THE CHINESE UNIVERSITY OF HONG KONG
DEPT OF MECHANICAL & AUTOMATION ENG

ENGG5403 Linear System Theory & Design

Assignment #3

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

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Problem 1

It was showed earlier that the invariant zeros of linear systems are invariant under state feedback. More specifically, for a system characterized by

$$\dot{x} = A\,x + B\,u$$
$$y = C\,x + D\,u$$

with a state feedback $u = Fx + v$, it gives a closed-loop system

$$\dot{x} = (A + BF)\,x + B\,v$$
$$y = (C + DF)\,x + D\,v$$

We have showed that if a scalar β is an invariant zero of the original system, it is also an invariant zero of the new one as well.

(a) Show that the state feedback law does not change the controllability property of the given system either.

(b) Show by a simple example that the state feedback law, however, may change the observability property of the given system.

Solution:

(a) The controllability matrix of the system is

$$Q_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{1}$$

After applied the state feedback $u = Fx + v$, the controllability matrix becomes

$$\begin{aligned} Q_c' &= \begin{bmatrix} B & (A+BF)\,B & (A+BF)^2\,B & \cdots & (A+BF)^{n-1}\,B \end{bmatrix} \\ &= \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} I & FB & F\,(A+BF)\,B & \cdots & F\,(A+BF)^{n-2}\,B \\ 0 & I & FB & \cdots & F\,(A+BF)^{n-3}\,B \\ 0 & 0 & I & \cdots & F\,(A+BF)^{n-4}\,B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix} \end{aligned} \tag{2}$$

Because two matrices in Equation above are both non-singular, the resulting Q_c' is full rank. Therefore, the state feedback law does not change the controllability property of the given system either.

(b) Taking the state feedback coefficient $F = -D^{-1}C$, the observability matrix becomes $Q_o = \mathbf{0}$. The rank of the matrix is 0, which is not full rank. Therefore, the system is unobservable. That means the state feedback law may change the observability property of the given system.

Problem 2

Verify that the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x,$$

is left invertible. Given an output

$$y(t) = \begin{pmatrix} \cos\omega t + \omega \sin\omega t \\ \mathrm{e}^t - \cos\omega t \end{pmatrix}, \quad t \geq 0,$$

which is produced by the given system with an initial condition,

$$x(0) = \begin{pmatrix} 0 \\ 1 \\ \omega^2 \end{pmatrix},$$

determine the corresponding control input, $u(t)$, which generates the above output, $y(t)$. Also, show that such a control input is unique.

Solution:

The transfer function G of the system is

$$G(s) = C(sI - A)^{-1} B + D = \begin{bmatrix} \frac{s-1}{s^2(s-2)} \\ \frac{1}{s^2(s-2)} \end{bmatrix} \tag{3}$$

Let $L(s) = \begin{bmatrix} s^2 & -s^2 \end{bmatrix}$. We can get that $L(s) \cdot G(s) = 1$. Therefore, the system is left invertible.

Substituting $y(t)$ into system equation obtains

$$y(t) = \begin{bmatrix} \cos\omega t + \omega \sin\omega t \\ e^t - \cos\omega t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^t - \cos\omega t \\ \cos\omega t + \omega \sin\omega t \end{bmatrix} \tag{4}$$

Substituting it back to the state-space form of system yields that

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1 + x_2 + x_3 + u \end{cases} \tag{5}$$

Therefore,

$$x_3 = \dot{x}_2 = -\omega \sin\omega t + \omega^2 \cos\omega t \tag{6}$$

$$u = \dot{x}_3 - x_1 - x_2 - x_3 = -2\omega^2 \cos\omega t - \omega^3 \sin\omega t - e^t \tag{7}$$

From the derivation above, it is obvious that the control input is unique.

Problem 3

Verify that the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u, \quad y = [0 \quad 1 \quad 0]\, x,$$

is right invertible. Find an initial condition, $x(0)$, and a control input, $u(t)$, which together produce an output

$$y(t) = \alpha \cos \omega t, \quad t \geq 0.$$

Show that the solutions are nonunique.

Solution:

The transfer function G of the system is

$$G(s) = C(sI - A)^{-1} B + D = \begin{bmatrix} \frac{s-1}{s^2(s-2)} & \frac{1}{s^2(s-2)} \end{bmatrix} \tag{8}$$

Let $R(s) = \begin{bmatrix} s^2 \\ -s^2 \end{bmatrix}$. We can get that $G(s) \cdot R(s) = 1$. Therefore, the system is right invertible.

Substituting $y(t)$ into system equation obtains

$$y(t) = \alpha \cos \omega t = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longrightarrow x_2 = \alpha \cos \omega t \tag{9}$$

Substituting it back to the state-space form of system yields that

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + u_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1 + x_2 + x_3 + u_1 \end{cases} \tag{10}$$

Therefore,

$$x_3 = \dot{x}_2 = -\alpha\omega \sin \omega t \tag{11}$$

$$\begin{cases} \dot{x}_1 = x_1 + \alpha \cos \omega t + u_2 \\ -\alpha\omega^2 \cos \omega t = x_1 + \alpha \cos \omega t - \alpha\omega \sin \omega t + u_1 \end{cases} \tag{12}$$

Because there are three unknowns (x_1, u_1, and u_2) and two equations, the solutions are non-unique.

Problem 4

Given an unforced system

$$\dot{x} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} x, \quad y = [\alpha \quad \star \quad \cdots \quad \star] x,$$

where $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, show that the system is observable if and only if $\alpha \neq 0$.

Solution:

The eigenvalues of matrix A are λ.

There is a theorem stating that:

The given system :E of (3.1. 1) is observable if and only if the following statement is true: For every eigenvalue of A, λ_i, $i = 1, 2, ..., n$,

$$\text{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n \tag{13}$$

In this equation,

$$\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \\ \alpha & * & \cdots & * \end{bmatrix} \tag{14}$$

Therefore, when $\alpha = 0$, the rank of matrix $\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix}$ is $n-1$, and when $\alpha \neq 0$, the rank of this matrix is n, which means the system is observable if and only if $\alpha \neq 0$.

Problem 5

Given an unsensed system characterized by a matrix pair in the CSD form

$$\dot{x} = Ax + Bu, \quad \text{with } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let the output equation be $y = Cx$. Verify that the resulting system has

(a) No invariant zero if $C = [\,1 \quad 0 \quad 0\,]$;

(b) One invariant zero if $C = [\,0 \quad 1 \quad 0\,]$; and

(c) Two invariant zero if $C = [\,0 \quad 0 \quad 1\,]$.

Solution:

(a) The transfer function G of the system is

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{s^3 - s^2 - s - 2} \tag{15}$$

Therefore, the resulting systems has no invaritant zero.

(b) The transfer function G of the system is

$$G(s) = C(sI - A)^{-1}B + D = \frac{s}{s^3 - s^2 - s - 2} \tag{16}$$

Therefore, the resulting systems has one invaritant zero.

(c) The transfer function G of the system is

$$G(s) = C(sI - A)^{-1}B + D = \frac{s^2}{s^3 - s^2 - s - 2} \tag{17}$$

Therefore, the resulting systems has two invaritant zeros.

Problem 6

Given the matrix pair (A, B) as that in Q.5, determine an appropriate state feedback gain matrix F such that $A+BF$ has its eigenvalues at $-1, -1 \pm j$, respectively. Show that such an F is unique.

Show by an example that solutions to the pole placement problem for a multiple input system is non-unique. **Hint:** put the pair in the CSD form.

Solution:

$$(s+1)(s+1+j)(s+1-j) = (s+1)\left(s^2+2s+2\right) = s^3+3s^2+4s+2 \tag{18}$$

Let $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$. Therefore,

$$A+BF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2+f_1 & 1+f_2 & 1+f_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \Longrightarrow F = \begin{bmatrix} -4 & -5 & -4 \end{bmatrix} \tag{19}$$

which is unique.

Suppose there are 2 inputs and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{20}$$

and $A+BF$ has its eigenvalues at $-1, -1 \pm j$. Let

$$F = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \end{bmatrix} \tag{21}$$

Then

$$A+BF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2+f_1+f_4 & 1+f_2+f_5 & 1+f_3+f_6 \end{bmatrix} \tag{22}$$

which gives

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2+f_1+f_4 & 1+f_2+f_5 & 1+f_3+f_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \tag{23}$$

We can get following equation:

$$\begin{cases} 2+f_1+f_4 = -2 \\ 1+f_2+f_5 = -4 \\ 1+f_3+f_6 = -3 \end{cases} \tag{24}$$

Because there are three unknown variables (f_1, f_2, f_3, f_4, f_5, and f_6) and three equations, the solutions are non-unique.