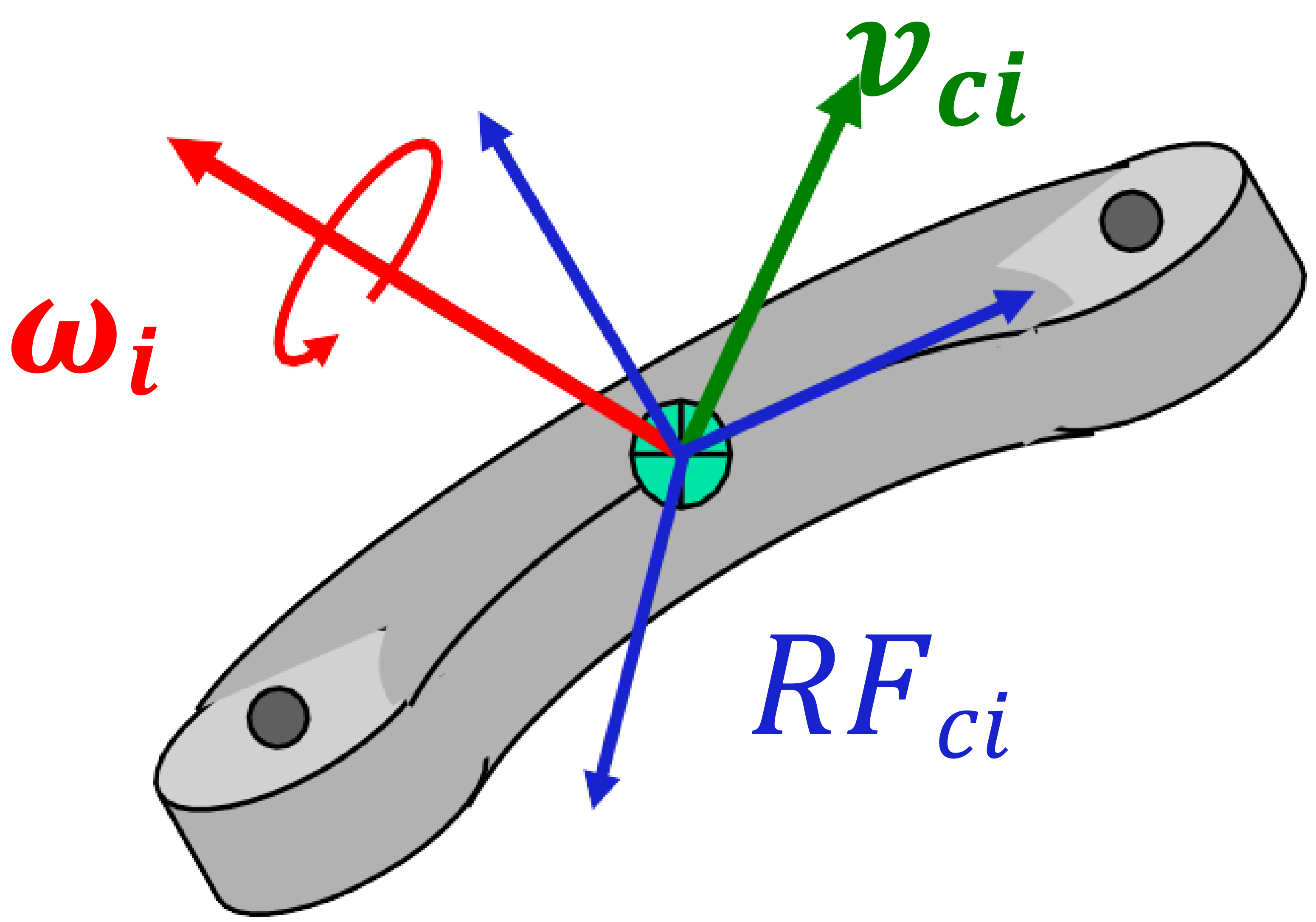


Robot Kinetic Energy

Robot kinetic energy

$$T = \sum_{i=1}^{N} T_i$$ ← N rigid bodies (+ fixed base)

$$T_i = T_i(q_j, \dot{q}_j; \underbrace{j \leq i})$$ ← open kinematic chain



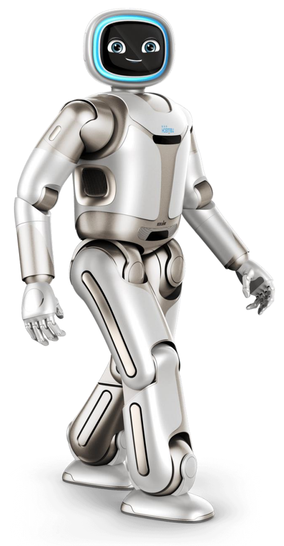
$i-th$ link (body) of the robot

König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

absolute velocity of the center of mass (CoM)

absolute angular velocity of whole body



Robot Kinetic Energy

Kinetic energy of a robot link

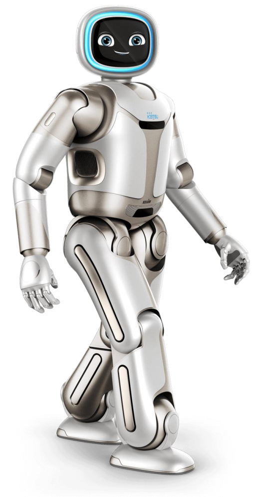
$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

ω_i, I_{ci} should be expressed in the **same reference frame**,
but the product $\omega_i^T I_{ci} \omega_i$ is **invariant** w.r.t. any chosen frame

in frame RF_{ci} attached to (the center of mass of) link i

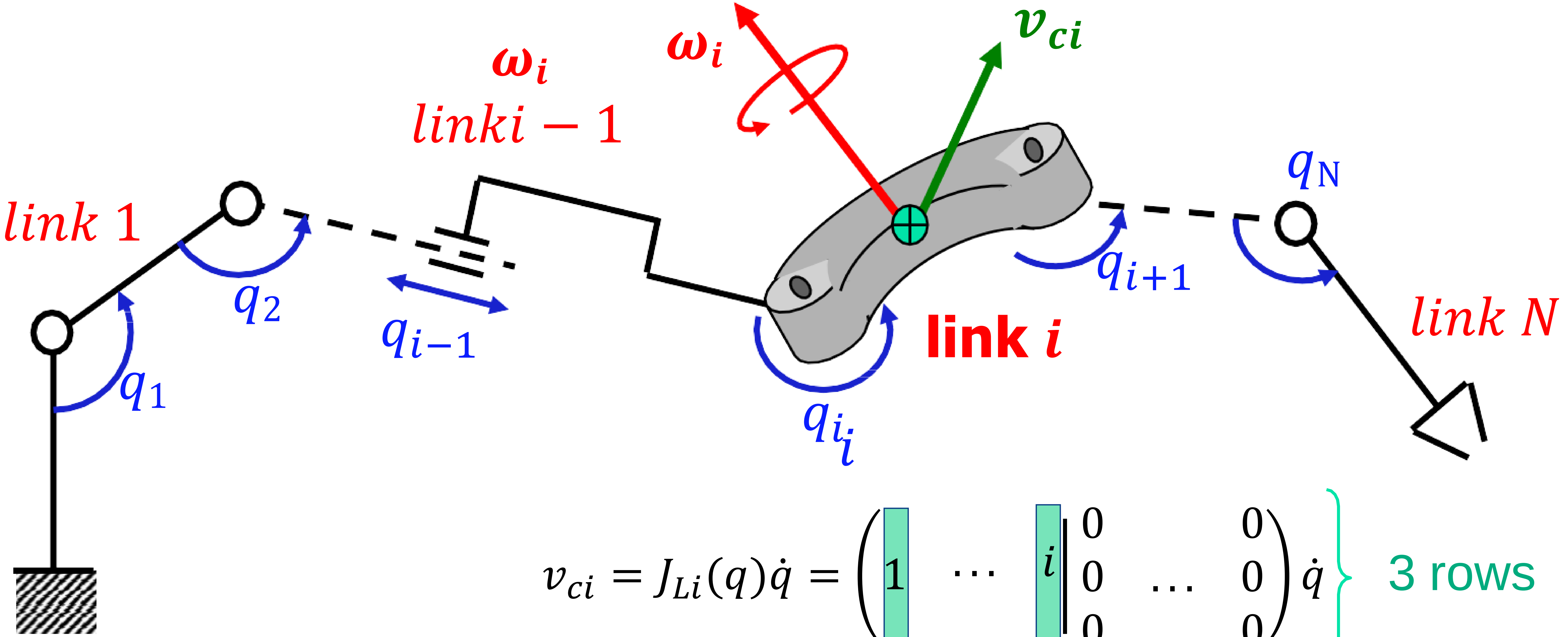
$$ {}^{i}I_{ci} = \begin{pmatrix} \int (y^2+z^2)\, dm & -\int x\, y dm & -\int x\, z dm \\ & \int (x^2+z^2)\, dm & -\int y\, z dm \\ \text{symm} & & \int (x^2+y^2)\, dm \end{pmatrix}$$

constant!



Robot Kinetic Energy

Dependence of $\boldsymbol{T}$ from $\boldsymbol{q}$ and $\boldsymbol{q}$



$$v_{ci} = J_{Li}(q)\dot{q} = \left.\begin{pmatrix} & & & 0 & & 0 \\ 1 & \cdots & i & 0 & \dots & 0 \\ & & & 0 & & 0 \end{pmatrix}\dot{q}\ \right\}\ \text{3 rows}$$

(partial) Jacobians
typically expresse in RF_0

$$\omega_i = J_{Ai}(q)\dot{q} = \left.\begin{pmatrix} & & & 0 & & 0 \\ 1 & \cdots & i & 0 & \dots & 0 \\ & & & 0 & & 0 \end{pmatrix}\dot{q}\ \right\}\ \text{3 rows}$$

Robot Kinetic Energy

Final expression of T

$$\boxed{T=\frac{1}{2}\sum_{i=1}^{N}\left(m_i v_{ci}^T v_{ci}+\omega_i^T I_{ci}\omega_i\right)}$$

$$=\frac{1}{2}\dot{q}^T\left(\sum_{i=1}^{N} m_i J_{Li}^T(q)J_{Li}(q)+J_{Ai}^T(q)I_{ci}J_{Ai}(q)\right)\dot{q}$$

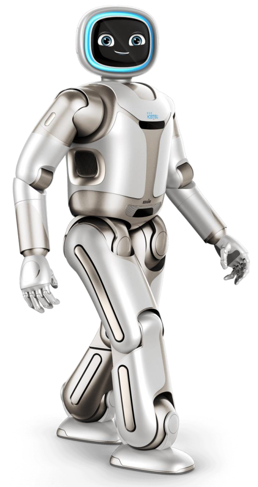
constant if ω_i is expressed in RF_{ci} else

$$\boxed{T=\frac{1}{2}\dot{q}^T M(q)\dot{q}}$$

$$\boxed{{}^0I_{ci}(q)={}^0R_i(q)\,{}^iI_{ci}\,{}^0R_i^T(q)}$$

robot (generalized) inertia matrix

- symmetric
- positive definite, $\forall q$ $\Rightarrow$ **always invertible**



Robot Potential Energy

Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^{N} U_i$$ N rigid bodies (+ fixed base)

$$U_i = U_i(\underbrace{q_j; j \le i}_{})$$ open kinematic chain

$$\boxed{U_i = -m_i g^T r_{0,ci}}$$

gravity acceleration vector; position of the center of mass of link i — typically expressed in RF_0

dependence on q

$$\boxed{\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^0A_1(q_1)\,{}^1A_2(q_2)\cdots{}^{i-1}A_i(q_i)\begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}}$$

constant in RF_i

NOTE: need to work with homogeneous coordinates

Summarizing

kinetic energy $T = \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j$

positive definite quadratic form

potential energy $U = U(q)$

Lagrangian $L = T(q,\dot{q}) - U(q)$

$T \geq 0$

T=0 ⬌ $\dot{q} = 0$

Euler-Lagrange equations $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k}$ $k = 1, \dots, N$

non-conservative (active/dissipative) generalized forces **performing work** on q_k coordinate

Applying

Applying Euler-Lagrange equations
(the scalar derivation-see Appendix for vector format)

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_i m_{kj}\,\dot{q}_j \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\,\ddot{q}_j + \sum_{i,j}\frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

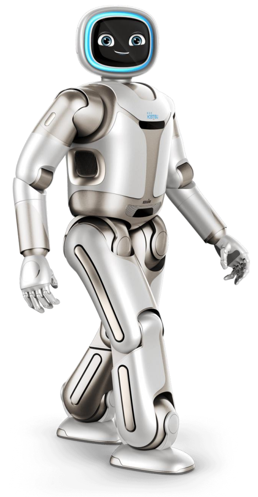
(dependences of elements on q are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION $\ddot{q}$

QUADRATIC terms in VELOCITY $\dot{q}$

NONLINEAR terms in CONFIGURATION q



Applying

***k*-th dynamic equation …**

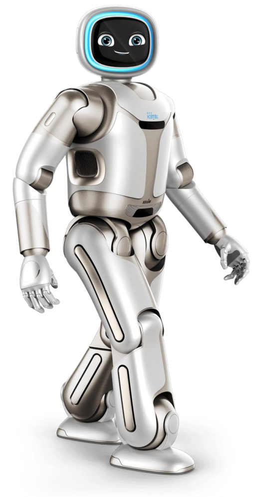
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$\sum_j m_{kj}\,\ddot{q}_j + \sum_{i,j}\left(\frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging
"mute" indices i, j

$$\cdots + \sum_{i,j}\underbrace{\frac{1}{2}\left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)}_{c_{kij} = c_{kji}}\dot{q}_i\dot{q}_j + \cdots$$

Christoffel symbols
of the first kind



Applying

… and interpretation of dynamic terms

$$\boxed{\sum_j m_{kj}(q)\ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q)\dot{q}_i\dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \qquad k = 1, \ldots, N$$

INERTIAL terms

CENTRIFUGAL $(i = j)$ and **CORIOLIS** $(i \neq j)$ terms

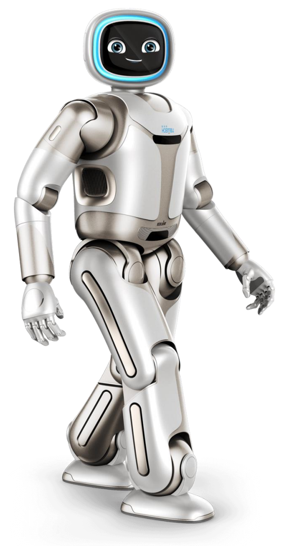
GRAVITY terms $g_k(q)$

$m_{kk}(q)$ = inertia at joint k when joint k accelerates $(m_{kk} > 0!!)$

$m_{kj}(q)$ = inertia "seen" at joint k when joint j accelerates

$c_{kii}(q)$ = coefficient of the centrifugal force at joint k when joint i is moving $(ciii = 0, \forall i)$

$c_{kij}(q)$ =coefficient of the Coriolis force at joint k when joint i and joint j are both moving



Robot Dynamic Model

Robot dynamic model
(in vector formats)

1. $$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

$k - th$ component of vector c

$$C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

$k - th$ column of matrix $M(q)$

symmetric matrix!

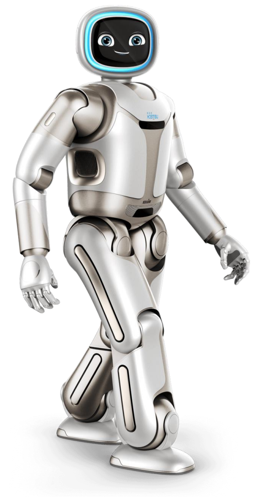
2. $$M(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u$$

NOTE: the model is in the form $\Phi(q,\dot{q},\ddot{q}) = u$ as expected

NOT a symmetric matrix in general

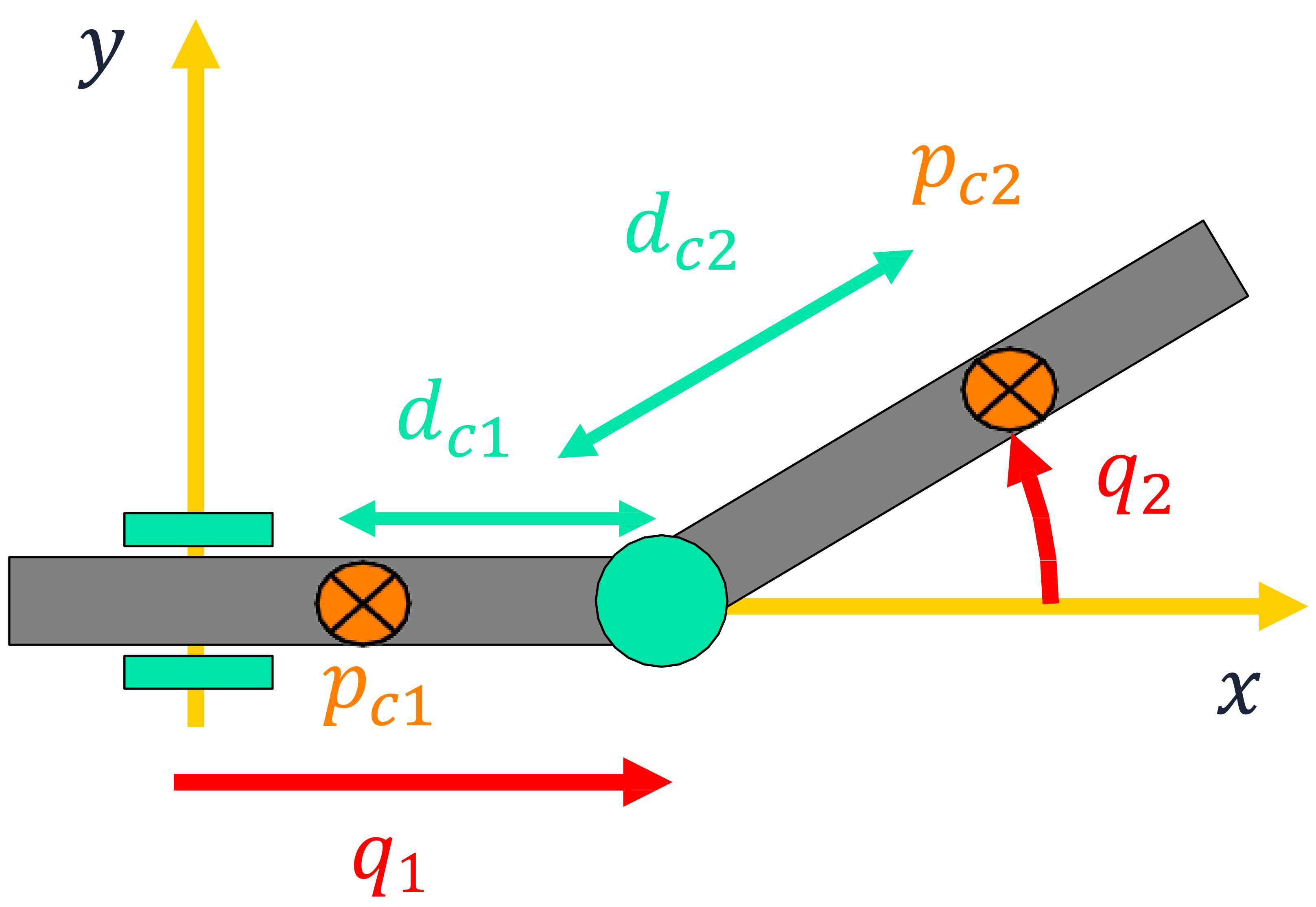
$$s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i$$

factorization of c by S is **not unique!**



An Example – PR Robot

Dynamic model of a PR robot



$$T = T_1 + T_2$$

$U = \text{constant} \Rightarrow g(q) \equiv 0$
(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

$$\boxed{T_1 = \frac{1}{2} m_1 \dot{q}_1^2}$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \Rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \, \dot{q}_2 \\ d_{c2} \cos q_2 \, \dot{q}_2 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

$$\boxed{T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2 d_{c2} \sin q_2 \, \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2}$$

An Example – PR Robot

Dynamic model of a PR robot (cont)

$$M(q) = \begin{pmatrix} \boxed{\begin{matrix} m_1 + m_2 \\ -m_2 d_{c2} \sin q_2 \end{matrix}} & \boxed{\begin{matrix} -m_2 d_{c2} \sin q_2 \\ I_{c2,zz} + m_2 d_{c2}^2 \end{matrix}} \end{pmatrix}$$

M_1 $\quad$ M_2

$$c(q,\dot{q}) = \begin{pmatrix} c_1(q,\dot{q}) \\ c_2(q,\dot{q}) \end{pmatrix}$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

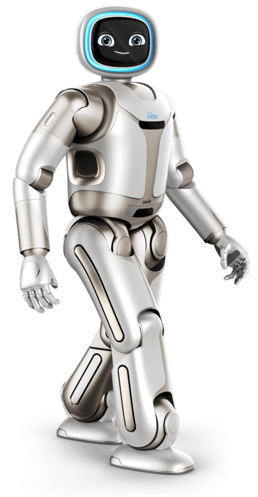
where $$C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

$$C_1(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$c_1(q,\dot{q}) = -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2$$

$$\mathrm{C}_2(\mathrm{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix}\right) = 0$$

$$c_2(q,\dot{q}) = 0$$



An Example – PR Robot

Dynamic model of a PR robot (cont)

$$\boxed{M(q)\ddot{q} + c(q,\dot{q}) = u}$$

$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

NOTE: the $m_{>>}$ element (here, for $N = 2$) of $M(q)$ is always constant!

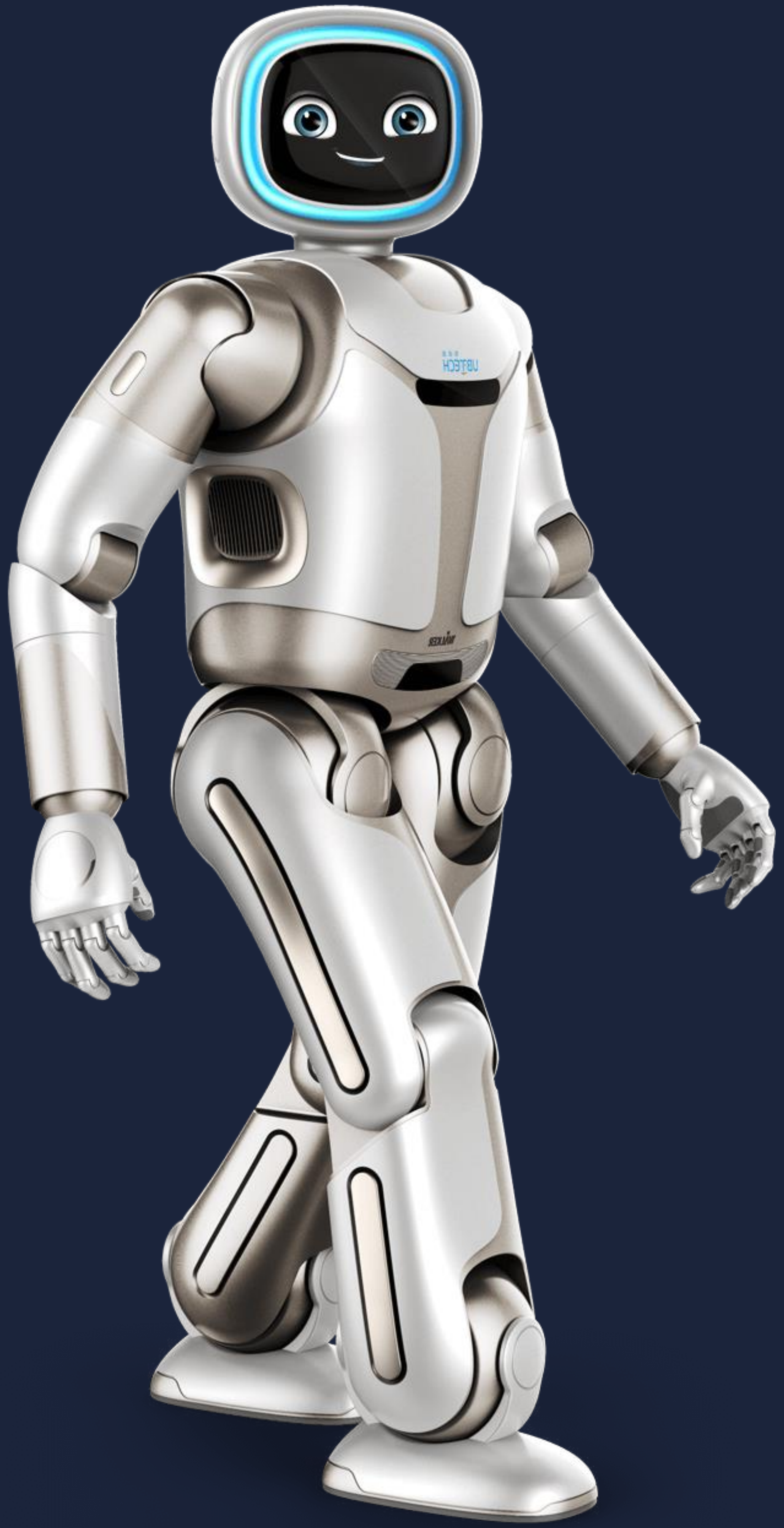
Q1: why does variable q_1 not appear in $M(q)$? ... this is a general property!

Q2: why Coriolis terms are not present?

Q3: when applying a force u_1, does the second joint accelerate? … always?

Q4: what is the expression of a factorization matrix S? ... is it unique here?

Q5: which is the configuration with “maximum inertia”?



Q&A