The Chinese University of Hong Kong
Dept of Mechanical & Automation Eng

ENGG5403 Linear System Theory & Design

Assignment #1

by

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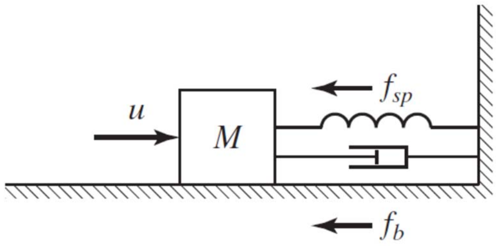
Problem 1

Consider the mechanical system shown in the figure below. Here $u(t)$ is an external force applied to the mass M, $y(t)$ is the displacement of the mass with respect to the position when the spring is relaxed. The spring force and friction force are given respectively by

$$f_{sp}(t) = k(1 + ay^2(t))y(t), \qquad f_b(t) = b\dot{y}(t).$$



1. Write the differential equation model of this system.
2. Write a state space description of the system.
3. Is the system linear? If it is not linear, linearize it around the operating point with $u_0 = 0$.
4. Find the transfer function of the linearized system.

Solution:

1. Using Newton's second law, the differential equation model of the system is derived as follows:

$$M\ddot{y} + f_b(t) + f_{sp}(t) = u \tag{1}$$

that is

$$M\ddot{y}(t) + b\dot{y}(t) + k\left(1 + ay^2(t)\right)y(t) = u \tag{2}$$

2. Let states be $x_1 = y(t)$ and $x_2 = \dot{y}(t)$. Therefore, the system can be described by these states:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{M}\left(1 + ax_1^2\right)x_1 - \frac{b}{M}x_2 + \frac{1}{M}u \end{cases} \tag{3}$$

Hence, the state space description of the system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k\left(1+ax_1^2\right)}{M} & -\frac{b}{M} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \tag{4}$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{5}$$

3. The system is not linear. The Jacobian matrix of the nonlinear equation is

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k\left(1+3ax_1^2\right)}{M} & -\frac{b}{M} \end{bmatrix} \tag{6}$$

The linearized system around the operating point with $u_0 = 0$ is

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{7}$$

where

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} \tag{8}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0} = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \tag{9}$$

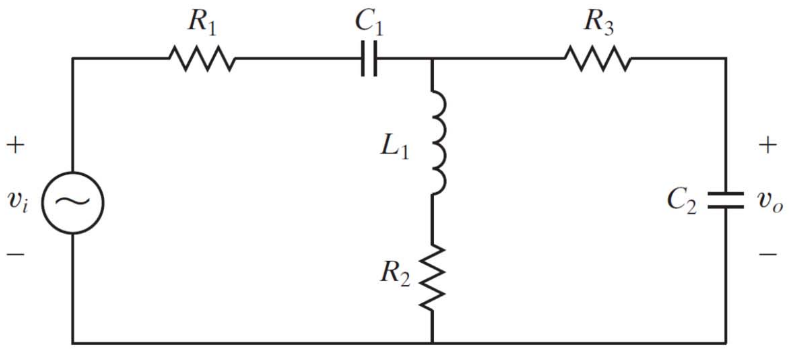
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{10}$$

4. The transfer function of the system is

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{M} & s+\frac{b}{M} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} = \frac{1}{Ms^2+bs+k} \tag{11}$$

Problem 2

Consider the electric circuit network in the figure below. Let the input be $v_i(t)$ and output be $v_o(t)$.



1. Derive the state and output equation of the network.
2. Find the transfer function of the network.

Assuming that $R_1 = R_2 = R_3 = 1\,\Omega$, $C_1 = C_2 = 1\,\text{F}$ and $L_1 = 1\,\text{H}$,

3. Find the unit step response of the network.
4. Find the unit impulse response of the network.

Solution:

1. Assume the voltages across the capacities C_1 and C_2 are equal to v_1 and v_2, respectively and the current across the inductor is equal to i.

   Applying KCL to the node near C_1, R_3, and L_1 yields that

$$C_1\frac{dv_1}{dt} = i + C_2\frac{dv_2}{dt} \tag{12}$$

   Applying KVL to the loop around v_i, R_1, C_1, L_1, and R_2 yields that

$$R_1C_1\frac{dv_1}{dt} + v_1 + L_1\frac{di}{dt} + R_2 i - v_i = 0 \tag{13}$$

   Applying KVL to the loop around v_o, R_2, L_1, and R_3 yields that

$$R_3C_2\frac{dv_2}{dt} + v_2 - L_1\frac{di}{dt} - R_2 i = 0 \tag{14}$$

   Combining Equation (12), (13), and (14) obtains that

$$\frac{dv_1}{dt} = -\frac{1}{(R_1+R_3)\,C_1}v_1 - \frac{1}{(R_1+R_3)\,C_1}v_2 + \frac{R_3}{(R_1+R_3)\,C_1}i + \frac{1}{(R_1+R_3)\,C_1}v_i \tag{15}$$

$$\frac{dv_2}{dt} = -\frac{1}{(R_1+R_3)C_2}v_1 - \frac{1}{(R_1+R_3)C_2}v_2 - \frac{R_1}{(R_1+R_3)C_2}i + \frac{1}{(R_1+R_3)C_2}v_i \quad (16)$$

$$\frac{di}{dt} = -\frac{R_3}{(R_1+R_3)L_1}v_1 + \frac{R_1}{(R_1+R_3)L_1}v_2 - \frac{R_1R_2+R_1R_3+R_2R_3}{(R_1+R_3)L_1}i + \frac{R_3}{(R_1+R_3)L_1}v_i \quad (17)$$

Therefore, the state and output equation of the network is

$$\begin{aligned}\dot{x} &= \begin{bmatrix}\frac{dv_1}{dt}\\ \frac{dv_2}{dt}\\ \frac{di}{dt}\end{bmatrix} = \begin{bmatrix}-\frac{1}{(R_1+R_3)C_1} & -\frac{1}{(R_1+R_3)C_1} & \frac{R_3}{(R_1+R_3)C_1}\\ -\frac{1}{(R_1+R_3)C_2} & -\frac{1}{(R_1+R_3)C_2} & -\frac{R_1}{(R_1+R_3)C_2}\\ -\frac{R_3}{(R_1+R_3)L_1} & \frac{R_1}{(R_1+R_3)L_1} & -\frac{R_1R_2+R_1R_3+R_2R_3}{(R_1+R_3)L_1}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ i\end{bmatrix} + \begin{bmatrix}\frac{1}{(R_1+R_3)C_1}\\ \frac{1}{(R_1+R_3)C_2}\\ \frac{R_3}{(R_1+R_3)L_1}\end{bmatrix}v_i\\ &= Ax + Bu\end{aligned} \quad (18)$$

$$y = v_o = v_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ i\end{bmatrix} = Cx \quad (19)$$

2. The transfer function of the system is equal to

$$\begin{aligned}G(s) &= C(sI-A)^{-1}B + D\\ &= \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}s+\frac{1}{(R_1+R_3)C_1} & \frac{1}{(R_1+R_3)C_1} & -\frac{R_3}{(R_1+R_3)C_1}\\ \frac{1}{(R_1+R_3)C_2} & s+\frac{1}{(R_1+R_3)C_2} & \frac{R_1}{(R_1+R_3)C_2}\\ \frac{R_3}{(R_1+R_3)L_1} & -\frac{R_1}{(R_1+R_3)L_1} & s+\frac{R_1R_2+R_1R_3+R_2R_3}{(R_1+R_3)L_1}\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{(R_1+R_3)C_1}\\ \frac{1}{(R_1+R_3)C_2}\\ \frac{R_3}{(R_1+R_3)L_1}\end{bmatrix}\\ &= \left[C_1 s(R_2+L_1 s)\right]/\big[C_1C_2L_1(R_1+R_3)s^3\\ &\quad + ((C_1+C_2)L_1 + C_1C_2(R_1R_2+R_1R_3+R_2R_3))s^2\\ &\quad + (C_1(R_1+R_2)+C_2(R_2+R_3))s + 1\big]\end{aligned} \quad (20)$$

3. When $R_1 = R_2 = R_3 = 1\ \Omega$, $C_1 = C_2 = 1$ F and $L_1 = 1$ H, the transfer function becomes

$$G(s) = \frac{s(s+1)}{2s^3+5s^2+4s+1} \quad (21)$$

Therefore, when subjected to the step input, the response of the system becomes

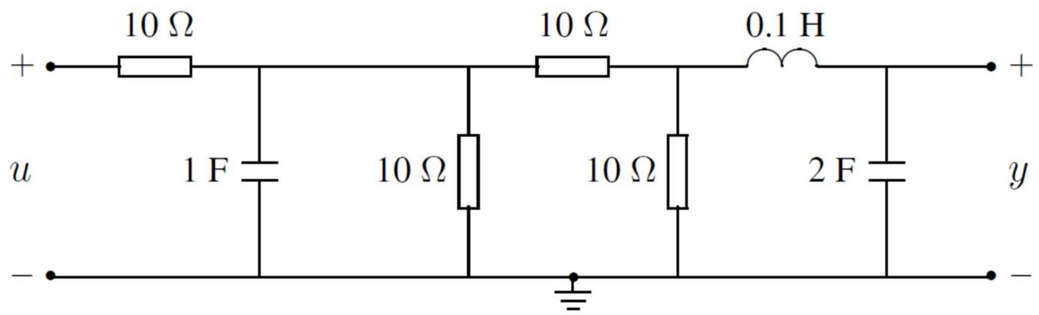
$$\begin{aligned}y(t) &= \mathscr{L}^{-1}\left\{G(s)\cdot\frac{1}{s}\right\}\\ &= \mathscr{L}^{-1}\left\{\frac{s+1}{2s^3+5s^2+4s+1}\right\}\\ &= \mathscr{L}^{-1}\left\{\frac{1}{(s+1)(2s+1)}\right\}\\ &= e^{-\frac{1}{2}t} - e^{-t}\end{aligned} \quad (22)$$

4. When subjected to the unit impulse input, the response of the system becomes

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{G(s)\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s(s+1)}{2s^3+5s^2+4s+1}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s}{(s+1)(2s+1)}\right\} \\
&= e^{-t}-\frac{1}{2}e^{-\frac{1}{2}t}
\end{aligned} \tag{23}$$

Problem 3

Consider an electric network shown in the circuit below with its input, u, being a voltage source, and output, y, being the voltage across the 2 F capacitor. Assume that the initial voltages across the 1 F and 2 F capacitors are 1 V and 2 V, respectively, and that the inductor is initially uncharged.



(a) Derive the state and output equations of the network.

(b) Find the unit step response of the network.

(c) Find the unit impulse response of the network.

(d) Determine the stability of the network.

Solution:

1. Assume the voltages across the capacities 1 F and 2 F are equal to v_1 and v_2, respectively and the current across the inductor and fourth resistor are equal to i and i_R. Analyzing the current across the inductor and the second capacitor yields that

$$2\frac{dv_2}{dt} = i \tag{24}$$

Applying KVL to the loop-the inductor, the second capacitor, and the fourth resistor-yields that

$$0.1\frac{di}{dt} + v_2 - 10i_R = 0 \tag{25}$$

Applying KVL to the loop-the second, third, and fourth resistor-yields that

$$10\left(i + i_R\right) + 10i_R - v_1 = 0 \tag{26}$$

Applying KVL to the loop-the input voltage source, the first resistor, and the first capacitor-yields that

$$-u + v_1 + 10\left(i + i_R + \frac{v_1}{10} + \frac{dv_1}{dt}\right) = 0 \tag{27}$$

Observing Equation (26), i_R can be represented by other parameters:

$$i_R = \frac{1}{20}v_1 - \frac{1}{2}i \tag{28}$$

Substituting Equation (28) into Equation (24), (25), and (27) yields that

$$\frac{dv_1}{dt} = -\frac{1}{4}v_1 - \frac{1}{2}i + \frac{1}{10}u \tag{29}$$

$$\frac{dv_2}{dt} = \frac{1}{2}i \tag{30}$$

$$\frac{di}{dt} = 5v_1 - 10v_2 - 50i \tag{31}$$

Therefore, the state and output equation of the network is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 5 & -10 & -50 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} + \begin{bmatrix} \frac{1}{10} \\ 0 \\ 0 \end{bmatrix} v_i \\ &= Ax + Bu \end{aligned} \tag{32}$$

$$y = v_o = v_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} = Cx \tag{33}$$

2. The Laplace transfer of the output of the system when subjected to the step input ($u(s) = \frac{1}{s}$) with the initial condition $x(0^-) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is equal to

$$\begin{aligned} Y(s) &= \left[C(sI - A)^{-1} B + D\right] u(s) + C(sI - A)^{-1} x(0^-) \\ &= \frac{8s^3 + 402s^2 + 130s + 1}{s\left(4s^3 + 201s^2 + 80s + 5\right)} \end{aligned} \tag{34}$$

Applying the inverse Laplace transform to Equation (34) obtains the response of output:

$$y(t) = 1.7589e^{-0.07761t} + 0.0441e^{-0.3231t} - 0.003e^{-49.8493t} + 0.2 \tag{35}$$

3. The Laplace transfer of the output of the system when subjected to the unit impulse ($u(s) = 1$) with the initial condition $x(0^-) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is equal to

$$\begin{aligned} Y(s) &= \left[C(sI - A)^{-1} B + D\right] u(s) + C(sI - A)^{-1} x(0^-) \\ &= \frac{8s^2 + 402s + 131}{4s^3 + 201s^2 + 80s + 5} \end{aligned} \tag{36}$$

Applying the inverse Laplace transform to Equation (36) obtains the response of output:

$$y(t) = -0.0401e^{-0.3231t} + 2.043e^{-0.07761t} - 0.0029^{-49.8493t} \tag{37}$$

4. The eigenvalues of matrix A are -49.8493, -0.3231, and -0.0776. They all have a negative real part. Therefore, the system is both BIBO stable and internal stable.

Problem 4

Given a linear system, $\dot{x} = Ax + Bu$, with $x(t_1) = x_1$ and $x(t_2) = x_2$ for some $t_1 > 0$ and $t_2 > 0$, show that

$$\int_{t_1}^{t_2} \mathrm{e}^{-A\tau} Bu(\tau) d\tau = \mathrm{e}^{-At_2} x_2 - \mathrm{e}^{-At_1} x_1.$$

Solution:

$$\begin{aligned}
\int_{t_1}^{t_2} e^{-A\tau} Bu(\tau)\, d\tau &= \int_{t_1}^{t_2} \left\{ -e^{-A\tau} Ax(\tau) + \left[e^{-A\tau} \left(Ax(\tau) + Bu(\tau) \right) \right] \right\} d\tau \\
&= \int_{t_1}^{t_2} \left\{ \left[e^{-A\tau} \right]' x(\tau) + e^{-A\tau} \dot{x}(\tau) \right\} d\tau \\
&= \int_{t_1}^{t_2} \left[e^{-A\tau} x(\tau) \right]' d\tau \\
&= e^{-A\tau} x(\tau) \Big|_{\tau=t_1}^{\tau=t_2} \\
&= e^{-At_2} x_2 - e^{-At_1} x_1
\end{aligned} \tag{38}$$

Problem 5

Given

$$e^{At} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix},$$

determine the values of the scalars α and β, and the matrices A and A^{100}.

Solution:

When $t = 0$, we have $e^{At} = e^0 = I$. Therefore, we can know that $\alpha = 2$ and $\beta = 1$.

Because $\mathscr{L}\left(e^{At}\right) = (sI - A)^{-1}$,

$$\mathscr{L}\left(e^{At}\right) = \mathscr{L}\left\{\begin{bmatrix} -e^{-t} + 2e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}\right\} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \tag{39}$$

Therefore,

$$sI - A = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \tag{40}$$

Hence,

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \tag{41}$$

The eigenvalues of matrix A is -2 and -1. Therefore, the matrix A can be decomposed to

$$A = XJX^{-1} = \begin{bmatrix} -1 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \tag{42}$$

Therefore,

$$A^{100} = XJ^{100}X^{-1} = \begin{bmatrix} -1 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2^{101} - 1 & 2^{100} - 1 \\ -2^{101} + 2 & -2^{100} + 2 \end{bmatrix} \tag{43}$$

Problem 6

Show that the pendulum system is a BIBO unstable system even though it was proved to be internally marginally stable. Identify a bounded input signal such that when it is applied to the pendulum, the resulting output response will go unbounded.

For simplicity, you can assume that $ML^2 = 1$ and $g = L$.

Solution:

The inverse pendulum operating point is unstable evidently. The inverted pendulum is a classic example in control theory and robotics that involves stabilizing a pendulum in an inverted position by controlling the base on which it stands. The inverted pendulum has an operating point where the pendulum is perfectly balanced in the upright position, and the objective is to maintain the pendulum in that position.

It is well-known that the inverted pendulum's operating point is marginally stable, meaning that small perturbations from the upright position will cause the pendulum to oscillate around the operating point. However, while the operating point may be internally marginally stable, it is actually BIBO (bounded-input bounded-output) unstable.

BIBO stability is concerned with the response of a system to arbitrary inputs, rather than just small perturbations around an operating point. A system is BIBO stable if, for any bounded input signal, the output signal remains bounded. In the case of the inverted pendulum, if an arbitrary input is applied to the system, the pendulum will eventually fall over, regardless of how small the initial perturbation was.

Near the point $\theta = 0$, the system can be modelled as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ML^2} \end{bmatrix} u \tag{44}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{45}$$

where $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$.

Because the real part of the eigenvalues of matrix A is 0, the system is internal stable.

Assume $ML^2 = 1$ and $g = L$, the system can be simplified as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{46}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{47}$$

The transfer function of the system is

$$G(s) = C(sI - A)^{-1} B + D = \frac{1}{s^2 + 1} \tag{48}$$

So, the impulse response of the system is

$$y(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t \tag{49}$$

Because

$$\int_0^\infty |y(t)|\, dt \to \infty \tag{50}$$

Therefore, the system is BIBO unstable by "BIBO stability criterion" (Kotsios & Kalouptsidis, 1993) and it states that a single-input single-output (SISO) system is absolutely bounded-input bounded-output (BIBO) stable if and only if its impulse response is absolutely integrable on the interval $(0, \infty]$.

This theorem has its roots in the field of control theory and systems engineering, where it's important to understand the behavior of a system in response to various inputs. The concept of BIBO stability is a basic requirement for a system to be considered well-behaved, as it ensures that the output of the system remains bounded when a bounded input is applied.

The theorem is a fundamental result that has been widely used in the analysis and design of control systems, and it's a cornerstone of modern control theory. The proof of the theorem involves the application of Laplace transforms and the use of the convolution theorem.

References

Kotsios, S. & Kalouptsidis, N. (1993). Bibo stability criteria for a certain class of discrete nonlinear systems. *International Journal of Control*, 58(3), 707–730.