THE CHINESE UNIVERSITY OF HONG KONG
DEPT OF MECHANICAL & AUTOMATION ENG

ENGG5403 Linear System Theory & Design

Assignment #6

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

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Problem 1

Write the system to be controlled in Homework Assignment 5 in the following form

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,\tilde{w} \\ y = C_1\,x + D_1\,\tilde{w} \\ z = C_2\,x + D_2\,u \end{cases}$$

with

$$\tilde{w} = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix},$$ the combination of the input and measurement noises.

1. Determine the best achievable H_∞-norm of the closed-loop system from $\tilde{w}$ to z?
2. Design an H_∞ suboptimal control law such that the H_∞-norm of the resulting closed-loop system is reasonably close to the optimal value.
3. Plot the singular value of the closed-loop system and find its H_∞-norm.
4. Find the resulting gain and phase margins of the system under the control law.
5. Assume that there is an unstructured but stable perturbation, Δ, presented in the given plant. Give the range of $\|\Delta\|_\infty$ so that the closed-loop would remain stable.

Solution:

Writing the system to be controlled in Homework Assignment 5 yields that

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} \\ z = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases} \tag{1}$$

1. Try gm8s_sc for many times, we find $\gamma_\infty^* = 0.75259$ is the best achievable H_∞ norm of the closed-loop system from $\tilde{w}$ to z.

2. When designing the H_∞ optimal controller, we use $\gamma = 0.805$. Choose $\epsilon = 0.01$ and solve h8care function in MATLAB to calculate P and Q in following equation

$$A^T P + PA + \tilde{C}_2^T \tilde{C}_2 + \gamma^{-2} P \tilde{E} \tilde{E}^T P - \left(PB + \tilde{C}_2^T \tilde{D}_2\right) \left(\tilde{D}_2^T \tilde{D}_2\right)^{-1} \left(\tilde{D}_2^T \tilde{C}_2 + B^T P\right) = 0 \tag{2}$$

and

$$QA^T + AQ + \tilde{E}\tilde{E}^T + \gamma^{-2}Q\tilde{C}_2^T\tilde{C}_2Q - \left(Q\tilde{C}_1^T + \tilde{E}\tilde{D}_1^T\right)\left(\tilde{D}_1\tilde{D}_1^T\right)^{-1}\left(\tilde{D}_1\tilde{E}^T + \tilde{C}_1Q\right) = 0 \quad (3)$$

The results of P and Q are

$$P = \begin{bmatrix} 0.1420 & 0.0099 & 0.0001 & -0.0001 \\ 0.0099 & 0.0013 & 0.0001 & 0.0001 \\ 0.0001 & 0.0001 & 0.0002 & 0.0001 \\ -0.0001 & 0.0001 & 0.0001 & 0.0002 \end{bmatrix} \quad (4)$$

and

$$Q = \begin{bmatrix} 3.4975 & 1.4166 & 3.0796 & 1.6504 \\ 1.4166 & 0.9522 & 1.1241 & 0.8309 \\ 3.0796 & 1.1241 & 2.8024 & 1.3486 \\ 1.6504 & 0.8309 & 1.3486 & 0.9637 \end{bmatrix} \quad (5)$$

F and K can be gotten from following equations

$$F = -\left(\tilde{D}_2^T\tilde{D}_2\right)^{-1}\left(\tilde{D}_2^T\tilde{C}_2 + B^TP\right) = \begin{bmatrix} -69.7284 & -10.9821 & -1.0077 & -1.0377 \end{bmatrix} \quad (6)$$

and

$$K = -\left(Q\tilde{C}_1^T + \tilde{E}\tilde{D}_1^T\right)\left(\tilde{D}_1\tilde{D}_1^T\right)^{-1} = \begin{bmatrix} -3.4999 & -3.0815 \\ -1.4172 & -1.1246 \\ -3.0815 & -2.8040 \\ -1.6513 & -1.3494 \end{bmatrix} \quad (7)$$

Also, the eigenvalues of the closed-loop system are verified

$$\lambda = \begin{cases} -369.08 \\ -5.99 + 5.91i \\ -5.99 - 5.91i \\ -1.03 + 1.02i \\ -1.03 - 1.02i \\ -0.50 + 0.87i \\ -0.50 - 0.87i \\ -0.47 \end{cases} \quad (8)$$

which are all in the left-half plane. Therefore, the H_∞-suboptimal output feedback law is then given by

$$\begin{cases} \dot{x}_{cmp} = \begin{bmatrix} -208.7362 & 1.0000 & -183.1922 & -0.0000 \\ -157.0601 & -11.9787 & -75.6699 & -0.0374 \\ -183.1922 & 0.0000 & -160.8651 & 1.0000 \\ -98.2744 & 1.0000 & -88.0212 & -1.0000 \end{bmatrix} x_{cmp} + \begin{bmatrix} 208.7362 & 183.1922 \\ 86.3534 & 75.6626 \\ 183.1922 & 160.8651 \\ 99.2744 & 87.0212 \end{bmatrix} y \\ u = \begin{bmatrix} -69.7282 & -10.9820 & -1.0077 & -1.0377 \end{bmatrix} x_{cmp} \end{cases} \quad (9)$$

3. Using Matlab, the singular value of the closed-loop system can be plotted as shown in Figure 1.

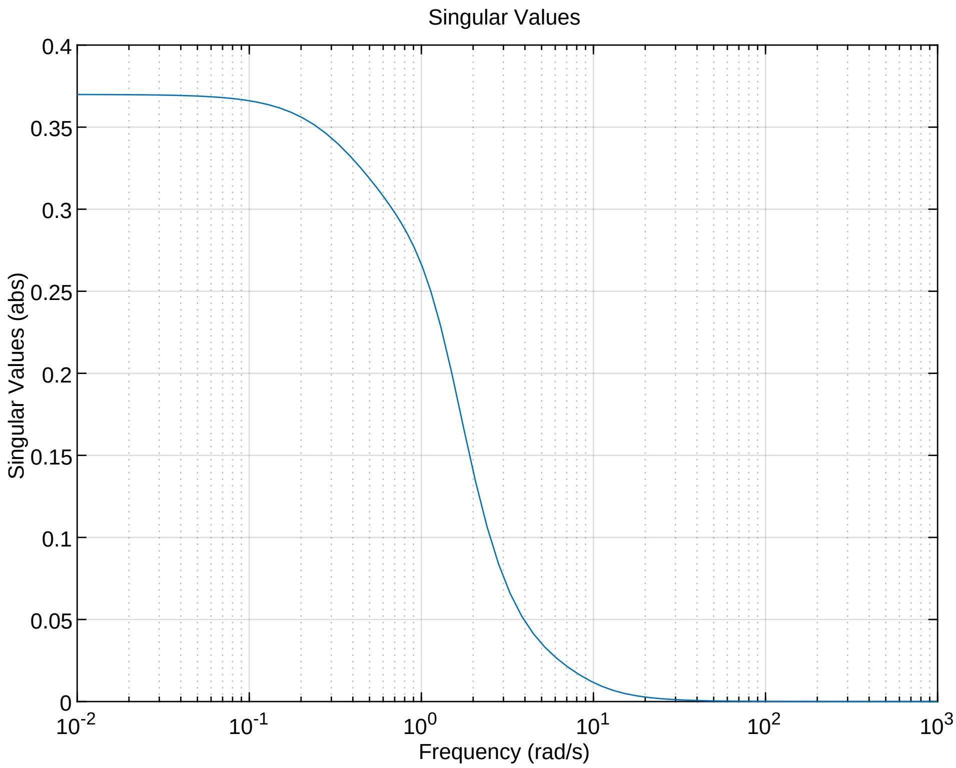

Figure 1: singular value of the closed-loop system.

And the H_∞ norm of the system is equal to 0.7264.

4. The resulting gain and phase margins of the system under the control law are all equal to ∞.

5. The range of the unstructured but stable perturbation is

$$\|\Delta\|_\infty < \frac{1}{\gamma} = 1.3766 \tag{10}$$

The codes for this question are listed below:

```
%% Q6-1
clc; clf; clear all; close all;
A = [
    0 1 0 0;
    -1 -1 1 1;
    0 0 0 1;
    1 1 -1 -1;
    ];
B = [
    0;
```

```
    1;
    0;
    0;
    ];
C1 = [1 0 0 0;
    0 0 1 0;];
C2 = [1 0 0 0];
D1 = [0 1 0;
    0 0 1;];
D2 = 0;
E = [0 0 0;
     1 0 0;
     0 0 0;
     0 0 0;];
epsilon = 0.01;
% C2 = [C2; epsilon*eye(size(C2,2)); zeros(size(D2,2),size(C2,2));];
% D2 = [D2; zeros(size(C2,2),size(D2,2)); epsilon*eye(size(D2,2));];
% E = [E epsilon*eye(size(E,1)) zeros(size(E,1),size(D1,1))];
% gms8 = gm8star(A,B,C2,D2,E);
gamma = 0.805;
% gm8s_sc(A,B,E,C1,D1,C2,D2,gamma);
C2 = [C2; epsilon*eye(size(C2,2)); zeros(size(D2,2),size(C2,2));];
D2 = [D2; zeros(size(C2,2),size(D2,2)); epsilon*eye(size(D2,2));];
E = [E epsilon*eye(size(E,1)) zeros(size(E,1),size(D1,1))];
D1 = [D1 zeros(size(D1,1),size(E,1)) epsilon*eye(size(D1,1))];
% gms8 = gm8star(A,B,C2,D2,E);
P = h8care(A,B,C2,D2,E,gamma);
Q = h8care(A',C1',E',D1',C2',gamma);
% % F = -((D2'*D2)^-1)*(D2'*C2+B'*P);
% % K = -(Q*C1'+E*D1')*((D1*D1')^-1);
% % epsilon = 0;
[F,K,Acmp,Bcmp,Ccmp,Dcmp,EigCL] = h8out(A,B,E,C1,D1,C2,D2,gamma,epsilon);
% % EigCL
% % gm8 = sqrt(max(eig(P*Q)))
C1 = [1 0 0 0;
    0 0 1 0;];
C2 = [1 0 0 0];
D1 = [0 1 0;
    0 0 1;];
D2 = 0;
E = [0 0 0;
     1 0 0;
     0 0 0;
     0 0 0;];
Acl = [A+B*Dcmp*C1 B*Ccmp; Bcmp*C1 Acmp];
Bcl = [E+B*Dcmp*D1; Bcmp*D1];
```

```
Ccl = [C2+D2*Dcmp*C1 D2*Ccmp];
Dcl = D2*Dcmp*D1;
[num,den] = ss2tf(Acl,Bcl,Ccl,Dcl,1);
sys = tf(num,den);
p= sigmaoptions;
p. MagUnits='abs';
fig1 = figure(1);
sigma(sys,p);
grid on;
% a = get(gca,'XTickLabel');
% set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q6_SV"+".pdf";
saveas(gcf,filename);
close(fig1);
[Gm,Pm,Wcg,Wcp] = margin(sys);
```

Problem 2

Consider a linear time-invariant system characterized by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew \\ z = C_2 x + D_2 u \end{cases} \tag{11}$$

where $C_2 = 0_{m\times n}$, $D_2 = I_m$, and where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$, are the state, control input, disturbance input and controlled output, respectively. Assume that the state variable x is available for feedback, i.e, the measurement output $y = x$, and assume that (A, B) is stabilizable and (A, B, C_2, D_2) has no invariant zeros on the imaginary axis.

(a) Show that the subsystem (A, B, C_2, D_2) has a total of n invariant zeros and are given by $\lambda(A)$, i.e., the eigenvalues of A.

(b) Show that there exist an $n \times n$ nonsingular transformation T such that

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tag{12}$$

where A_- and A_+ are stable and unstable matrices, respectively.

(c) Let us define a state transformation $x = T\tilde{x}$, where T as given in Part (b). It is easy to verify that the given system Σ can be transformed into the following:

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tilde{x} + \begin{bmatrix} B_- \\ B_+ \end{bmatrix} u + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w \\ z = \begin{bmatrix} 0 & 0 \end{bmatrix} \tilde{x} + Iu \end{cases} \tag{13}$$

where B_-, B_+, E_-, and E_+ are respectively appropriate constant matrices. Show that (A, B) is stabilizable if and only if (A_+, B_+) is controllable.

(d) Show that the solution to the corresponding H_2 Riccati equation for the transformed system in Part (c), if existent, can be partitioned as follows

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_+ \end{bmatrix}, P_+ > 0 \tag{14}$$

Find the H_2 optimal state feedback control law $u = F\tilde{x}$ for the transformed system in terms of P_+. Show that the resulting closed-loop system has poles at $\lambda(A_-)$ and $\lambda(-A_+)$.

(e) Show that $\gamma_2^* = 0$, i.e., the disturbance can be totally rejected from the controlled output, if and only if $E_+ = 0$, i.e., the disturbance is not allowed to enter the unstable invariant zero subspace.

Solution:

(a) or a linear time-invariant system, the invariant zeros are defined as the values of s for which the system has no response to the input. We can find the invariant zeros by computing the transfer function from u to z and looking for the zeros of the determinant of the system matrix. In our case, the transfer function is given by:

$$G(s) = C_2(sI - A)^{-1}B + D_2 = I_m. \tag{15}$$

Since C_2 is a zero matrix and D_2 is the identity matrix, the transfer function is always equal to I_m and does not depend on the system's poles or zeros. However, we can still find the invariant zeros by computing the transfer function from w to z. In this case, the transfer function is given by:

$$H(s) = C_2(sI - A)^{-1}E. \tag{16}$$

To find the invariant zeros, we want to find the values of s for which the determinant of $(sI - A)$ is zero:

$$\det(sI - A) = 0. \tag{17}$$

This equation gives us the eigenvalues of A, which are the invariant zeros of the subsystem (A, B, C_2, D_2). Thus, the subsystem has a total of n invariant zeros, given by the eigenvalues of A.

(b) Since (A, B) is stabilizable, there exists a nonsingular transformation T such that the transformed A matrix, $\tilde{A}$, is in the block upper-triangular form with stable and unstable parts A_- and A_+, respectively. This transformation is obtained through modal decomposition. To show this, we first find the eigenvectors of A corresponding to its eigenvalues. Let v_i be the eigenvector corresponding to the eigenvalue λ_i of A:

$$Av_i = \lambda_i v_i. \tag{18}$$

We can then construct the transformation matrix T by placing the eigenvectors as its columns:

$$T = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}. \tag{19}$$

Applying this transformation, we obtain the block upper-triangular form for $\tilde{A}$:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}, \tag{20}$$

where A_- and A_+ are stable and unstable matrices, respectively.

(c) We have already defined the state transformation $x = T\tilde{x}$. Let's rewrite the system Σ in terms of $\tilde{x}$:

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu + Ew) = T^{-1}A(T\tilde{x}) + T^{-1}Bu + T^{-1}Ew. \tag{21}$$

Using the definition of $\tilde{A}$ and noting that $T^{-1}A = \tilde{A}T^{-1}$, we can write

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + T^{-1}Bu + T^{-1}Ew. \tag{22}$$

Now, let's define $\tilde{B} = T^{-1}B$ and $\tilde{E} = T^{-1}E$. Then, we can partition these matrices as:

$$\tilde{B} = \begin{bmatrix} B_- \\ B_+ \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} E_- \\ E_+ \end{bmatrix}. \tag{23}$$

With these definitions, the transformed system takes the form:

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}\tilde{x} + \begin{bmatrix} B_- \\ B_+ \end{bmatrix}u + \begin{bmatrix} E_- \\ E_+ \end{bmatrix}w \\ z = \begin{bmatrix} 0 & 0 \end{bmatrix}\tilde{x} + Iu \end{cases} \tag{24}$$

The pair (A, B) is stabilizable if and only if the unstable part of the system, (A_+, B_+), is controllable. This is because stabilizability implies that we can place the closed-loop poles in the left-half plane, and this can only be achieved if we can control the unstable part of the system.

(d) The H_2 optimal control problem is related to minimizing the H_2 norm of the transfer function from w to z. The optimal control law can be found by solving the H_2 Riccati equation:

$$A^T P + PA - PBB^T P + Q = 0, \tag{25}$$

where $Q = C_2^T C_2$ and $R = D_2^T D_2$. In our case, $C_2 = 0_{m\times n}$ and $D_2 = I_m$, so $Q = 0_{n\times n}$ and $R = I_m$. Now, if the solution to the Riccati equation exists, we can partition the matrix P as:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_+ \end{bmatrix}, \tag{26}$$

with $P_+ > 0$. The H_2 optimal state feedback control law is given by $u = F\tilde{x}$, where $F = -(R + B^T PB)^{-1} B^T PA$. Using the partitioned form of P, the feedback control law becomes:

$$u = -(I + B_+^T P_+ B_+)^{-1} B_+^T P_+ A_+ \tilde{x}_+. \tag{27}$$

The resulting closed-loop system has the state-space representation:

$$\dot{\tilde{x}} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ + B_+ F \end{bmatrix} \tilde{x} + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w. \tag{28}$$

The poles of the closed-loop system are given by the eigenvalues of the matrix in the above equation. Since A_- is stable and the feedback law stabilizes the unstable part $A_+ + B_+ F$, the closed-loop poles are given by $\lambda(A_-)$ and $\lambda(-A_+)$.

(e) The disturbance can be totally rejected from the controlled output, i.e., $\gamma_2^* = 0$, if and only if the transfer function from w to z is identically zero. This can be achieved if and only if $E_+ = 0$, meaning the disturbance is not allowed to enter the unstable invariant zero subspace. If $E_+ = 0$, then the disturbance only affects the stable part of the system, and the feedback control law can completely reject the disturbance from the controlled output.