The Chinese University of Hong Kong
Dept of Mechanical & Automation Eng

ENGG5403 Linear System Theory & Design

Assignment #5

by

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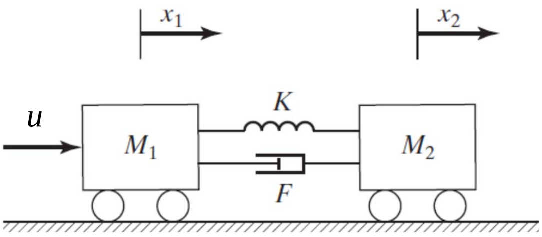
Problem 1

Reconsider the two-cart system as in Homework Assignment 4, i.e.,



The carts, assumed to have masses M_1 and M_2, respectively, are connected by a spring and a damper. A force $u(t)$ is applied to Cart M_1 and we wish to control the displacement of Cart M_1, i.e., $z = x_1$. For simplicity, we assume $M_1=1$, $M_2=1$, $F=1$ and $K=1$.

1. Assume all state variables of the plant are available for feedback. Find an LQR control law, which minimizes the following performance index:

$$J = \int_0^\infty \left(x^\mathrm{T} Q x + u^\mathrm{T} R u \right) dt, \quad Q = I_4, \ R = 1$$

What are the resulting gain and phase margins of your LQR design?

2. Assume that there is an input noise (or disturbance) entering the system as:

$$\dot{x} = Ax + Bu + Bv(t),$$

and the system measurement output is

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + w(t)$$

where $w(t)$ is the measurement noise. Assume both $v(t)$ and $w(t)$ have zero means and

$$E[v(t)v^\mathrm{T}(t)] = Q_v \delta(t-\tau), \ Q_v = 1, \quad E[w(t)w^\mathrm{T}(t)] = R_e \delta(t-\tau), \ R_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Design an appropriate Kalman filter.

3. Derive the corresponding LQG control law, which combines the LQR law in Step 1 and the Kalman filter in Step 2. What are the closed-loop eigenvalues? What are the resulting gain and phase margins of your LQG control law? Simulate your design using SIMULINK with

$$r = 1, \quad x_1(0) = x_2(0) = 1, \ \dot{x}_1(0) = \dot{x}_2(0) = 0,$$

and the initial condition for the Kalman filter being 0.

Solution:

1. The given plant can be characterized as a linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

MATLAB is used to calculate the LQR control law, which gives

$$K = \begin{bmatrix} 0.8231 & 1.5068 & 0.5912 & 1.3189 \end{bmatrix} \tag{2}$$

Also, the gain margin and phase margin are calculated via MATLAB, which are equal to 0 and $74.9506°$.

2. From the measurement output, it can be seen that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = 0$$

MATLAB is used to design the Kalman filter, which gives following results:

$$L = \begin{bmatrix} 0.7199 & 0.5557 \\ 0.4135 & 0.2272 \\ 0.5557 & 0.4977 \\ 0.4494 & 0.2782 \end{bmatrix} \tag{3}$$

3. The corresponding LQG control law is

$$\begin{cases} \dot{\hat{x}} = (A - BK - LC)\,\hat{x} + Ly \\ u = -K\hat{x} - Gr - BGr \end{cases} \tag{4}$$

where

$$G = \left[C_2 (A - BK)^{-1} B\right]^{-1} = -1.4142$$

Therefore, the numerical representation of LQG control law is

$$\begin{cases} \dot{\hat{x}} = \begin{bmatrix} -0.7199 & 1.0000 & -0.5557 & 0 \\ -2.2365 & -2.5068 & 0.1816 & -0.3189 \\ -0.5557 & 0 & -0.4977 & 1.0000 \\ 0.5506 & 1.0000 & -1.2782 & -1.0000 \end{bmatrix} \hat{x} + \begin{bmatrix} 0.7199 & 0.5557 \\ 0.4135 & 0.2272 \\ 0.5557 & 0.4977 \\ 0.4494 & 0.2782 \end{bmatrix} y - \begin{bmatrix} 0 \\ -1.4142 \\ 0 \\ 0 \end{bmatrix} \\ u = -\begin{bmatrix} 0.8231 & 1.5068 & 0.5912 & 1.3189 \end{bmatrix} \hat{x} + 1.4142 \end{cases} \tag{5}$$

The closed-loop system is characterized by the following state space equation.

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} - \begin{bmatrix} BG \\ 0 \end{bmatrix} r + \tilde{v}, \quad \tilde{v} = \begin{bmatrix} v \\ v - Lw \end{bmatrix} \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + w \end{cases} \tag{6}$$

The closed-loop eigenvalues are

$$\begin{cases} -1.0890 + 0.9414i \\ -1.0890 - 0.9414i \\ -0.6644 + 0.4910i \\ -0.6644 - 0.4910i \\ -1.0334 + 1.0334i \\ -1.0334 - 1.0334i \\ -0.5754 + 0.5754i \\ -0.5754 - 0.5754i \end{cases} \tag{7}$$

Mathematically, the open loop transfer function for an LQG control system can be represented as:

$$G_{ol}(s) = G(s)\,G_c(s) = \left[C(sI-A)^{-1}B\right]\left[K(sI-A+BK+LC)^{-1}L\right] \tag{8}$$

where $G(s)$ is the transfer function of the plant and $G_c(s)$ is the transfer function of the controller.

The resulting gain margin and phase margin are equal to 7.8614 and $29.1886°$ for state x_1 and 2.1608 and $24.0074°$ for state x_2.

The code for all above calculation is shown below:

```
clc; clf; clear all;
A = [
    0 1 0 0;
    -1 -1 1 1;
    0 0 0 1;
    1 1 -1 -1;
    ];
B = [
    0;
    1;
    0;
    0;
    ];
% C = [0 0 1 0];
C = [1 0 0 0;
    0 0 1 0;];
```

```
D = 0;
Q = eye(4);
R = 1;
[K,S,CLP] = lqr(A,B,Q,R);
syms s
% G = K*inv(s*eye(4)-A)*B;
[num,den] = ss2tf(A,B,K,D,1);
sys = tf(num,den);
[Gm,Pm] = margin(sys);
Qv = 1;
Re = eye(2);
[L,P,E] = lqe(A,B,C,Qv,Re);
C2 = [1 0 0 0];
G = inv(C2*inv(A+B*K)*B);
Ad = [A-B*K B*K;
    zeros(4) A-L*C;];
eigrA = eig(A-B*K-L*C);
eigrcA = eig(Ad);
eigABK = eig(A-B*K);
eigACL = eig(A-L*C);

[num2,den2] = ss2tf(A,B,C,[0;0],1);
sys21 = tf(num2(1,:),den2(1,:));
sys22 = tf(num2(2,:),den2(1,:));
[num3,den3] = ss2tf(A-B*K-L*C,L,K,[0 0],1);
sys3 = tf(num3,den3);
[Gm2,Pm2] = margin(sys21*sys3);
[Gm3,Pm3] = margin(sys22*sys3);
```

The block diagram shown in Figure 1 is used to get the overall closed-loop system state responses and the estimation errors of the state variables.

And the MATLAB code to plot the response is shown below:

```
clc; clf; clear all; close all;
% Loop through all 4 files

figx1 = openfig("Q5_x1.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_1$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_x1"+".pdf";
saveas(gcf,filename);
close(figx1);
```

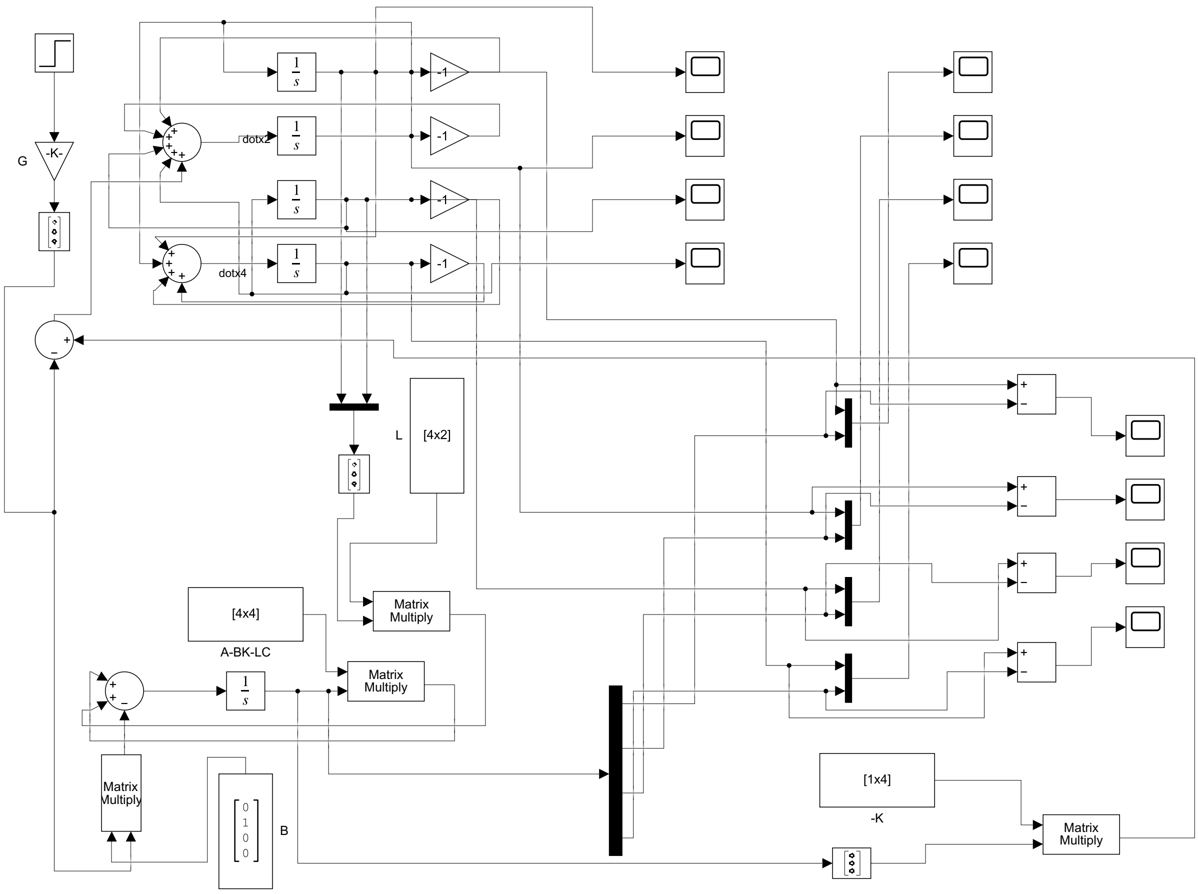


Figure 1: Block diagram for the simulation.

```

figx2 = openfig("Q5_x2.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_2$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_x2"+".pdf";
saveas(gcf,filename);
close(figx2);

figx3 = openfig("Q5_x3.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_3$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_x3"+".pdf";
saveas(gcf,filename);
close(figx3);
```

```

figx4 = openfig("Q5_x4.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_4$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_x4"+".pdf";
saveas(gcf,filename);
close(figx4);

fige1 = openfig("Q5_e1.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_1$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_e1"+".pdf";
saveas(gcf,filename);
close(fige1);

fige2 = openfig("Q5_e2.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_2$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_e2"+".pdf";
saveas(gcf,filename);
close(fige2);

fige3 = openfig("Q5_e3.fig",'reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_3$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_e3"+".pdf";
saveas(gcf,filename);
close(fige3);

fige4 = openfig("Q5_e4.fig",'reuse');
```

```
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$e_4$', 'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q5_e4"+".pdf";
saveas(gcf,filename);
close(fige4);
```

The simulation of the overall closed-loop system state responses is shown in Figure 2. And the simulation of the estimation errors of the state variables for the overall closed-loop system is shown in Figure 3.

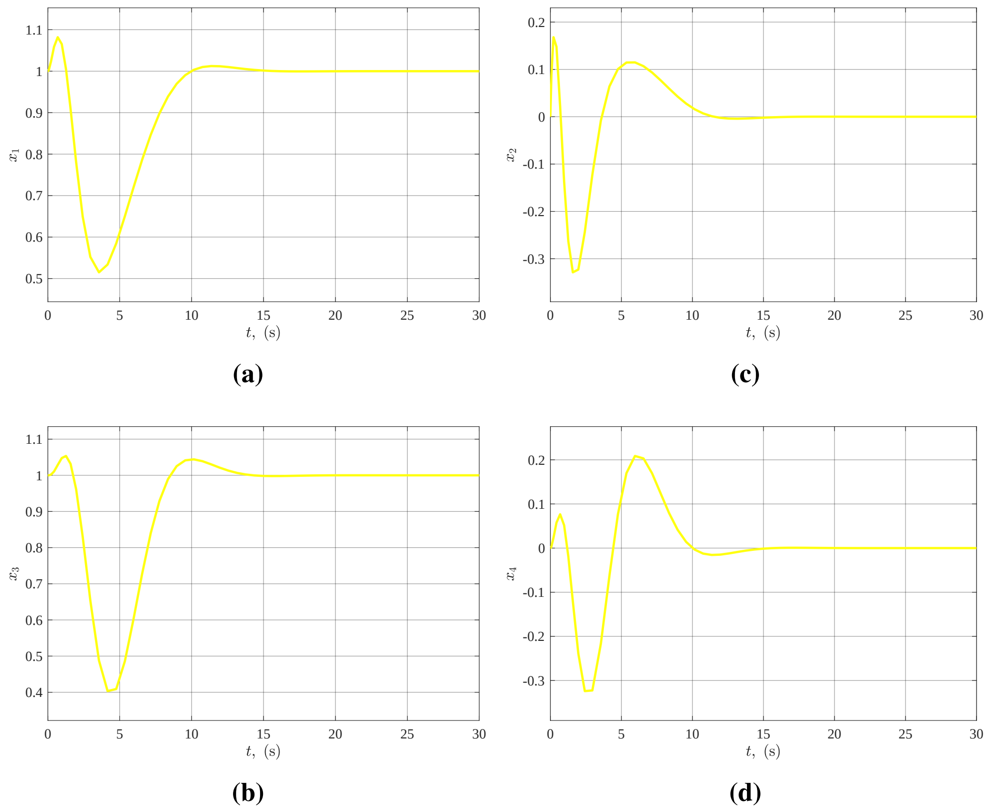
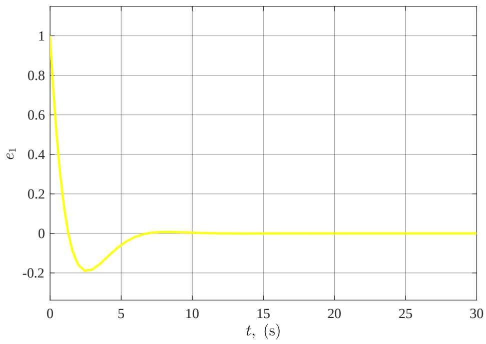
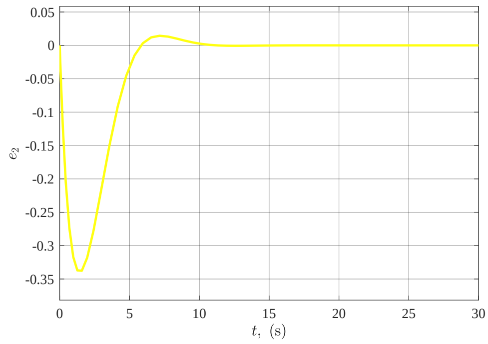


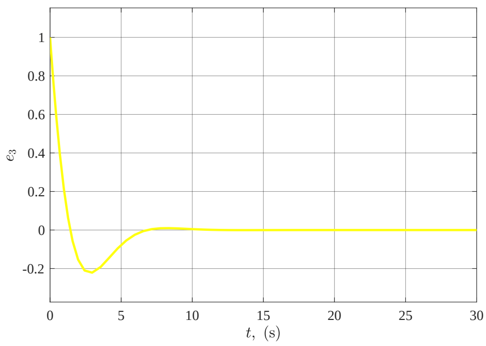
Figure 2: Simulation of the overall closed-loop system state responses.



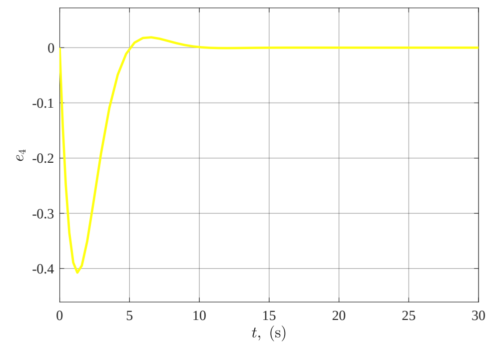
(a)



(c)



(b)



(d)

Figure 3: Simulation errors of the state variables for the overall closed-loop system.