THE CHINESE UNIVERSITY OF HONG KONG
DEPT OF MECHANICAL & AUTOMATION ENG

ENGG5403 Linear System Theory & Design

Assignment #4

by

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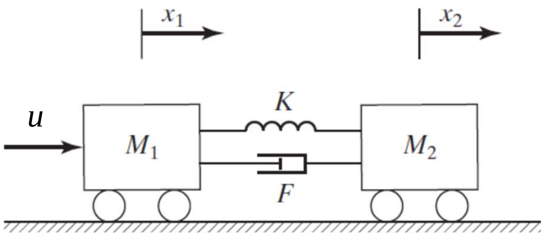
Problem 1

Consider a two-cart system as depicted in the figure below



The carts, assumed to have masses M_1 and M_2, respectively, are connected by a spring and a damper. A force $u(t)$ is applied to Cart M_1 and the position of Cart M_2 can be observed, i.e., $y = x_2$. For simplicity, we assume $M_1 = 1$, $M_2 = 1$, $F = 1$ and $K = 1$.

It was derived in Part 1 of this course that the given plant can be characterized as a linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

where the system data are given as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = 0.$$

1. Design an observer-based controller to stabilize the system and to maintain the position of Cart M_1, i.e.,

$$z = x_1 = C_2 x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

at 2 m, i.e., $r = 2$. Place all the closed-loop poles at -1, i.e., select a state feedback gain matrix F and an observer gain matrix K such that the eigenvalues of $A+BF$ and $A+KC$ are all at -1. Express the corresponding observer-based controller in the usual state-space form. Simulate the overall closed-loop system state responses and the estimation errors of the state variables (set the initial condition of the controller to 0).

2. Comment if it is possible to control the system using PID and/or lead/lag compensators. Why or why not?

Solution:

1. Assume that all the state variables are available for feedback, we design a state feedback

control law

$$u = Fx = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{1}$$

such that $A + BF$ has eigenvalues at -1, i.e., the desired characteristic polynomial $s^4 + 4s^3 + 6s^2 + 4s + 1$ and

$$\begin{aligned} |sI - A - BF| &= \begin{vmatrix} s & -1 & 0 & 0 \\ 1 - f_1 & s + 1 - f_2 & -1 - f_3 & -1 - f_4 \\ 0 & 0 & s & -1 \\ -1 & -1 & 1 & s + 1 \end{vmatrix} \\ &= 2s^2 - f_3 - f_1 s - f_2 s - f_3 s - f_4 s - f_1 s^2 - f_2 s^2 \\ &\quad - f_2 s^3 - f_4 s^2 - f_1 + 2s^3 + s^4 \\ &= s^4 + (-f_2 + 2)\, s^3 + (-f_1 - f_2 - f_4 + 2)\, s^2 \\ &\quad + (-f_1 - f_2 - f_3 - f_4)\, s + (-f_1 - f_3) \end{aligned} \tag{2}$$

Hence,

$$\begin{cases} -f_2 + 2 = 4 \\ -f_1 - f_2 - f_4 + 2 = 6 \\ -f_1 - f_2 - f_3 - f_4 = 4 \\ -f_1 - f_3 = 1 \end{cases} \tag{3}$$

Solving the equation above yields that

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 & -1 \end{bmatrix} \tag{4}$$

Then, design an observer,

$$\dot{\hat{x}} = A\hat{x} + Bu - K\,(y - C\hat{x}) = A\hat{x} + BBu - \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} (y - C\hat{x}) \tag{5}$$

such that $A + KC$ has eigenvalues at -1, i.e., the desired characteristic polynomial $s^4 +$

$4s^3 + 6s^2 + 4s + 1$ and

$$|sI - A - KC| = \begin{vmatrix} s & -1 & -k_1 & 0 \\ 1 & s+1 & -1-k_2 & -1 \\ 0 & 0 & s-k_3 & -1 \\ -1 & -1 & 1-k_4 & s+1 \end{vmatrix}$$
$$= 2s^2 - k_4 - k_1 s - k_2 s - k_3 s - k_4 s - 2k_3 s^2 - k_3 s^3 - k_4 s^2 - k_2 + 2s^3 + s^4$$
$$= s^4 + (-k_3 + 2)\, s^3 + (-2k_3 - k_4 + 2)\, s^2 + (-k_1 - k_2 - k_3 - k_4)\, s + (-k_2 - k_4) \tag{6}$$

Hence,

$$\begin{cases} -k_3 + 2 = 4 \\ -2k_3 - k_4 + 2 = 6 \\ -k_1 - k_2 - k_3 - k_4 = 4 \\ -k_2 - k_4 = 1 \end{cases} \tag{7}$$

Solving the equation above yields that

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \tag{8}$$

And the control gain to make the DC gain from r to z unity is equal to

$$G = \left[C_2 (A + BF)^{-1} B\right]^{-1} = -1 \tag{9}$$

The measurement feedback controller is given as

$$\begin{cases} \dot{\hat{x}} = (A + BF + KC + KDF)\,\hat{x} - Ky - BGr = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \\ u = F\hat{x} - Gr = \begin{bmatrix} -1 & -2 & 0 & -1 \end{bmatrix} \hat{x} + 2 \end{cases} \tag{10}$$

The block diagram shown in Figure 1 is used to get the overall closed-loop system state responses and the estimation errors of the state variables.

And the MATLAB code to plot the response is shown below:

```
clc; clf; clear all; close all;
% Loop through all 4 files

figx1 = openfig("Q4_x1.fig",'reuse');
```

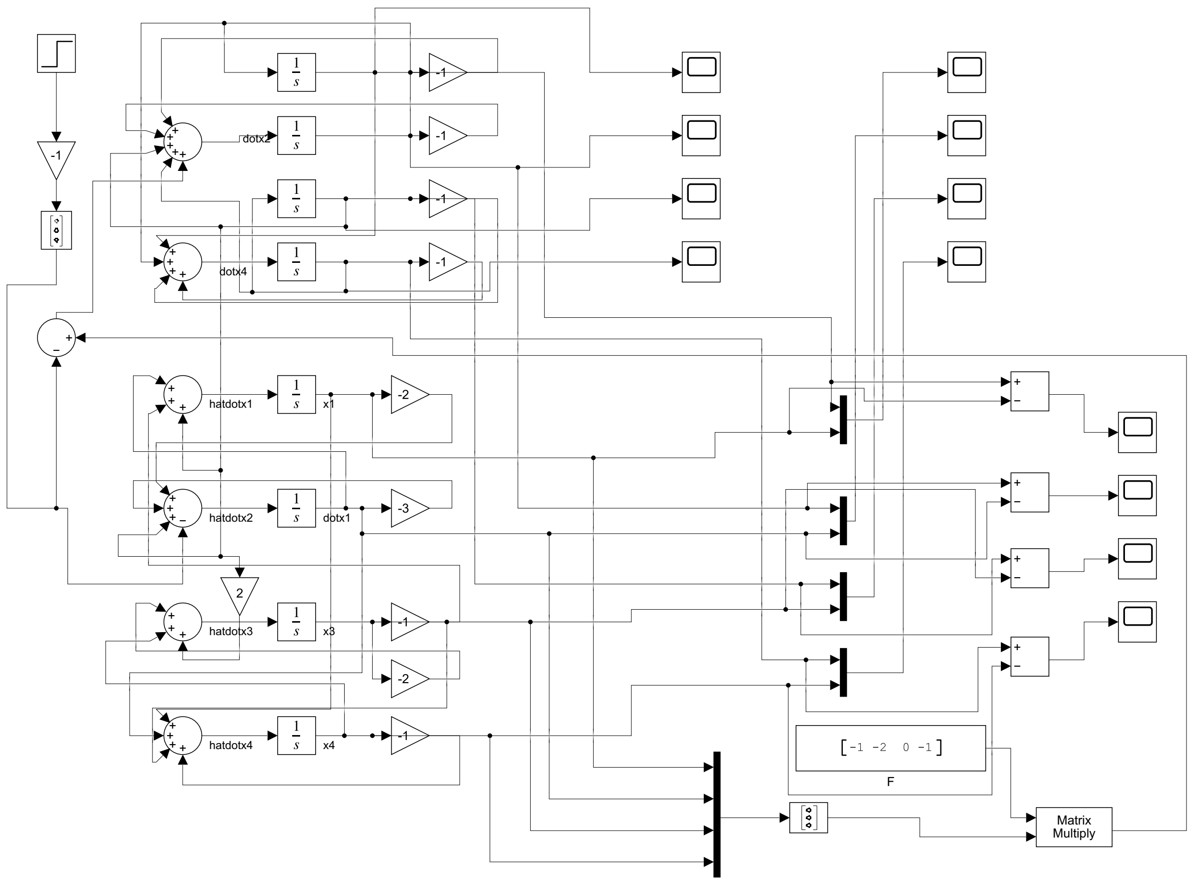


Figure 1: Block diagram for the simulation.

```
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_1$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_x1"+".pdf";
saveas(gcf,filename);
close(figx1);

figx2 = openfig("Q4_x2.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_2$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_x2"+".pdf";
saveas(gcf,filename);
close(figx2);

figx3 = openfig("Q4_x3.fig",'reuse');
```

```
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_3$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_x3"+".pdf";
saveas(gcf,filename);
close(figx3);

figx4 = openfig("Q4_x4.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_4$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_x4"+".pdf";
saveas(gcf,filename);
close(figx4);

fige1 = openfig("Q4_e1.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_1$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_e1"+".pdf";
saveas(gcf,filename);
close(fige1);

fige2 = openfig("Q4_e2.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_2$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_e2"+".pdf";
saveas(gcf,filename);
close(fige2);

fige3 = openfig("Q4_e3.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
```

```
ylabel('$e_3$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_e3"+".pdf";
saveas(gcf,filename);
close(fige3);

fige4 = openfig("Q4_e4.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_4$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q4_e4"+".pdf";
saveas(gcf,filename);
close(fige4);
```

The simulation of the overall closed-loop system state responses is shown in Figure 2. And the simulation of the estimation errors of the state variables for the overall closed-loop system is shown in Figure 3.

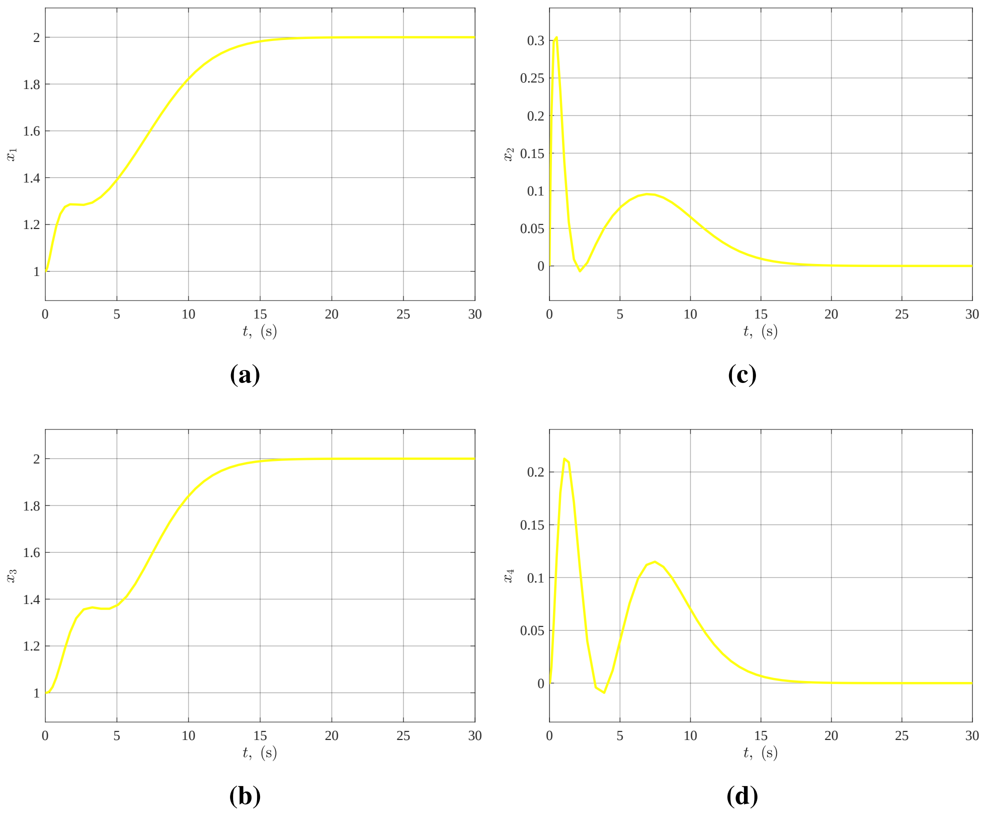


Figure 2: Simulation of the overall closed-loop system state responses.

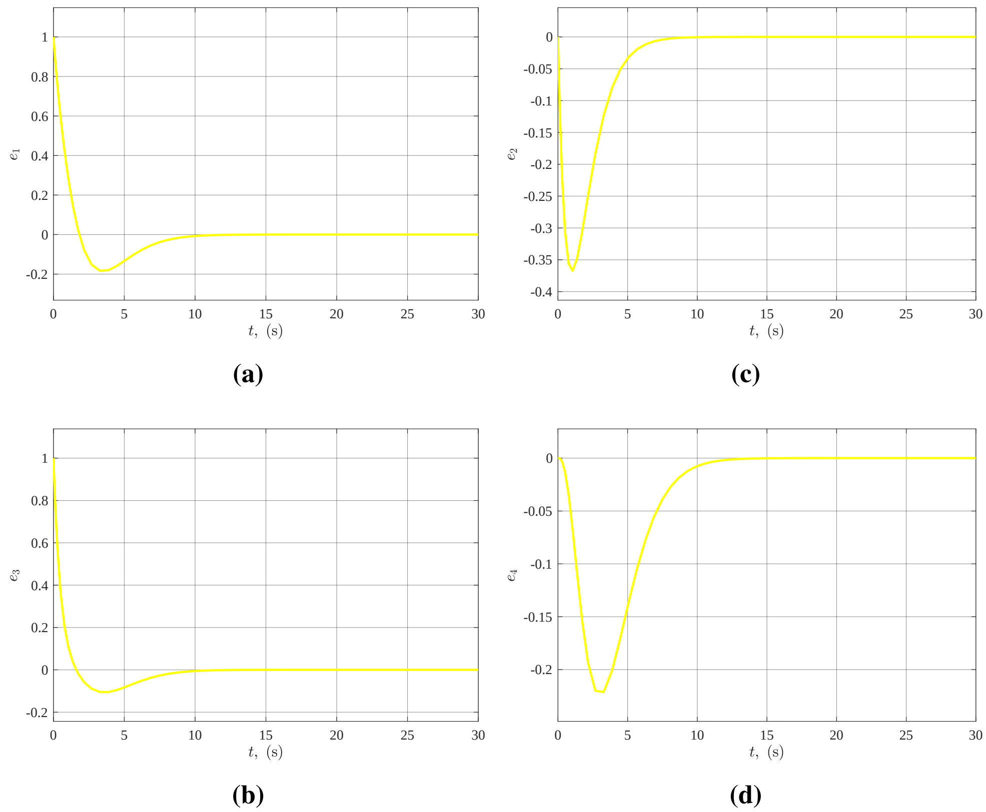


Figure 3: Simulation errors of the state variables forthe overall closed-loop system.

2. First, the system is marginal stable because the eigenvalues of matrix are $-1 \pm j$ and 0. It is almost impossible and impracticable to control a marginal stable system using both PID and lead/lag compensators, but the effectiveness of these control methods will depend on the specific characteristics of the system.

   PID control is a widely used method for controlling a variety of systems, including marginal stable ones. PID controllers use feedback to adjust the system's input based on the difference between the desired output and the actual output. The proportional, integral, and derivative components of the controller can be tuned to achieve stable and responsive control.

   Lead/lag compensators are another method of control that can be used to stabilize a marginal stable system. Lead compensators introduce additional phase lead to the system, which can improve stability and increase the system's response time. Lag compensators, on the other hand, introduce phase lag to the system, which can improve stability and reduce overshoot.

   However, the effectiveness of these control methods will depend on the specific characteristics of the system, such as its dynamics, transfer function, and time constants. It is important to carefully tune the control parameters and compensator gains to achieve the desired performance and stability. Additionally, in some cases, more advanced control techniques such as model predictive control or adaptive control may be necessary to achieve optimal control of a marginal stable system.