THE CHINESE UNIVERSITY OF HONG KONG
DEPT OF MECHANICAL & AUTOMATION ENG

ENGG5403 Linear System Theory & Design

Assignment #2

by

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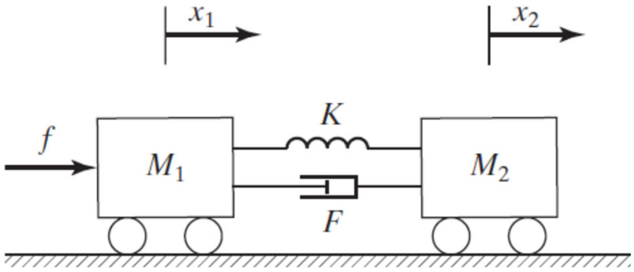
Problem 1

It was shown in the section of dynamic modeling that the two-cart system can be described by the following state space model:



$$\begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K & -F & K & F \\ 0 & 0 & 0 & 1 \\ K & F & -K & -F \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} f, \quad y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix}$$

(a) Determine the stability of the network.

(b) Determine the controllability and observability of the network.

Solution:

(a) The matrix A is a fundamental component of a dynamic system, and its eigenvalues play a vital role in determining the behavior of the system. In this case, Matrix A has two eigenvalues, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \sqrt{F^2 - 2K} - F$, and $\lambda_4 = -\sqrt{F^2 - 2K} - F$, with positive values of F and K. The poles on the imaginary axis are not simple, indicating that the system exhibits some form of oscillation. Moreover, the poles located on the imaginary axis are not simple. Furthermore, the existence of only one eigenvector corresponding to the eigenvalue 0 implies that the system is **not internally stable**.

The transfer function of a system represents the relationship between its input and output, and its stability is a crucial aspect of its behavior. In this case, the transfer function G is

$$G(s) = C(sI - A)^{-1} B + D = \frac{Fs + K}{s^2 \left(s^2 + 2Fs + 2K\right)} \tag{1}$$

which indicates that not all poles of the denominator have a negative part. This means that the system is **not BIBO stable**, which implies that its output may become unbounded if the input is not well-controlled.

(b) The controllability matrix of the system is

$$Q_c = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -F & 2F^2 - K \\ 1 & -F & 2F^2 - K & 2F\left(-2F^2 + K\right) + 2FK \\ 0 & 0 & F & -2F^2 + K \\ 0 & F & -2F^2 + K & -2F\left(-2F^2 + K\right) - 2FK \end{bmatrix} \quad (2)$$

The determinant of the matrix above is

$$\det Q_c = K^2 \quad (3)$$

Therefore, when $K \neq 0$, the rank of Q_c is full, which indicates that the system is controllable.

The observability matrix of the system is

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K & F & -K & -F \\ -2FK & -2F^2 + K & 2FK & 2F^2 - K \end{bmatrix} \quad (4)$$

The determinant of the matrix above is

$$\det Q_o = K^2 \quad (5)$$

Therefore, when $K \neq 0$, the rank of Q_o is full, which indicates that the system is observable.

Problem 2

Given a linear time-invariant system, $\dot{x} = Ax + Bu$, let

$$\tilde{A} := \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix}.$$

(a) Verify that $\mathrm{e}^{\tilde{A}t}$ has the form

$$\mathrm{e}^{\tilde{A}t} = \begin{bmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{bmatrix}.$$

(b) Show that the controllability grammian of the system is given by

$$W_c(t) = \int_0^t \mathrm{e}^{-A\tau} BB' \mathrm{e}^{-A'\tau} d\tau = E_3'(t) E_2(t).$$

Solution:

(a) The definition of e^A is

$$e^{\tilde{A}t} = I + \tilde{A}t + \frac{\tilde{A}^2 t^2}{2} + \cdots \tag{6}$$

where

$$\tilde{A} = \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} \tag{7}$$

$$\tilde{A}^2 = \begin{bmatrix} A^2 & ABB' - BB'A' \\ 0 & A'^2 \end{bmatrix} \tag{8}$$

$$\tilde{A}^3 = \begin{bmatrix} A^3 & A^2BB' - ABB'A' + BB'A'^2 \\ 0 & -A'^3 \end{bmatrix} \tag{9}$$

$$\vdots$$

Therefore,

$$\begin{aligned} e^{\tilde{A}t} &= I + \tilde{A}t + \frac{\tilde{A}^2 t^2}{2} + \cdots \\ &= \begin{bmatrix} I + At + \frac{A^2t^2}{2} + \cdots & BB't + \frac{ABB' - BB'A'}{2} t^2 + \cdots \\ 0 & I - A't + \frac{A'^2 t^2}{2} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{bmatrix} \end{aligned} \tag{10}$$

(b) On one hand, taking the time derivative to the left of Equation (10) yields that

$$\begin{aligned}\frac{d}{dt}e^{\tilde{A}t} = \tilde{A}e^{\tilde{A}t} &= \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix}\begin{bmatrix} e^{At} & Xt \\ 0 & e^{-A't} \end{bmatrix} \\ &= \begin{bmatrix} Ae^{At} & AXt + BB'e^{-A't} \\ 0 & -A'e^{-A't} \end{bmatrix}\end{aligned} \tag{11}$$

where $X = BB' + \frac{ABB'-BB'A'}{2}t + \cdots$. On the other hand, taking the time derivative to the right of Equation (10) yields that

$$\frac{d}{dt}\begin{bmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{bmatrix} = \begin{bmatrix} \frac{dE_1(t)}{dt} & \frac{dE_2(t)}{dt} \\ 0 & \frac{dE_3(t)}{dt} \end{bmatrix} \tag{12}$$

Therefore,

$$\frac{dE_2(t)}{dt} = AXt + BB'e^{-A't} \tag{13}$$

$$\frac{dE_3(t)}{dt} = -A'e^{-A't} \tag{14}$$

$$\frac{dE_3'(t)}{dt} = -e^{-At}A \tag{15}$$

Combining equations above obtains that

$$\begin{aligned}\frac{d}{dt}\left(E_3'(t)E_2(t)\right) &= \frac{dE_3'(t)}{dt}E_2(t)E_3'(t)\frac{dE_2(t)}{dt} \\ &= -e^{-At}AXt + e^{-At}\left(AXt + BB'e^{-A't}\right) \\ &= e^{-At}BB'e^{-A't}\end{aligned} \tag{16}$$

Hence,

$$\begin{aligned}W_c(t) &= \int_0^t e^{-A\tau}BB'e^{-A'\tau}d\tau \\ &= \int_0^t \frac{d}{dt}\left(E_3'(\tau)E_2(\tau)\right)d\tau \\ &= E_3'(\tau)E_2(\tau)\Big|_0^t \\ &= E_3'(t)E_2(t) - E_3'(0)E_2(0) \\ &= E_3'(t)E_2(t) - 0E_3'(0) \\ &= E_3'(t)E_2(t)\end{aligned} \tag{17}$$

Problem 3

Show that if (A, B) is uncontrollable, then $(A + \alpha I, B)$ is also uncontrollable for any $\alpha \in \mathbb{R}$.

Solution:

To show that if (A, B) is uncontrollable, then $(A + \alpha I, B)$ is also uncontrollable for any $\alpha \in \mathbb{R}$, we will use the following definition of uncontrollability:

Definition: A system (A, B) is uncontrollable if and only if the controllability matrix

$$Q_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{18}$$

doesn't have full rank, i.e. $\operatorname{rank}(Q_c) < n$.

Now, let (A, B) be an uncontrollable system. We want to show that the system $(A + \alpha I, B)$ is also uncontrollable for any $\alpha \in \mathbb{R}$.

Let Q_c' be the controllability matrix of $(A + \alpha I, B)$. Then, by definition, Q_c' is given by

$$Q_c' = \begin{bmatrix} B & (A + \alpha I) B & (A + \alpha I)^2 B & \cdots & (A + \alpha I)^{n-1} B \end{bmatrix} \tag{19}$$

We can expand $(A + \alpha I)^k$ using the binomial theorem, and write it as:

$$(A + \alpha I)^n = \sum_{k=0}^{n} C_n^k \alpha^{n-k} A^k \tag{20}$$

Using this expression, we can rewrite Q_c' as:

$$\begin{aligned} Q_c' &= \begin{bmatrix} B & (A + \alpha I) B & (A + \alpha I)^2 B & \cdots & (A + \alpha I)^{n-1} B \end{bmatrix} \\ &= \begin{bmatrix} B & \sum_{k=0}^{1} C_1^k \alpha^{1-k} A^k B & \sum_{k=0}^{2} C_2^k \alpha^{2-k} A^k B & \cdots & \sum_{k=0}^{n-1} C_{n-1}^k \alpha^{n-1-k} A^k B \end{bmatrix} \end{aligned} \tag{21}$$

Note that Q_c' can be expressed as a linear combination of the columns of Q_c. Since Q_c has rank less than n, the matrix Q_c' has rank less than n, since it can be written as a linear combination of matrices whose columns are linear combinations of the columns of Q_c. Therefore, we have shown that if (A, B) is uncontrollable, then $(A + \alpha I, B)$ is also uncontrollable for any $\alpha \in \mathbb{R}$.

Problem 4

Consider an uncontrollable system, $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume that

$$\text{rank}\,(Q_c) = \text{rank}\,([\,B \quad AB \quad \cdots \quad A^{n-1}B\,]) = r < n.$$

Let $\{q_1, q_2, \ldots, q_r\}$ be a basis for the range space of the controllability matrix, Q_c, and let $\{q_{r+1}, \ldots, q_n\}$ be any vectors such that

$$T = [\,q_1 \quad q_1 \quad \cdots \quad q_r \quad q_{r+1} \quad \cdots \quad q_n\,]$$

is nonsingular. Show that the state transformation

$$x = T\tilde{x} = T\begin{pmatrix} \tilde{x}_c \\ \tilde{x}_{\bar{c}} \end{pmatrix}, \quad \tilde{x}_c \in \mathbb{R}^r, \quad \tilde{x}_{\bar{c}} \in \mathbb{R}^{n-r},$$

transforms the given system into the form

$$\begin{pmatrix} \dot{\tilde{x}}_c \\ \dot{\tilde{x}}_{\bar{c}} \end{pmatrix} = \begin{bmatrix} A_{cc} & A_{c\bar{c}} \\ 0 & A_{\bar{c}\bar{c}} \end{bmatrix} \begin{pmatrix} \tilde{x}_c \\ \tilde{x}_{\bar{c}} \end{pmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u,$$

where (A_{cc}, B_c) is controllable. Show that the uncontrollable modes of the system are given by $\lambda(A_{\bar{c}\bar{c}})$.

Solution:

Because

$$Q_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{22}$$

Aq_i $(i = 1, 2, ..., r)$ is also in the range space of Q_c, which means

$$Aq_i = \sum_{j=1}^{r} \bar{a}_{ji} q_j, i = 1, 2, ..., r \tag{23}$$

This indicates that

$$A\begin{bmatrix} q_1 & q_2 & \cdots & q_r \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_r \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1r} \\ \vdots & \ddots & \vdots \\ \bar{a}_{r1} & \cdots & \bar{a}_{rr} \end{bmatrix} \tag{24}$$

Let $\begin{bmatrix} q_{r+1} & \cdots & q_n \end{bmatrix}$ be any vectors such that

$$T = \begin{bmatrix} q_1 & q_2 & \cdots & q_r & q_{r+1} & \cdots & q_n \end{bmatrix} \tag{25}$$

is nonsingular. Similar to Equation (23),

$$Aq_i = \sum_{j=1}^{n} \bar{a}_{ji} q_j, i = r+1, ..., n \tag{26}$$

which gives that

$$A \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1r} & \bar{a}_{1,r+1} & \cdots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{r1} & \cdots & \bar{a}_{rr} & \bar{a}_{r,r+1} & \cdots & \bar{a}_{rn} \\ 0 & \cdots & 0 & \bar{a}_{r+1,r+1} & \cdots & \bar{a}_{r+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \bar{a}_{n,r+1} & \cdots & \bar{a}_{nn} \end{bmatrix} \tag{27}$$

Because

$$x = T\bar{x} \tag{28}$$

we can know that

$$\dot{\bar{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\bar{x} + T^{-1}Bu \tag{29}$$

Therefore,

$$AT = T\bar{A} \tag{30}$$

Combining Equation (27) and (30) obtains that

$$\bar{A} = \begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1r} & \bar{a}_{1,r+1} & \cdots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{r1} & \cdots & \bar{a}_{rr} & \bar{a}_{r,r+1} & \cdots & \bar{a}_{rn} \\ 0 & \cdots & 0 & \bar{a}_{r+1,r+1} & \cdots & \bar{a}_{r+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \bar{a}_{n,r+1} & \cdots & \bar{a}_{nn} \end{bmatrix} = \begin{bmatrix} A_{CC} & A_{C\bar{C}} \\ 0 & A_{\bar{C}\bar{C}} \end{bmatrix} \tag{31}$$

In the same way, we can get that

$$B = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \bar{b}_{11} & \cdots & \bar{b}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{b}_{r1} & \cdots & \bar{b}_{rn} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \tag{32}$$

and

$$B = T\bar{B} \tag{33}$$

Therefore, we can know that

$$\bar{B} = \begin{bmatrix} \bar{b}_{11} & \cdots & \bar{b}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{b}_{r1} & \cdots & \bar{b}_{rn} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} B_C \\ 0 \end{bmatrix} \tag{34}$$

Hence, the state transformation

$$x = T\bar{x} = T \begin{bmatrix} \bar{x}_C \\ \bar{x}_{\bar{C}} \end{bmatrix} \tag{35}$$

transforms the given system into the form

$$\begin{bmatrix} \dot{\bar{x}}_C \\ \dot{\bar{x}}_{\bar{C}} \end{bmatrix} = \begin{bmatrix} A_{CC} & A_{C\bar{C}} \\ 0 & A_{\bar{C}\bar{C}} \end{bmatrix} \begin{bmatrix} \bar{x}_C \\ \bar{x}_{\bar{C}} \end{bmatrix} + \begin{bmatrix} B_C \\ 0 \end{bmatrix} u \tag{36}$$

where (A_{CC}, B_C) is controllable. Because $\dot{\bar{x}}_{\bar{C}} = A_{\bar{C}\bar{C}}\bar{x}_{\bar{C}} + 0u$, u can not control $\bar{x}_{\bar{C}}$. So the uncontrollable modes of the system are given by $\lambda\left(A_{\bar{C}\bar{C}}\right)$.

Problem 5

Verify the result in Q.4 for the following systems:

$$\dot{x} = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 0 & 2 & -2 \\ -1 & -1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} u,$$

and

$$\dot{x} = \begin{bmatrix} -3 & -3 & 1 & 0 \\ 26 & 36 & -3 & -25 \\ 30 & 39 & -2 & -27 \\ 30 & 43 & -3 & -32 \end{bmatrix} x + \begin{bmatrix} 3 & 3 \\ -2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} u.$$

Solution:

1. The controllability matrix of the system is

$$Q_c = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 & 28 \\ 1 & 2 & 4 & 8 \\ -1 & -4 & -16 & -64 \\ 1 & 3 & 10 & 36 \end{bmatrix} \tag{37}$$

because $\mathrm{rank}\,(Q_c) = 2 < 4$, the system is uncontrollable. Define

$$T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix} \tag{38}$$

Thus, we can calculate that

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & -8 & -10 & -4 \\ 1 & 6 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix} \tag{39}$$

$$\bar{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

The controllability matrix of system (A_{CC}, B_C) is

$$Q_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

The rank of Q_c is 2, which is full rank. Therefore, the system is controllable.

The eigenvalues of $A_{\bar{C}\bar{C}}$ are $\lambda_1 = 2$ and $\lambda_2 = 4$. Hence,

$$\operatorname{rank}\left(\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ -1 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\right) = 3 < 4 \tag{42}$$

$$\operatorname{rank}\left(\begin{bmatrix} \lambda_2 I - A & B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -3 & -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 & 1 \\ 2 & 0 & 0 & 2 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix}\right) = 3 < 4 \tag{43}$$

Therefore, all the uncontrollable modes of the system are given by $\lambda\left(A_{\bar{C}\bar{C}}\right)$.

2. The controllability matrix of the system is

$$Q_c = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 3 & 3 & -3 & -3 & 3 & 3 & -3 & -3 \\ -2 & -1 & 6 & 8 & 2 & 6 & 14 & 22 \\ 0 & 3 & 12 & 18 & 12 & 24 & 36 & 60 \\ 0 & 1 & 4 & 6 & 4 & 8 & 12 & 20 \end{bmatrix} \tag{44}$$

because $\operatorname{rank}(Q_c) = 2 < 4$, the system is uncontrollable. Define

$$T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & -4 & 0 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \tag{45}$$

Thus, we can calculate that

$$\bar{A} = T^{-1}AT = \begin{bmatrix} -5 & -7 & -28 & 3 \\ 4 & 6 & 30 & -3 \\ 0 & 0 & -9 & 1 \\ 0 & 0 & -60 & 7 \end{bmatrix} \tag{46}$$

$$\bar{B} = T^{-1}B = \begin{bmatrix} 1 & -0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{47}$$

The controllability matrix of system (A_{CC}, B_C) is

$$Q_c = \begin{bmatrix} 1 & -0 & -5 & -7 \\ 0 & 1 & 4 & 6 \end{bmatrix} \tag{48}$$

The rank of Q_c is 2, which is full rank. Therefore, the system is controllable.

The eigenvalues of $A_{\bar{C}\bar{C}}$ are $\lambda_1 = -3$ and $\lambda_2 = 1$. Hence,

$$\text{rank}\left(\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 0 & 3 & -1 & 0 & 3 & 3 \\ -26 & -39 & 3 & 25 & -2 & -1 \\ -30 & -39 & -1 & 27 & 0 & 3 \\ -30 & -43 & 3 & 29 & 0 & 1 \end{bmatrix}\right) = 3 < 4 \tag{49}$$

$$\text{rank}\left(\begin{bmatrix} \lambda_2 I - A & B \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 4 & 3 & -1 & 0 & 3 & 3 \\ -26 & -35 & 3 & 25 & -2 & -1 \\ -30 & -39 & 3 & 27 & 0 & 3 \\ -30 & -43 & 3 & 33 & 0 & 1 \end{bmatrix}\right) = 3 < 4 \tag{50}$$

Therefore, all the uncontrollable modes of the system are given by $\lambda\left(A_{\bar{C}\bar{C}}\right)$.

Problem 6

Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, show than if the pair (A, B) is controllable (detectable) if and only if $(A^{\mathrm{T}}, B^{\mathrm{T}})$ is observable (stabilizable).

Solution:

Let's first prove the duality of controllability and observability.

The duality of controllability and observability is a fundamental concept in control theory that relates the controllability and observability properties of a linear time-invariant (LTI) system.

In general, controllability refers to the ability to steer the system's state from an initial state to a desired final state using an appropriate control input. On the other hand, observability refers to the ability to infer the system's initial state from a set of available measurements of the system's outputs.

The duality of controllability and observability states that if a system is controllable, then its dual system (i.e., the system obtained by interchanging the roles of the state and output variables) is observable, and vice versa. More formally, the duality principle can be stated as follows:

Let (A, B, C, D) be the state-space representation of an LTI system. Then, the system is controllable if and only if its dual system (A', C', B', D') is observable, where $A' = A^T$, $B' = C^T$, $C' = B^T$, and $D' = D^T$.

This duality principle can be proven mathematically by considering the controllability and observability Gramians of the system and its dual, and showing that they are related by a matrix transposition operation. In particular, the controllability Gramian of the original system is equal to the observability Gramian of its dual, which are

$$W_c(t) = \int_0^t \int_0^t e^{A\tau} B B' e^{A'\tau} d\tau = W_o \tag{51}$$

and vice versa.

Then, the duality of detectability and stabilizability will be proved. A pair of matrices (A, B) is considered detectable if and only if it is possible to find a matrix K, of appropriate dimension, such that all eigenvalues of the matrix $A + BK$ have a negative real part. Similarly, a pair of matrices $\left(A^T, B^T\right)$ is considered stabilizable if and only if it is possible to find a matrix L, of appropriate dimension, such that all eigenvalues of the matrix $A' + LB'$ have negative real part.

It is worth noting that due to the property $\lambda_i(A + BK) = \lambda_i(A + BK)^T = \lambda_i\left(A^T + K^T B^T\right)$, the same matrix K used for detectability can be used for stabilizability by letting $L = K^T$. As such, it follows that the pair (A, B) is detectable if and only if the pair $\left(A^T, B^T\right)$ is stabilizable.

References