ENGG 5403

Linear System Theory and Design / Part 1: Theory

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What are we going to learn in this class?

How to describe a physical system by mathematical equations? (Dynamic modeling) ①

How does it behave? (System responses) ②

Is the system stable? (Stability issues) ③

Can the system be controlled? (Controllability & observability) ④

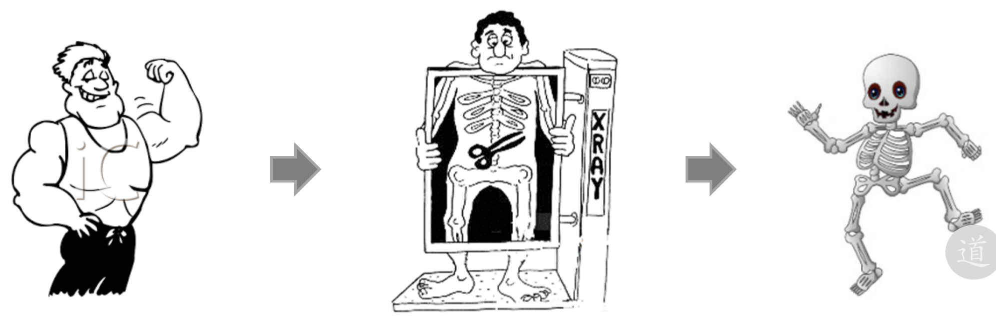
Is it a good system? (System structural properties) ⑤

How to control the physical system? (Control system design) ⑥

Questions 1 to 5 will be answered in Part 1, and the last one in Part 2…

My philosophy towards control systems design……

- Break the system to be controlled into essential pieces and examine their inherent properties.



- For a lousy system, it is better to re-design the system itself (instead of employing advanced techniques to control it, even it is possible).
- Do not push to the physical limits of the system (most of the so-called optimal performance measures do not mean anything in practice).
- Choose the simplest possible control law (if it is not for publication)

My philosophy towards control systems design……

your teaching style?

CSM

Ben: I have taught both undergraduate and graduate control classes, including classical feedback control, computer control systems, optimal control, and multivariable control systems. My favorite course is a graduate-level module on multivariable control, in which I need to cover topics ranging from classical techniques, such as LQR control, Kalman filter, LQG, and LTR control methods, to modern control theories, such as H_2 control, robust and H_∞ control, and disturbance decoupling problems. These topics happen to be in line with my research interests. Instead of focusing on mathematical details, I spend considerable time giving students the overall picture and development in the field by highlighting interesting history and milestones behind the theories. My homework assignments are pretty unique too. I challenge my students in the assignments to beat the designs in my monographs. This teaching method forces them to read and learn things beyond the class and textbooks to complete the assignments and familiarizes them with control system design for real and complicated problems. All my teaching materials can be freely accessed from my Web site at http://uav.ece.nus.edu.sg/~bmchen/.

Q. What are some of the most promising opportunities in the control field?

Ben: In my opinion, the area of control applications is full of opportunities, to tackle real and meaningful problems and to attract more research funding. Applications also challenge academic researchers to think more realistically. I personally believe that a good control system design should not start from differential equations but should be down to earth and start from the hardware level, including the selection and placement of sensors and actuators.

Q. You are the author of 11 books in the control field. What topics do these books cover?

Ben: I have authored or coauthored ten monographs and one textbook, of which eight are directly related to control theory and application. My earlier monographs were more on systems and control theory, including *Loop Transfer Recovery: Analysis and Design* (with A. Saberi and P. Sannuti, Springer, 1993), *H_2 Optimal Control* (with A. Saberi and P. Sannuti, Prentice Hall, 1995), *H_∞ Control and Its Applications* (Springer, 1998), *Robust and H_∞ Control* (Springer, 2000), and *Linear Systems Theory: A Structural Decomposition Approach* (with Z. Lin and Y. Shamash, Birkhäuser, 2004). My recent works focus more on control applications, which include *Hard Disk Drive Servo Systems* (first edition with T.H. Lee and V. Venkataramanan, Springer, 2002; second edition with T.H. Lee, K. Peng, and V. Venkataramanan, Springer, 2006) and *Unmanned Rotorcraft Systems* (with G. Cai and T.H. Lee, Springer, 2011). Even though my most recent monograph, *Stock Market*

Profile of Ben M. Chen

- *Current position:* professor and area director of Control, Intelligent Systems, and Robotics, Department of Electrical and Computer Engineering; head of Control Science Group, Temasek Laboratories, National University of Singapore.
- *Visiting and research positions:* Changjiang Chair professor, Nanjing University of Science and Technology.
- *Contact information:* Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117576, +011 65 6516 2289, bmchen@nus.edu.sg; http://uav.ece.nus.edu.sg/~bmchen/.
- *IEEE Control Systems Society experience highlights:* associate editor, Conference Editorial Board, 1997–1998; associate editor, *IEEE Transactions on Automatic Control*, 1999–2001; chair, IEEE Singapore Control Systems Chapter, 2002–2003.
- *Notable awards:* University Researcher Award, National University of Singapore, 2000; Prestigious Engineering Achievement Award, Institution of Engineers Singapore, 2001; Best Industrial Control Application Prize, Fifth Asian Control Conference, Melbourne, Australia, 2004; IEEE Fellow, 2007; Best Application Paper Award, 7th Asian Control Conference, Hong Kong, 2009.

I personally believe that a good control system design should not start from differential equations but should be down to earth and start from the hardware level, including the selection and placement of sensors and actuators *(to design a good system)*.

Course outline

- Introduction
- Mathematical background materials
- State space representation of systems
- Realization of linear systems
- Solution of state equations
- Stability analysis
- Controllability and observability
- Systems zeros and invertibility
- Some structural decomposition techniques
- Review of classical control system design
- State feedback design
- Observer and compensator design
- Modern control systems design
- Concluding remarks

Preparation

Fundamental

Design

Conclusion

Reading materials

- C.T. Chen, *Linear System Theory & Design*, Holt, Rinehart & Winston, 1984
- T. Kailath, *Linear Systems*, Prentice Hall, 1980
- B.M. Chen, Z. Lin, Y. Shamash, *Linear Systems Theory,* Birkhauser, 2004★

Systems

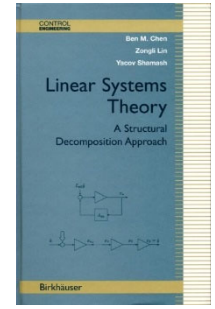
- L. Qiu, K. Zhou, *Feedback Control*, Prentice Hall, 2010★
- F.L. Lewis, *Applied Optimal Control & Estimation*, Texas Instruments, 1992
- B.M. Chen, *Robust and H_∞ Control,* Springer, 2000★
- A. Saberi, P. Sannuti, B.M. Chen, *H_2 Optimal Control,* Prentice Hall, 1995
- A. Saberi, B. M. Chen, P. Sannuti, *Loop Transfer Recovery,* Springer, 1993

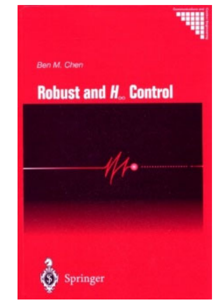
Control

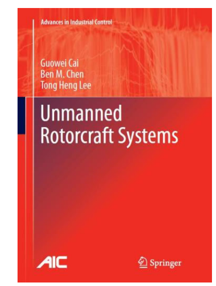
- G. Cai, B.M. Chen, T.H. Lee, *Unmanned Rotorcraft Systems*, Springer, 2011★
- B.M. Chen, *et al.*, *Hard Disk Drive Servo Systems*, 2nd Edn., Springer, 2006★

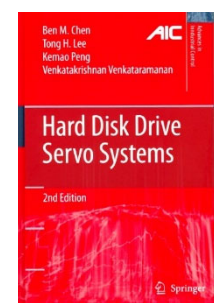
Applications

★ This text is available for downloading at SpingerLink Book through CUHK Library…









Important Notice:

We will focus only on continuous-time systems and control in this course. All results presented here, however, have discrete-time counterparts. Interested students are advised to take another class on digital/computer control systems if there is such a module at CUHK. Alternatively, one could grasp the ideas on discrete-time version from the references listed on the previous page.

Basically, there are two ways to design and implement a control system for real problems:

1. doing everything in the continuous-time setting to design an appropriate control law and then discretize it when implemented to the real system.

2. discretizing the plant first and preparing everything in the discrete-time setting to design a discrete-time controller for direct implementation.

The methods covered in this course are sufficient to handle the first case…

Homework assignments and design problems

There will be six (6) homework assignments and two (2) design problems to design controllers for real physical systems.

All students are expected to have knowledge in MATLAB™ (Control Toolbox and Robust Control Toolbox) and SIMULINK™ after completing these assignments. Homework assignments and projects are to be marked and counted towards your final grade.

⋆ Some problems might be solved by using a linear systems toolkit developed by the course instructor and his co-workers.

Final Grade = 30% ~ Midterm Exam

30% ~ Homework Assignments (6)

30% ~ Design Projects (2)

10% ~ Quizzes (to be randomly announced in the class)

Notice:

1. Lectures are to be conducted in the face-to-face mode. Online lectures might be arranged if necessary.
2. Midterm exam will be of open-book. It covers materials in the first part. The schedule will be announced in the class.
3. The following is the teaching assistant and his contact information. You can approach the course TA for help when needed…
   - XYZ, email: xyz@link.cuhk.edu.hk
4. There is no final exam for this course.

Course Material Flow (theory)…

- Introduction
- Mathematical Background
- Dynamic Modeling/State Space Representation/Realization
- System Dynamic Responses
- System Stability
- Controllability & Observability
- Systems Invertibility & Zeros
- Structural Decompositions
- Advanced Concepts of Linear Systems*
- Decomposition of Proper Systems*
- Linear Systems Toolkit

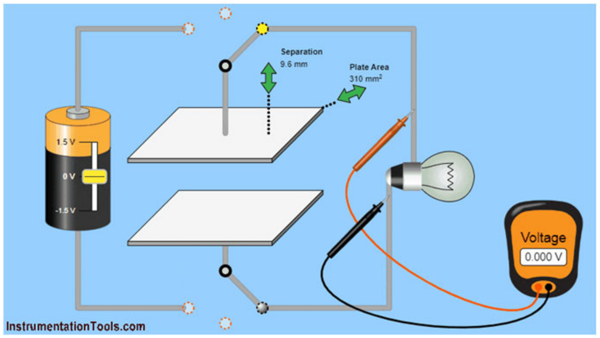
Course Material Flow (design)…

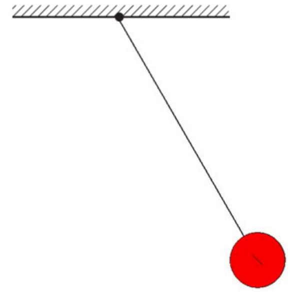
- Review of Classical Control
- Stabilization of Multivariable Systems
- LQR Control
- Kalman Filter
- LQG Control
- Introduction to Robust Control
- H_2 and H_∞ Control
- Loop Transfer Recovery Design*
- Robust and Perfect Tracking Control*
- Concluding Remarks

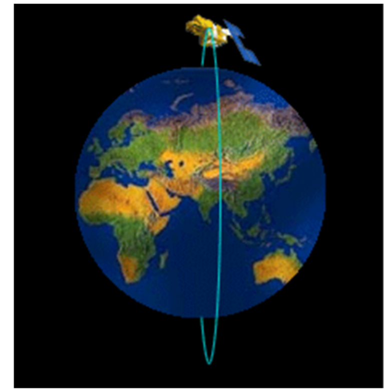
Introduction

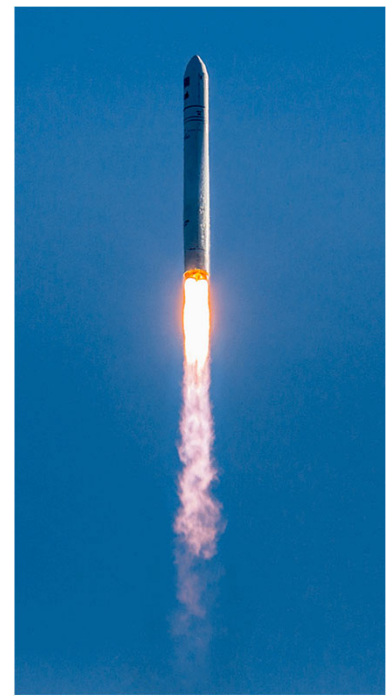
What is a system?

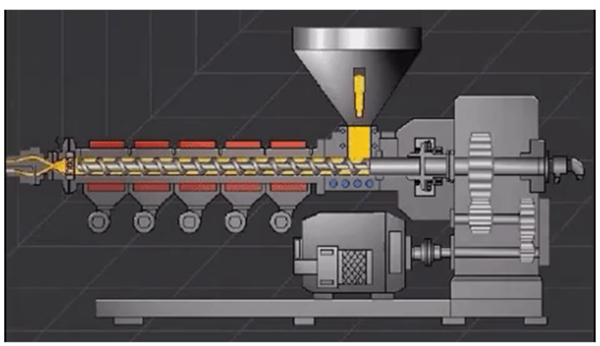
Examples: Some systems of interest…

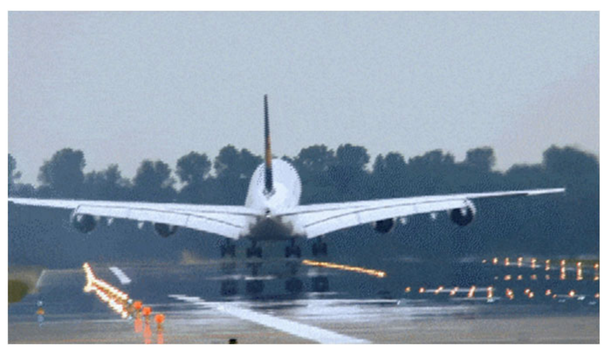






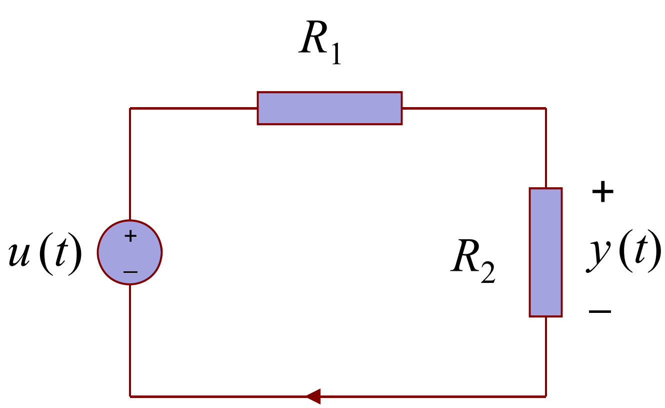






Block diagram representation of a system

$u(t)$ is a signal or certain information injected into the system, which is called the system input, whereas $y(t)$ is a signal or certain information produced by the system with respect to the input signal $u(t)$. $y(t)$ is called the system output. For example,



input: voltage source

output: voltage across R_2

$$y(t) = \frac{R_2}{R_1 + R_2} \cdot u(t)$$

Linear systems

Let $y_1(t)$ be the output produced by an input signal $u_1(t)$ and $y_2(t)$ be the output produced by another input signal $u_2(t)$. Then, the system is said to be linear if

a) the input is $\alpha\, u_1(t)$, the output is $\alpha\, y_1(t)$, where α is a scalar; and

b) the input is $u_1(t) + u_2(t)$, the output is $y_1(t) + y_2(t)$.

Or equivalently, the input is $\alpha\, u_1(t) + \beta\, u_2(t)$, the output is $\alpha\, y_1(t) + \beta\, y_2(t)$. Such a property is called **superposition**. For the circuit example on the previous page,

$$y(t) = \frac{R_2}{R_1 + R_2} \cdot \left[\alpha\, u_1(t) + \beta\, u_2(t)\right] = \alpha \frac{R_2}{R_1 + R_2} u_1(t) + \beta \frac{R_2}{R_1 + R_2} u_2(t) = \alpha\, y_1(t) + \beta\, y_2(t)$$

It is a linear system! We will mainly focus on linear systems in this course.

Time invariant systems

A system is said to be time-invariant if for a shift input signal $u(t-t_0)$, the output of the system is $y(t-t_0)$. To see if a system is time-invariant or not, we test

a) Find the output $y_1(t)$ that corresponds to the input $u_1(t)$.

b) Let $u_2(t)=u_1(t-t_0)$ and then find the corresponding output $y_2(t)$.

c) If $y_2(t)=y_1(t-t_0)$, then the system is time-invariant. Otherwise, it is not!

In common words, if a system is time-invariant, then for the same input signal, the output produced by the system today will be **exactly the same** as that produced by the system tomorrow or any other time.

Time variant systems examples

Example 1: Consider a system characterized by

$$\boxed{y(t) = \cos(t)u(t)}$$

Step One:

$$y_1(t) = \cos(t)\cdot u_1(t) \quad \Rightarrow \quad y_1(t-t_0) = \cos(t-t_0)\cdot u_1(t-t_0)$$

Step Two: Let $u_2(t) = u_1(t-t_0)$, we have

$$y_2(t) = \cos(t)\cdot u_2(t) = \cos(t)\cdot u_1(t-t_0) \neq y_1(t-t_0)$$

The system is not time-invariant. It is time-variant!

Example 2: Consider a financial system such as a stock market. Assume that you invest \$10,000 today in the market and make \$2,000. Is it guaranteed that you will make exactly another \$2,000 tomorrow if you invest the same amount of money? Is such a system time-invariant? You know the answer, don't you?

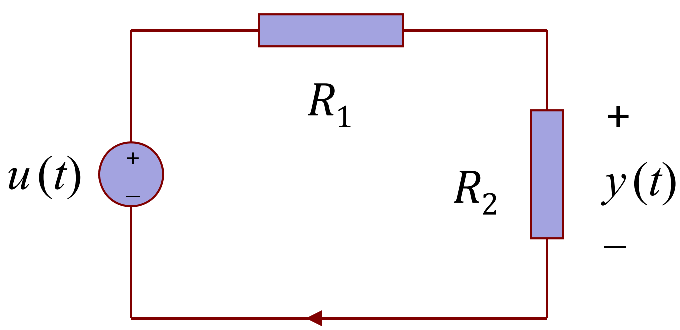
Systems with memory and without memory

A system is said to have memory if the value of $y(t)$ at any particular time t_1 depends on the time from $-\infty$ to t_1. For example,

$$u(t) = C\frac{dy(t)}{dt} \quad \Rightarrow \quad y(t) = \frac{1}{C}\int_{-\infty}^{t} u(t)dt$$

A dynamic system...

On the other hand, a system is said to have no memory if the value of $y(t)$ at any particular time t_1 depends only on t_1. For example,



$$y(t) = \frac{R_2}{R_1 + R_2} \cdot u(t)$$

A static system...

Causal systems

A causal system is a system where the output $y(t)$ at a particular time t_1 depends on the input for $t \le t_1$. For example,

$$\boxed{u(t) = C\frac{dy(t)}{dt} \quad \Rightarrow \quad y(t) = \frac{1}{C}\int_{-\infty}^{t} u(\tau)d\tau}$$

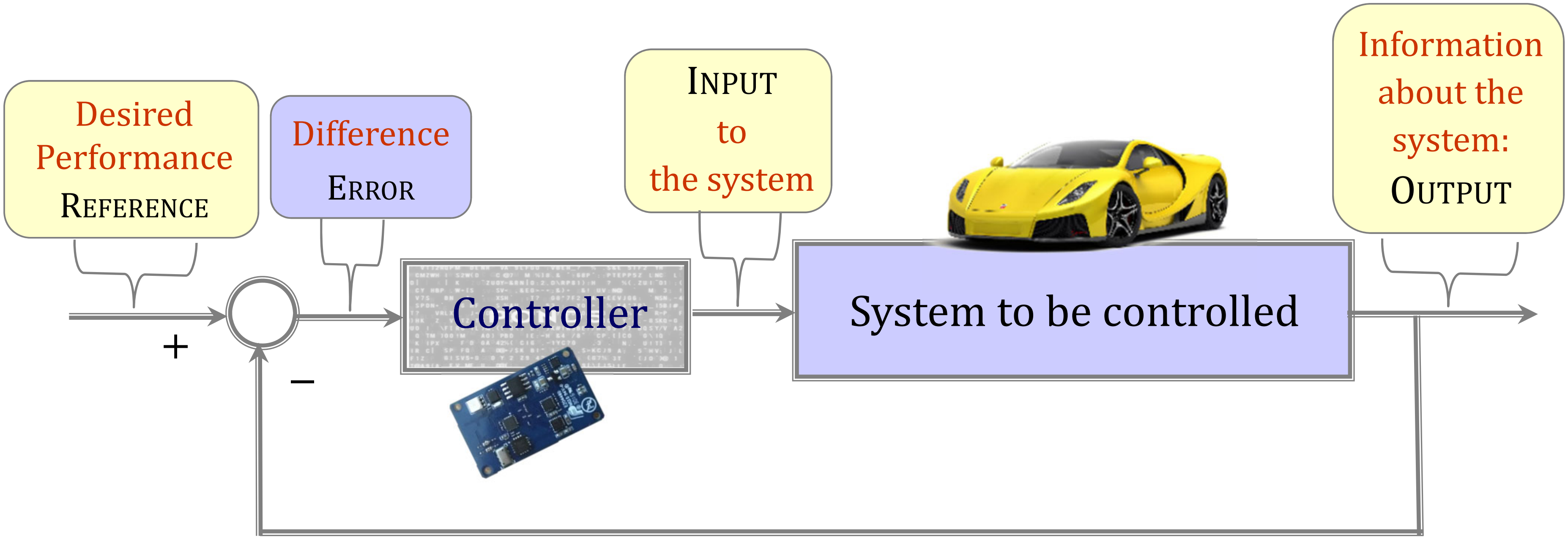
On the other hand, a system is said to be non-causal if the value of $y(t)$ at a particular time t_1 depends on the input $u(t)$ for some $t > t_1$. For example,

$$\boxed{y(t) = u(t+1)}$$

in which the value of $y(t)$ at $t = 0$ depends on the input at $t = 1$.

What is control?

Typical structure of a control system

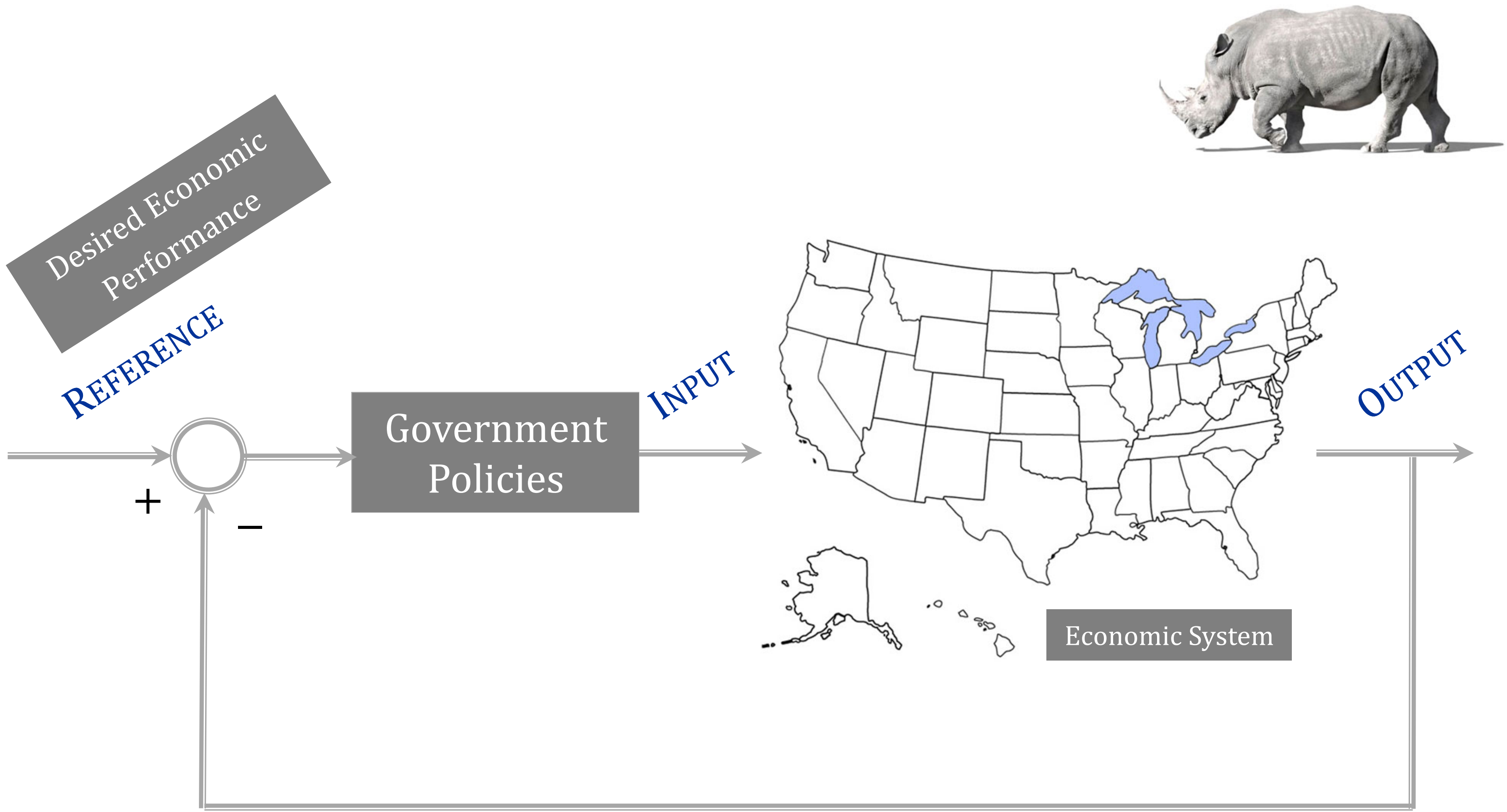


Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

Key issues: (1) How to describe the system to be controlled? **(Modeling)**

(2) How to design the controller? **(Control)**

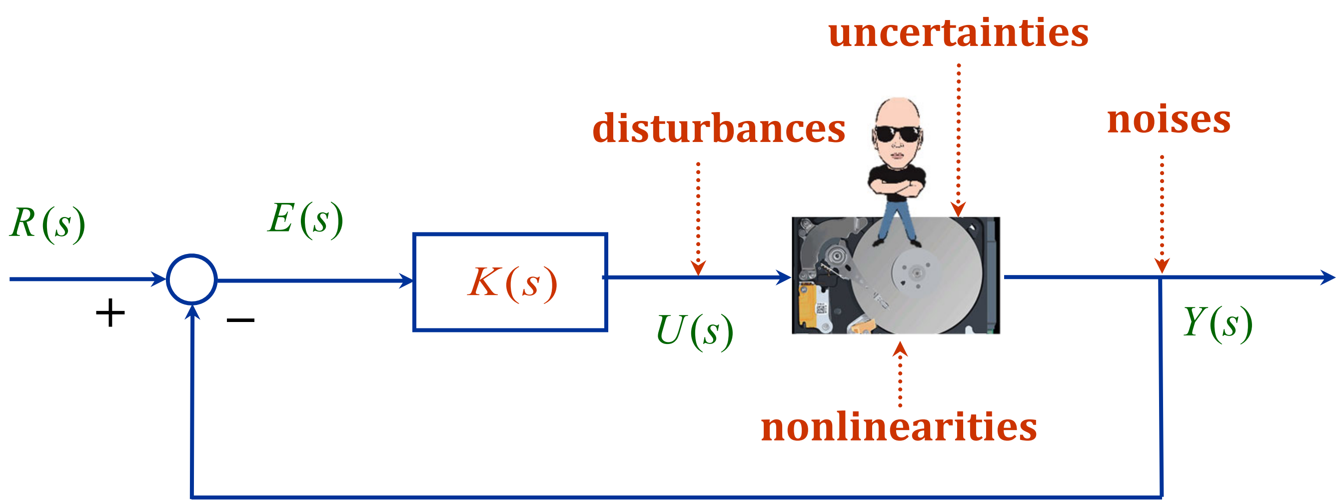
24



Some control systems examples…

Uncertainties, nonlinearities and disturbances

There are many other factors of life have to be carefully considered when dealing with real-life problems. These factors include:



? If you were the system, what would be your disturbances, noises, uncertainties, and nonlinearities?

A brief view on control design techniques

- **Classical control**

  PID control, developed in 1930s/40s and used heavily for in industrial applications.

- **Optimal control**

  Linear quadratic regulator control, Kalman filter, H_2 control, developed in 1960s to achieve certain optimal performance.

- **Robust control**

  H_∞ control, developed in 1980s & 90s to handle systems with uncertainties and disturbances and with high performances.

- **Nonlinear control**

  Developed to handle nonlinear systems with high performances.

- **Multi-agent systems & cooperative control**

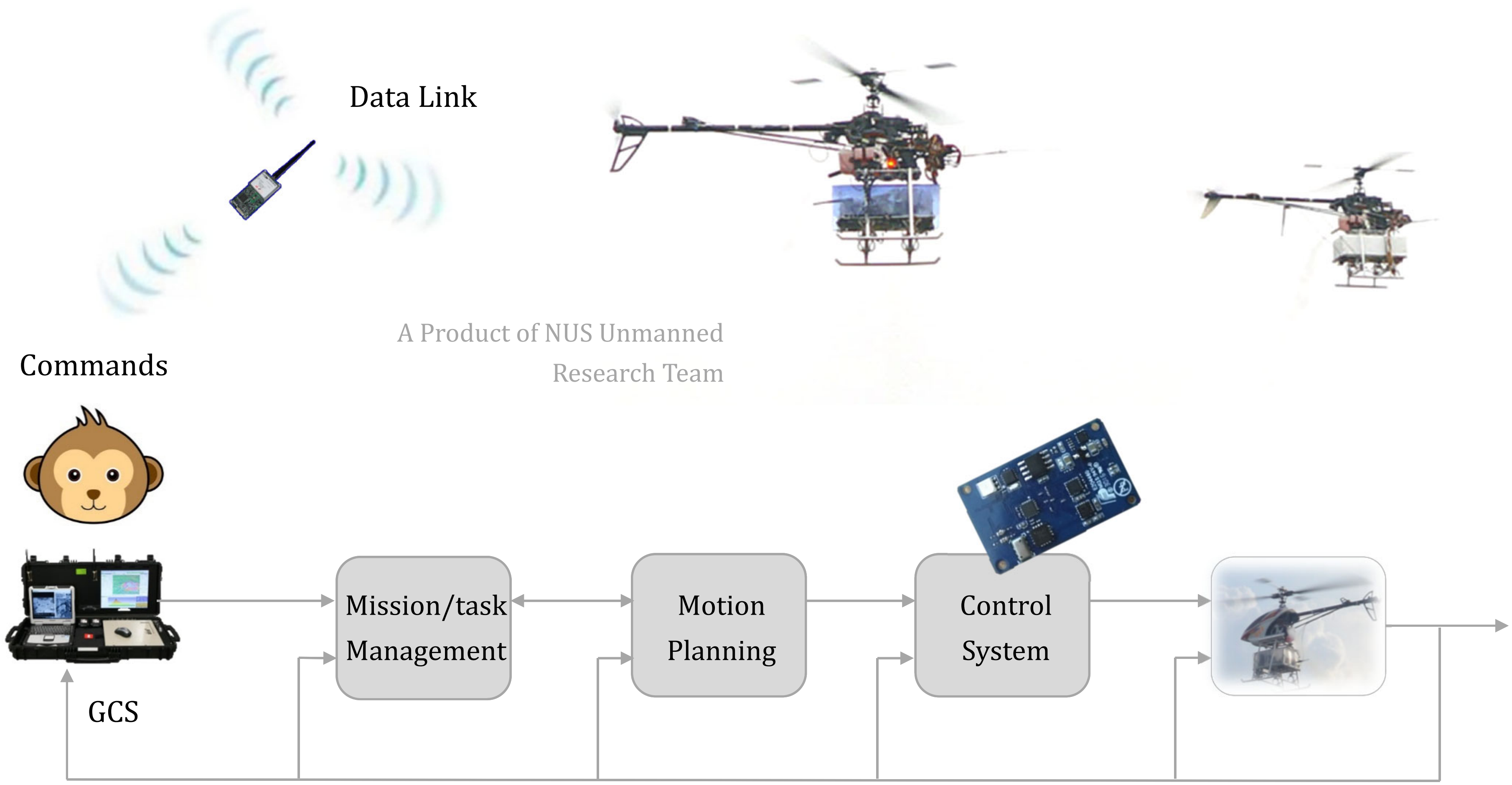
  It is a hot topic at moment.

- **Intelligent control** (with a possible link to deep learning…)

  Knowledge-based control, adaptive control, neural and fuzzy control, developed to handle systems with unknown models.

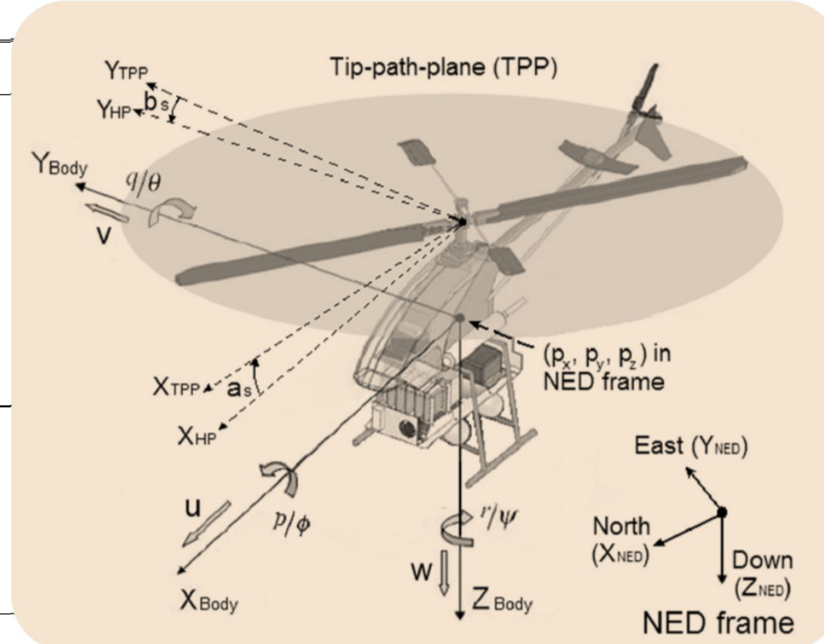
An actual control system example…

∞ Flight formation of fully autonomous unmanned helicopters ∞



Flight dynamics variable description

Variable	Physical description
p_x, p_y, p_z	Position vector along NED-frame x, y, and z axes
u, v, w	Velocity vector along body-frame x, y, and z axes
p, q, r	Roll, pitch, and yaw angular rates
ϕ, θ, ψ	Euler angles
a_s, b_s	Longitudinal and lateral tip-path-plane (TPP) flapping angle
$\delta_{ped,int}$	Intermediate state in yaw rate gyro dynamics
δ_{lat}	Normalized aileron servo input $(-1, 1)$
δ_{lon}	Normalized elevator servo input $(-1, 1)$
δ_{col}	Normalized collective pitch servo input $(-1, 1)$
δ_{ped}	Normalized rudder servo input $(-1, 1)$

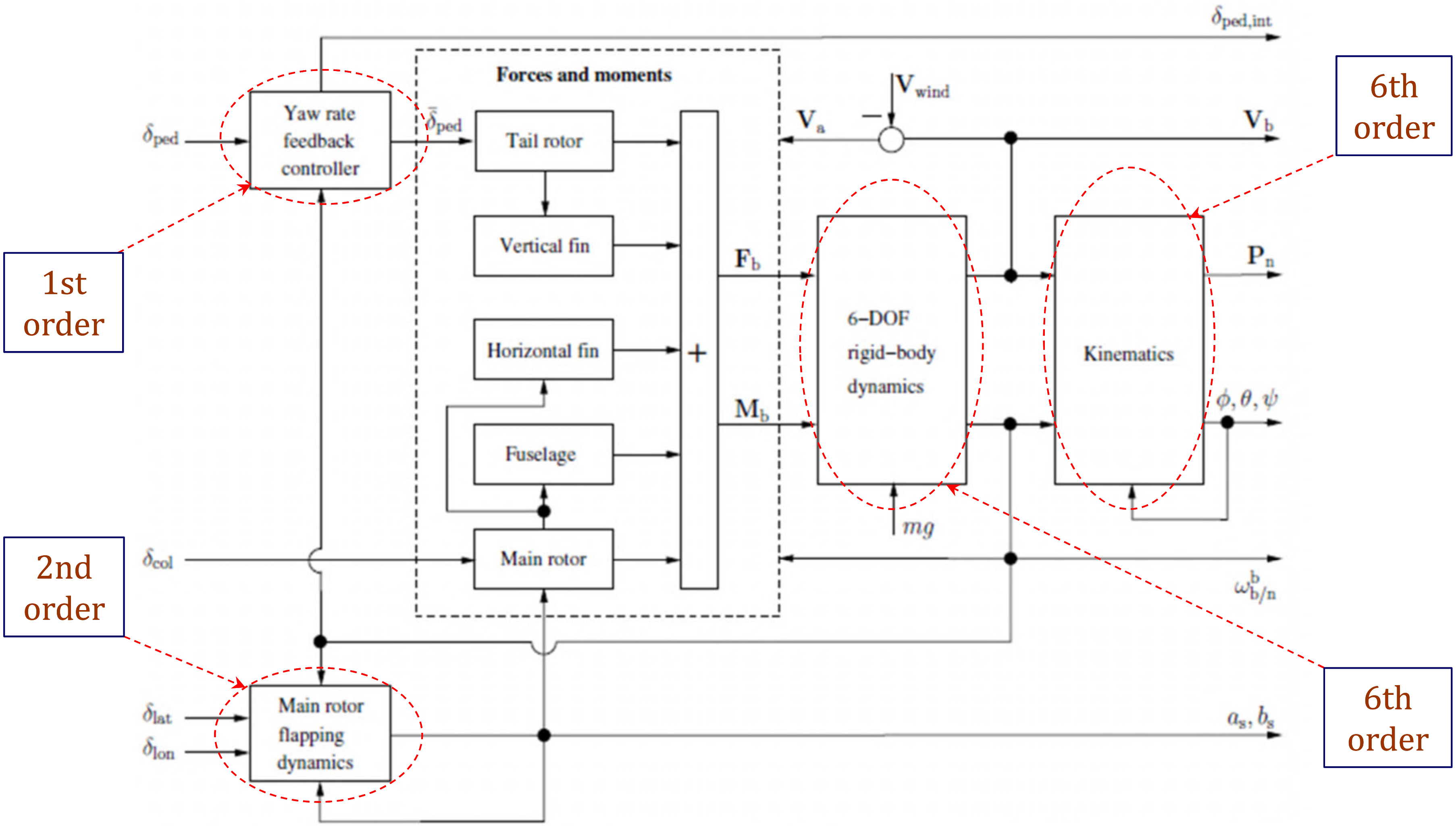


First-principles modeling approach is adopted to obtain an accurate nonlinear model in full envelope, which includes:

- kinematics
- 6 DOF rigid-body dynamics
- main rotor flapping dynamics
- yaw rate gyro dynamics

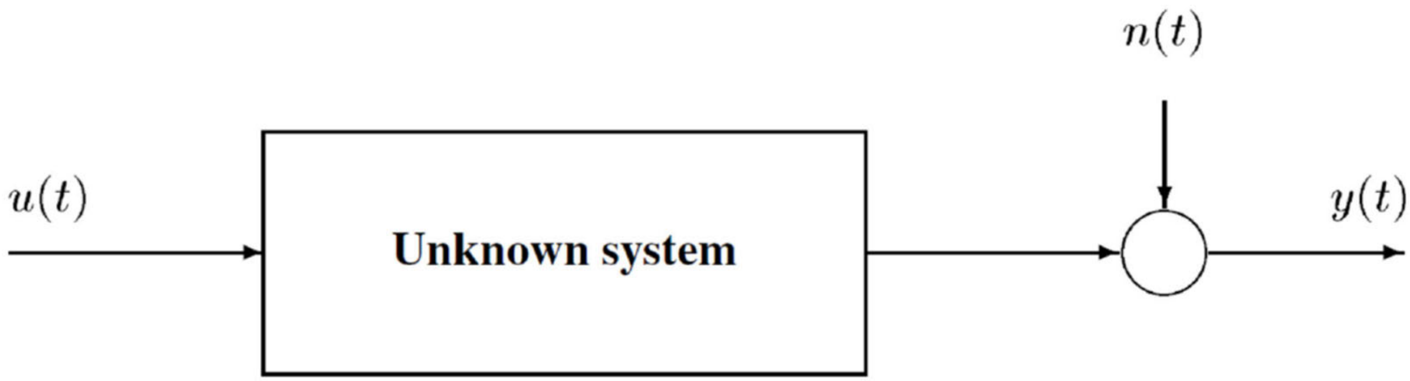
★ G. Cai, B. M. Chen, T. H. Lee and K. Y. Lum, Comprehensive nonlinear modeling of a miniature unmanned helicopter, *Journal of the American Helicopter Society*, Vol. 57, No. 1, pp. 012004-1~13, January 2012.

Flight dynamic model structure



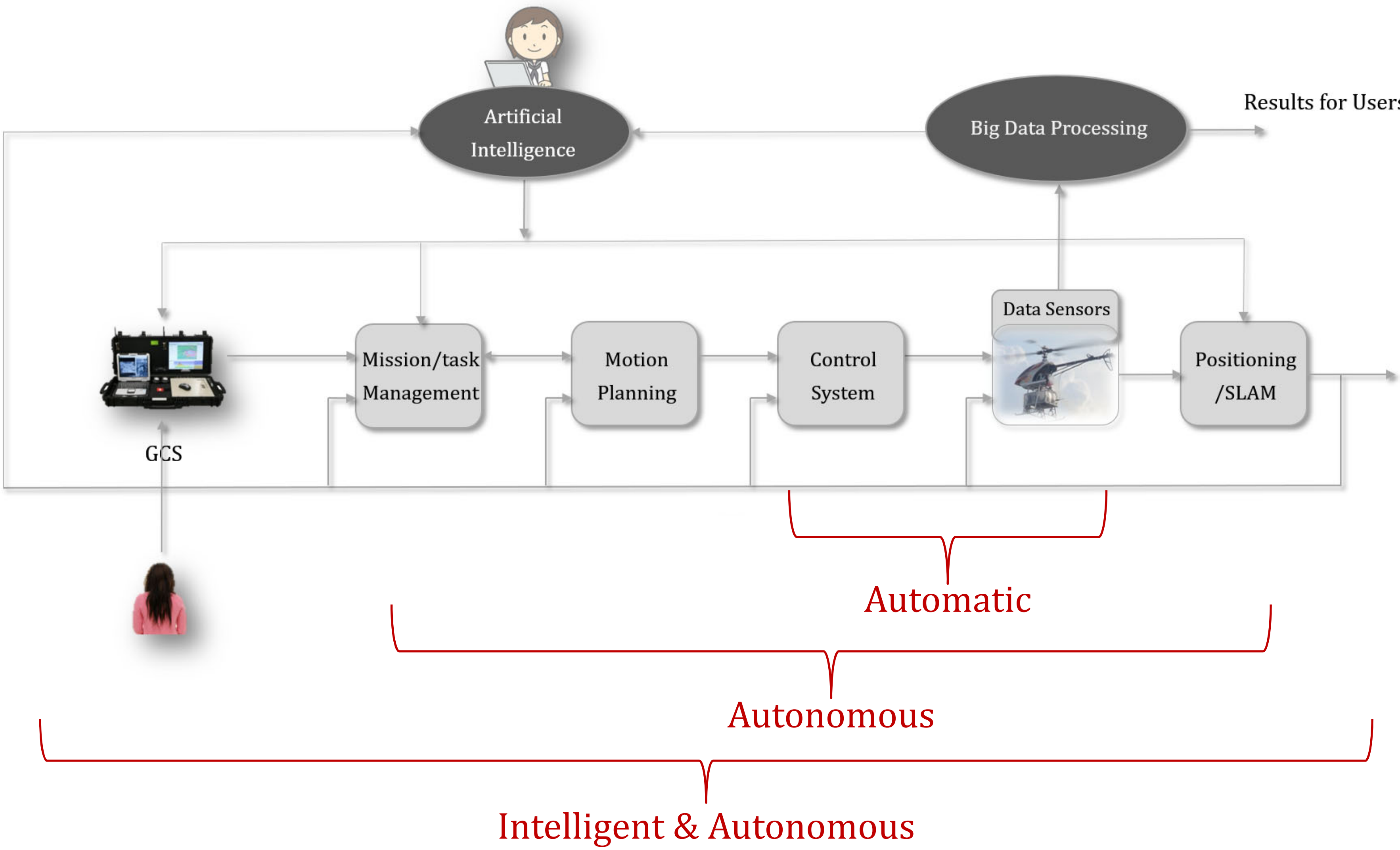
The model structure can be determined by the first-principles approach…

Some model parameters needed to be identified using a black box approach



Black box modelling and system identification

Automatic / Autonomous / Intelligent Autonomous Systems



*B. M. Chen, On the trends of autonomous unmanned systems research, *Engineering*, 2022. https://doi.org/10.1016/j.eng.2021.10.014

Mathematical Background

Vector spaces and subspaces

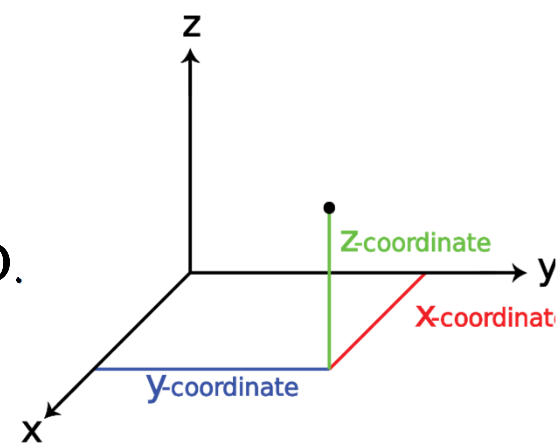
We assume that the reader is familiar with the basic definitions of scalar fields and vector spaces.

Let $\mathcal{X}$ be a *vector space* over a certain scalar field $\mathbb{K}$. A subset of $\mathcal{X}$, say $\mathcal{S}$, is said to be a *subspace* of $\mathcal{X}$ if $\mathcal{S}$ itself is a vector space over $\mathbb{K}$. The *dimension* of a subspace $\mathcal{S}$, denoted by $\dim \mathcal{S}$, is defined as the maximal possible number of linearly independent vectors in $\mathcal{S}$.

We say that vectors $s_1, s_2, \ldots, s_k \in \mathcal{S}$, $k = \dim \mathcal{S}$, form a *basis* for $\mathcal{S}$ if they are *linearly independent*, *i.e.*, $\sum_{i=1}^{k} \alpha_i s_i = 0$ holds only if $\alpha_i = 0$. Two subspaces $\mathcal{V}$ and $\mathcal{W}$ are said to be independent if $\mathcal{V} \cap \mathcal{W} = \{0\}$.

Example:

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for a basis for 3D.}$$



Definition 2.2.4 (Kernel and image of a matrix). *Given* $A \in \mathbb{C}^{m\times n}$ *(or* $\mathbb{R}^{m\times n}$*), a linear map from* $\mathcal{X} = \mathbb{C}^n$ *(or* $\mathbb{R}^n$*) to* $\mathcal{Y} = \mathbb{C}^m$ *(or* $\mathbb{R}^m$*), the* kernel *or* null space *of* A *is defined as*

$$\ker(A) := \{x \in \mathcal{X} \mid Ax = 0\}, \tag{2.2.8}$$

and the image *or* range space *of* A *is defined as*

$$\operatorname{im}(A) = A\mathcal{X} := \{Ax \mid x \in \mathcal{X}\}. \tag{2.2.9}$$

Obviously, $\ker(A)$ *is a subspace of* $\mathcal{X}$*, and* $\operatorname{im}(A)$ *is a subspace of* $\mathcal{Y}$*.*

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad \Rightarrow \quad \ker(A) = \left\{ \alpha_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right\}, \quad \operatorname{im}(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Definition 2.2.6 (Invariant subspace). *Given $A \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$), a linear map from $\mathcal{X} = \mathbb{C}^n$ (or $\mathbb{R}^n$) to $\mathcal{X}$, a subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be A-*invariant *if*

$$A\mathcal{V} \subset \mathcal{V}. \tag{2.2.11}$$

Such a $\mathcal{V}$ is also called an invariant subspace *of A.*

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \quad \mathcal{V} = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \text{ is an invariant subspace of } A.$$

$$\because \quad A\left[\alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \alpha\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = (5\alpha)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathcal{V}$$

Matrix inverse

If two square matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ satisfy $AB = BA = I$, then B is said to be the *inverse* of A and is denoted by A^{-1}. If the inverse of A exists, then A is said to be *nonsingular*; otherwise it is *singular*. We note that A is nonsingular if and only if $\det(A) \neq 0$. The following identities are useful.

$$(I + AB)^{-1}A = A(I + BA)^{-1}, \tag{2.3.14}$$

☞ $$[I + C(sI - A)^{-1}B]^{-1} = I - C(sI - A + BC)^{-1}B, \tag{2.3.15}$$

and

$$(I - BD)^{-1} = I + B(I - DB)^{-1}D. \tag{2.3.16}$$

If A and B are nonsingular, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{bmatrix} \tag{2.3.17}$$

$$\begin{bmatrix} A & D \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}DB^{-1} \\ 0 & B^{-1} \end{bmatrix}. \tag{2.3.18}$$

Eigenvalues: Given an $n \times n$ matrix A, a complex scalar λ is said to be an eigenvalue of A if

$$Ax = \lambda x \quad (\Longleftrightarrow \quad (\lambda I - A)x = 0), \tag{2.3.30}$$

for some nonzero vector $x \in \mathbb{C}^n$. Such an x is called a (right) *eigenvector* associated with the eigenvalue λ.

It then follows from (2.3.30) that, for an eigenvalue λ,

$$\text{rank}\,(\lambda I - A) < n \quad (\Longleftrightarrow \quad \det\,(\lambda I - A) = 0). \tag{2.3.31}$$

Thus, the eigenvalues of A are the roots of its *characteristic polynomial*,

$$\chi(\lambda) := \det\,(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \tag{2.3.32}$$

which has a total of n roots. The set of these roots or eigenvalues of A is denoted by $\lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The following property is the *Cayley–Hamilton theorem*,

$$\chi(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0. \tag{2.3.33}$$

We can show that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

68

Arthur Cayley
1821–1895
British Mathematician

has a characteristic polynomial of

$$\chi(\lambda) = \det\left(\lambda I - A\right) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_3 & a_2 & \lambda + a_1 \end{vmatrix} = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

William R. Hamilton
1805–1865
Irish Mathematician

Generally, we can show that

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}$$

This result is particularly useful for pole placement…

169

$$\Rightarrow \quad \chi(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

Spectral radius and trace

The *spectral radius* of A is defined as

$$\rho(A) := \max \{ |\lambda| \mid \lambda \in \lambda(A) \}, \tag{2.3.34}$$

and the *trace* of A, defined as

$$\operatorname{trace}(A) := \sum_{i=1}^{n} a_{ii}, \tag{2.3.35}$$

is related to the eigenvalues of A as

$$\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i. \tag{2.3.36}$$

Remark: Matrix trace be computed using an m-function TRACE and the roots of a polynomial can be computed using ROOTS in MATLAB.

Special matrices

The following are several important types of square matrices. We say that a matrix $A \in \mathbb{R}^{n\times n}$ is

1. *symmetric* if $A' = A$ (such a matrix has all eigenvalues on the real axis);
2. *skew-symmetric* if $A' = -A$ (such a matrix has all eigenvalues on the imaginary axis);
3. *orthogonal* if $A'A = AA' = I$ (such a matrix has all eigenvalues on the unit circle);
4. *nilpotent* if $A^k = 0$ for integer k (such a matrix has all eigenvalues at the origin);
5. *idempotent* if $A^2 = A$ (such a matrix has eigenvalues at either 1 or 0);
6. a *permutation matrix* if A is nonsingular and each one of its columns (or rows) has only one nonzero element, which is equal to 1.

Special matrices (cont.)

We say that a matrix $A \in \mathbb{C}^{n\times n}$ is

1. *Hermitian* if $A^{\mathrm{H}} = A$ (such a matrix has all eigenvalues on the real axis);
2. *unitary* if $A^{\mathrm{H}}A = AA^{\mathrm{H}} = I$ (such a matrix has all eigenvalues on the unit circle);
3. *positive definite* if $x^{\mathrm{H}}Ax > 0$ for every nonzero vector $x \in \mathbb{C}^n$;
4. *positive semi-definite* if $x^{\mathrm{H}}Ax \geq 0$ for every vector $x \in \mathbb{C}^n$;
5. *negative definite* if $x^{\mathrm{H}}Ax < 0$ for every nonzero vector $x \in \mathbb{C}^n$;
6. *negative semi-definite* if $x^{\mathrm{H}}Ax \leq 0$ for every vector $x \in \mathbb{C}^n$;
7. *indefinite* if A is neither positive nor negative semi-definite.

Note that Hermitian and symmetric matrices have all its eigenvalues being real scalars. Moreover, a Hermitian or symmetric matrix $A > 0$ (positive definite) iff all its eigenvalues are positive, $A \geq 0$ (positive semi-definite) iff all its eigenvalues are non-negative, $A < 0$ (negative definite) iff all its eigenvalues are negative.

Matrix norms

Given a matrix $A = [a_{ij}] \in \mathbb{C}^{m \times n}$, its *Frobenius norm* is defined as

$$\|A\|_{\mathrm{F}} := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \left(\sum_{i=1}^{\min\{m,n\}} \sigma_i(A) \right)^{1/2}. \tag{2.4.3}$$

The p-*norm* of A is a norm induced from the vector p-norm, *i.e.*,

$$\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p = 1} \|Ax\|_p. \tag{2.4.4}$$

In particular, for $p = 1, 2, \infty$, we have

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \tag{2.4.5}$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^{\mathrm{H}} A)} = \sigma_{\max}(A), \tag{2.4.6}$$

which is also called the *spectral norm* of A, and

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|. \tag{2.4.7}$$

Norms of continuous-time signals

For any $p \in [1, \infty)$, let L_p^m denote the linear space formed by all measurable signals $g : \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$\int_0^\infty |g(t)|^p dt < \infty.$$

For any $g \in L_p^m$, $p \in [1, \infty)$, its L_p-*norm* is defined as

$$\|g\|_p := \left(\int_0^\infty |g(t)|^p dt \right)^{1/p}, \quad 1 \le p < \infty. \qquad (2.4.9)$$

Let L_∞^m denote the linear space formed by all signals $g : \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$|g(t)| < \infty, \quad \forall t \in \mathbb{R}_+.$$

The L_∞-norm of a $g \in L_\infty^m$ is defined as

$$\|g\|_\infty := \sup_{t \ge 0} \ |g(t)|. \qquad (2.4.10)$$

Laplace transform

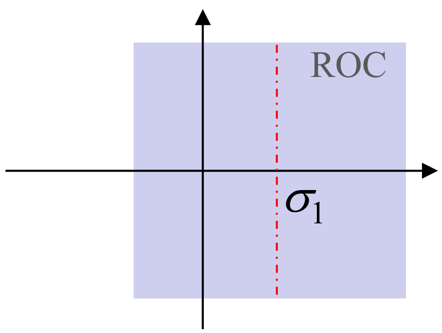
Given a time-domain function $f(t)$, the one-sided Laplace transform is defined as follows:

$$F(s) = L\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt, \quad s = \sigma + j\omega$$

where the lower limit of integration is set to 0^- to include the origin ($t = 0$) and to capture any discontinuities of the function at $t = 0$.

Given a frequency-domain function $F(s)$, the inverse Laplace transform is to convert it back to its original time-domain function $f(t)$:

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds$$



Laplace transform technique is invaluable in solving engineering problems!

Summary of Laplace transform properties

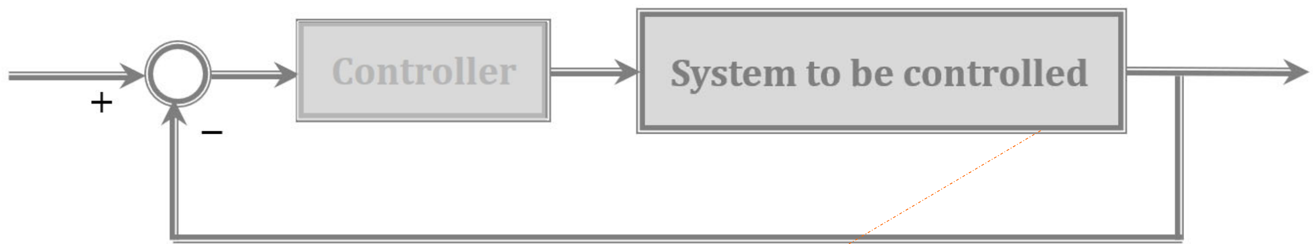
Property	***f (t)***	***F (s)***
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F(\frac{s}{a})$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time derivative	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - s^0 f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(\zeta) d\zeta$	$\frac{1}{s} F(s)$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^-)$	$\lim_{s \to \infty} [sF(s)]$
Final value	$f(\infty)$	$\lim_{s \to 0} [sF(s)]$
Convolution	$f_1(t) \otimes f_2(t)$	$F_1(s) F_2(s)$

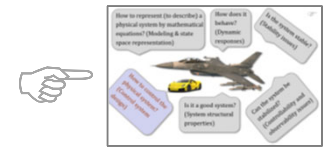
Pierre-Simon Laplace
1749-1827
French Scholar

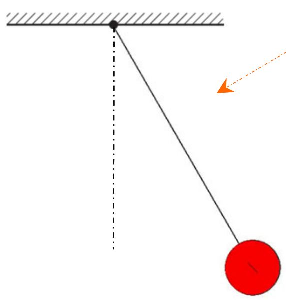
Some commonly used Laplace transform pairs

$f(t)$	$\Leftrightarrow$	$F(s)$	$f(t)$	$\Leftrightarrow$	$F(s)$
$\delta(t)$	$\Leftrightarrow$	1	$\sin \omega t$	$\Leftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
$1(t)$	$\Leftrightarrow$	$\dfrac{1}{s}$	$\cos \omega t$	$\Leftrightarrow$	$\dfrac{s}{s^2+\omega^2}$
t	$\Leftrightarrow$	$\dfrac{1}{s^2}$	$\sin(\omega t+\theta)$	$\Leftrightarrow$	$\dfrac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
t^n	$\Leftrightarrow$	$\dfrac{n!}{s^{n+1}}$	$\cos(\omega t+\theta)$	$\Leftrightarrow$	$\dfrac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
e^{-at}	$\Leftrightarrow$	$\dfrac{1}{s+a}$	$e^{-at}\sin\omega t$	$\Leftrightarrow$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
te^{-at}	$\Leftrightarrow$	$\dfrac{1}{(s+a)^2}$	$e^{-at}\cos\omega t$	$\Leftrightarrow$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

Dynamic Modeling & State Space Representation







$$G(s)$$

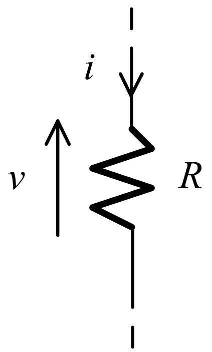
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Linear system

$$\dot{x}(t) = f\left(x(t), u(t)\right), \quad y(t) = g\left(x(t), u(t)\right)$$ nonlinear

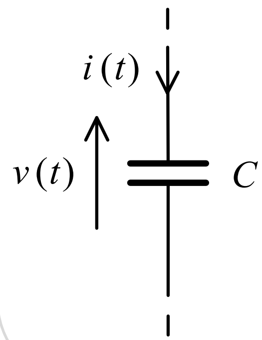
Basic laws for electrical systems

resistor



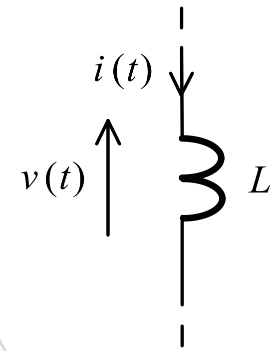
$$v = i\,R$$

capacitor



$$i = C\frac{dv}{dt}$$

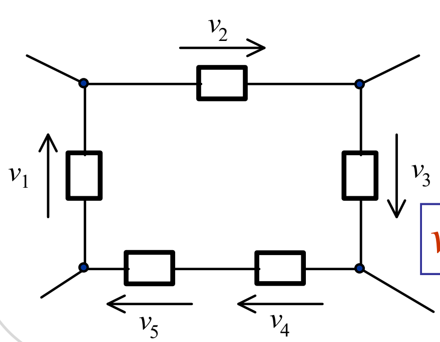
inductor



$$v = L\frac{di}{dt}$$

Kirchhoff's Voltage Law (KVL):

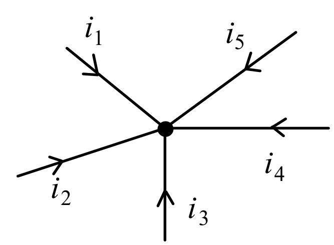
The sum of voltage drops around any close loop in a circuit is 0.

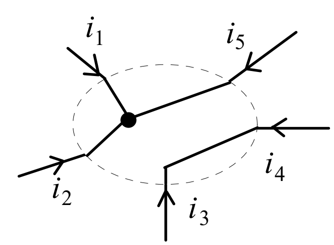


$$v_1+v_2+v_3+v_4+v_5 = 0$$

Kirchhoff's Current Law (KCL):

The sum of currents entering/ leaving a node/closed surface is 0.

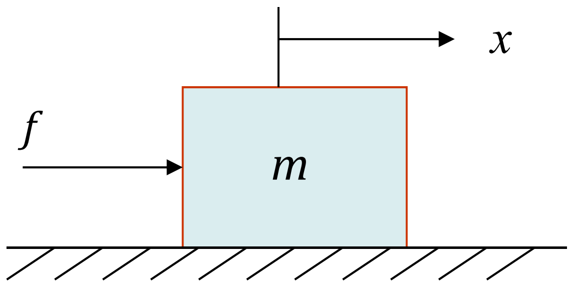




$$i_1 + i_2 + i_3 + i_4 + i_5 = 0$$

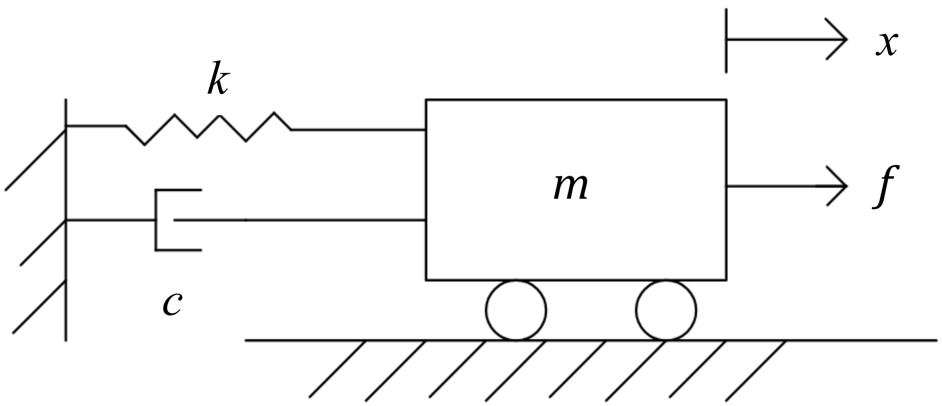
Some basic mechanical systems

Newton's law of motion $f = ma = m\ddot{x} = m\dot{v}$



Mass-spring-damper system

$$m\ddot{x} + c\dot{x} + kx = f$$



Isaac Newton
1642–1726
English…

Clarence de Silva★
University of
British Columbia

Gustav Kirchhoff
1824–1887
German Physicist

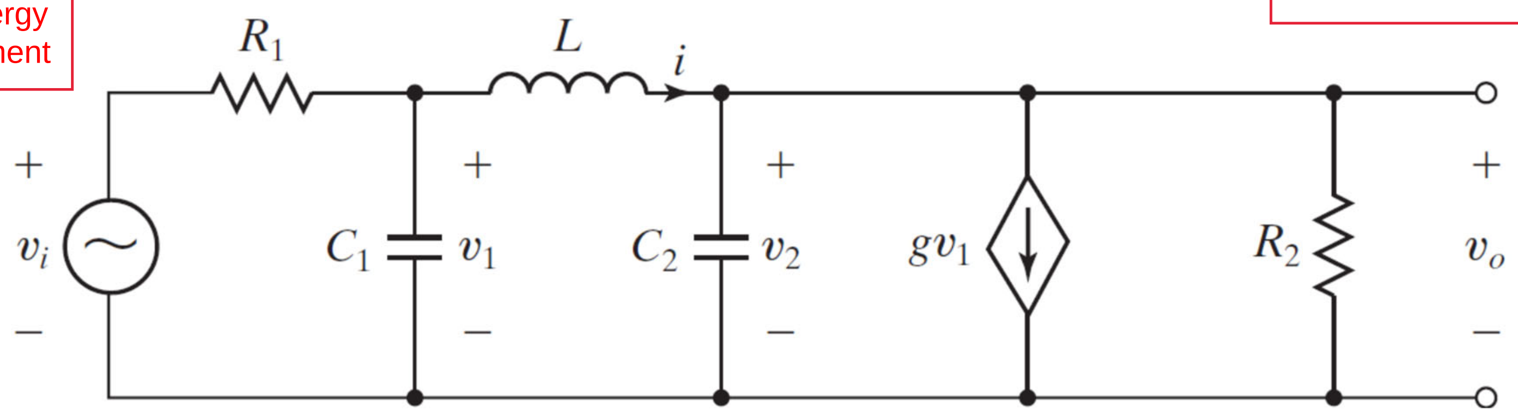
★C. W. de Silva, *Modeling of Dynamic Systems*, Taylor & Francis/CRC Press, 2017.

Dynamic modeling based on first principles

Example (Qiu and Zhou)**:** Consider an RLC circuit as shown in the figure below, where the diamond symbol labeled, gv_1 means a dependent current source whose current is proportional to v_1. The input and output of the system are $v_i(t)$ and $v_o(t)$, respectively. Find the dynamic model of the given system.

order = # energy storage equipment

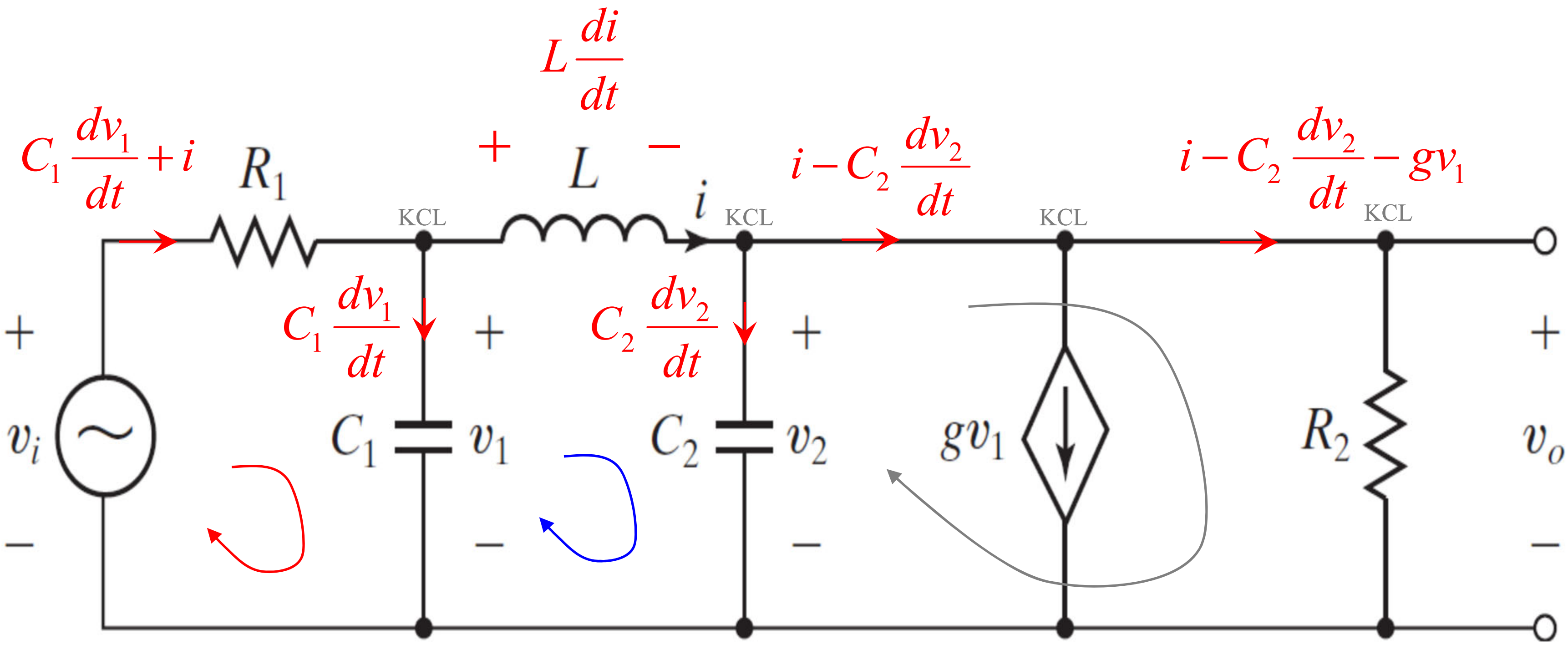
Third order system



The common practice in solving an electric circuit problem is to assign a voltage variable to a capacitor and a current variable to an inductor.

For the given circuit, we assign v_1 and v_2 as the voltages across C_1 and C_2 and i as the inductor current. The system is of 3rd order as it has 3 energy storing elements.

Assign all the branch currents and calculate their values using KCL…



As it is of a 3rd order system, find 3 equations from 3 independent loops using KVL

Red Loop:

$$\left(C_1\frac{dv_1}{dt}+i\right)R_1+v_1=v_i$$

Blue Loop:

$$L\frac{di}{dt}+v_2=v_1$$

Gray Loop:

$$\left(i-C_2\frac{dv_2}{dt}-gv_1\right)R_2=v_2$$

Red Loop: $\left(C_1\dfrac{dv_1}{dt}+i\right)R_1+v_1=v_i \quad\Rightarrow\quad \dot v_1=-\dfrac{1}{R_1C_1}v_1-\dfrac{1}{C_1}i+\dfrac{1}{R_1C_1}v_i$

Blue Loop: $L\dfrac{di}{dt}+v_2=v_1 \quad\Rightarrow\quad \dot i=\dfrac{1}{L}v_1-\dfrac{1}{L}v_2$

Gray Loop: $\left(i-C_2\dfrac{dv_2}{dt}-gv_1\right)R_2=v_2 \quad\Rightarrow\quad \dot v_2=-\dfrac{g}{C_2}v_1-\dfrac{1}{R_2C_2}v_2+\dfrac{1}{C_2}i$

Define a so-called state variable vector

$$x=\begin{pmatrix}v_1\\ v_2\\ i\end{pmatrix} \quad\Rightarrow\quad \dot x=\begin{pmatrix}\dot v_1\\ \dot v_2\\ \dot i\end{pmatrix}=\begin{pmatrix}-{}^1\!/\!_{R_1C_1} & 0 & -{}^1\!/\!_{C_1}\\ -{}^g\!/\!_{C_2} & -{}^1\!/\!_{R_2C_2} & {}^1\!/\!_{C_2}\\ {}^1\!/\!_{L} & -{}^1\!/\!_{L} & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ i\end{pmatrix}+\begin{pmatrix}{}^1\!/\!_{R_1C_1}\\ 0\\ 0\end{pmatrix}v_i$$

The output variable

$$v_o=v_2=\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ i\end{pmatrix}$$

Define

$$A=\begin{pmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ -\frac{g}{C_2} & -\frac{1}{R_2C_2} & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{pmatrix},\quad B=\begin{pmatrix} \frac{1}{R_1C_1} \\ 0 \\ 0 \end{pmatrix},\quad C=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Rudolf E. Kalman
1930–2016
Hungarian-American Scholar

The dynamic equation of the system can be expressed as

$$\dot{x}=Ax+Bv_i$$

and the system output $v_o=Cx=Cx+0\cdot v_i$.

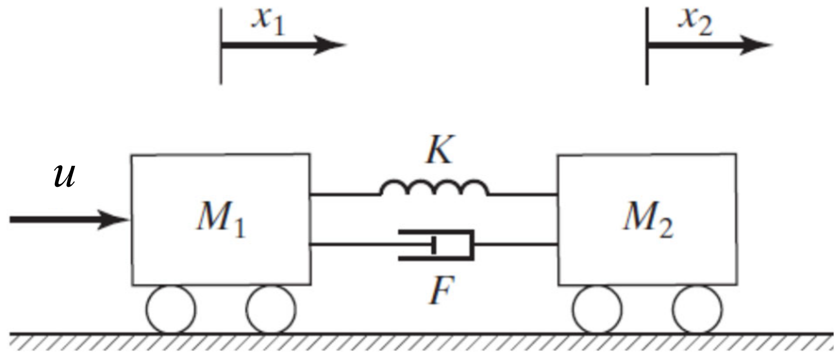
The dynamic equation together with the output equation form the so-called state space representation of the given electrical circuit or system.

In fact, all linear time-invariant systems can be expressed in the form of

$$\dot{x}=Ax+Bu,\quad y=Cx+Du$$

* R.E. Kalman, On the general theory of control systems, *Proceedings of 1st International IFAC Congress on Automatic and Remote Control*, Moscow, USSR, pp. 481–492, August 1960.

Example (Qiu and Zhou)**:** Consider a two-cart system as depicted in the figure below



The carts, assumed to have masses M_1 and M_2, respectively, are connected by a spring and a damper. A force $u(t)$ is applied to Cart M_1 and we wish to observe the position of Cart M_2, i.e., $y = x_2$.

Applying Newton's law of motion to M_1, we obtain

$$M_1\ddot{x}_1 = u(t) - K\left(x_1 - x_2\right) - F\left(\dot{x}_1 - \dot{x}_2\right)$$

Applying Newton's law of motion to M_2, we obtain

$$M_2\ddot{x}_2 = K\left(x_1 - x_2\right) + F\left(\dot{x}_1 - \dot{x}_2\right)$$

Define a state variable vector

$$x = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} \quad \Rightarrow \quad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \frac{1}{M_1}u - \frac{K}{M_1}x_1 + \frac{K}{M_1}x_2 - \frac{F}{M_1}\dot{x}_1 + \frac{F}{M_1}\dot{x}_2 \\ \dot{x}_2 \\ \frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{F}{M_2}\dot{x}_1 - \frac{F}{M_2}\dot{x}_2 \end{pmatrix}$$

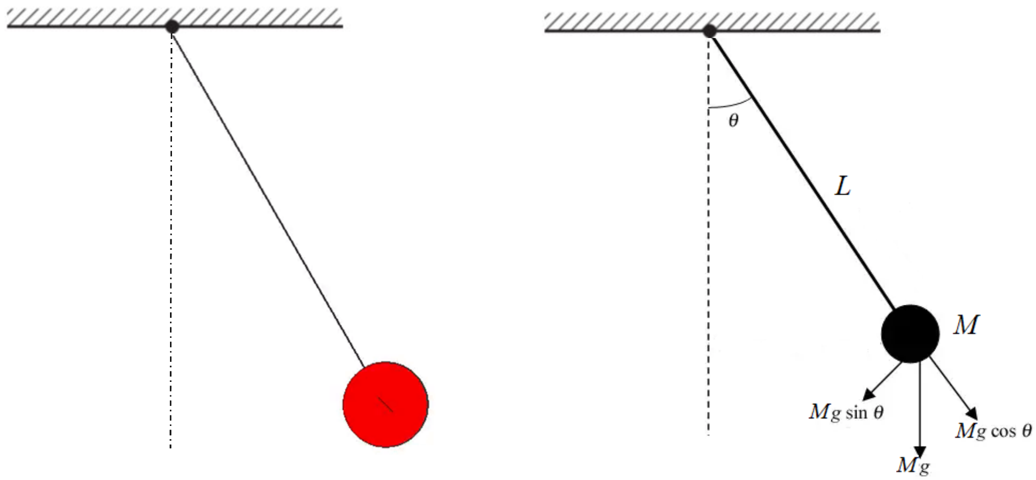
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M_1} & -\frac{F}{M_1} & \frac{K}{M_1} & \frac{F}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & \frac{F}{M_2} & -\frac{K}{M_2} & -\frac{F}{M_2} \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} u = Ax + Bu$$

The variable to be observed, i.e., the system output

$$y = x_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} = C\,x + 0 \cdot u$$

which together form the state space representation of the two-cart system.

Example (Qiu and Zhou)**:** Consider a pendulum system shown in the figure below



Li Qiu
HKUST

Kemin Zhou
Louisiana State University

A torque $u(t)$ can be applied around the pivot point and we are concerned with the angle $\theta(t)$. The length of the pendulum is L and the mass M of the pendulum is concentrated at its tip.

In a rotational motion, Newton's second law takes the form

$$J \frac{d^2\theta}{dt^2} = \tau(t)$$

where J is the moment of inertia and τ is the total torque applied.

For the pendulum system, the moment of inertia $J = ML^2$ and there are two torques applied to the system: the external torque $u(t)$ and the torque due to the gravity of the mass, which is $MgL\sin\theta(t)$. As such, the equation governing the motion is given by

$$ML^2\frac{d^2\theta}{dt^2} = u - MgL\sin\theta \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = \ddot{\theta} = -\frac{g}{L}\sin\theta + \frac{1}{ML^2}u$$

Question: Can we write this dynamic equation in the form of

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

with properly defined state variable?

Let us define

$$x = \begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix} \Rightarrow \dot{x} = \begin{pmatrix}\dot{\theta} \\ \ddot{\theta}\end{pmatrix} = \begin{pmatrix}\dot{\theta} \\ -\frac{g}{L}\sin\theta\end{pmatrix} + \begin{bmatrix}0 \\ \frac{1}{ML^2}\end{bmatrix}u = \begin{bmatrix}0 & 1 \\ ? & 0\end{bmatrix}\begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix} + \begin{bmatrix}0 \\ \frac{1}{ML^2}\end{bmatrix}u$$

Can or Cannot?

Why not?

Linearization

We now study how to approximate a nonlinear system by a linear model. Assume that a nonlinear system is described by the following dynamic equations:

$$\dot{x}(t) = f\big(x(t), u(t)\big), \quad y(t) = g\big(x(t), u(t)\big)$$

where $x \in \mathbb{R}^n$ is the state vector, u and y are respectively the input and output scalar variables, f and g are continuously differentiable functions.

A triple of constant vectors (u_0, x_0, y_0) is said to be an operating (equilibrium) point of the system if

$$0 = f\big(x_0, u_0\big), \quad y_0 = g\big(x_0, u_0\big)$$

The physical meaning of an operating point is that if the system has initial condition x_0 and a constant input u_0 is applied, then the state and output will stay at constant values x_0 and y_0, respectively, for all time, i.e.,

$$u(t) = u_0, \quad x(0) = x_0 \quad \Rightarrow \quad x(t) = x_0, \quad y(t) = y_0$$

Denote

$$\tilde{u}(t) = u(t) - u_0, \quad \tilde{x}(t) = x(t) - x_0, \quad \tilde{y}(t) = y(t) - y_0$$

It can be shown that

$$\dot{\tilde{x}}(t) = \left.\frac{\partial f}{\partial x}\right|_{x=x_0,\, u=u_0} \tilde{x}(t) + \left.\frac{\partial f}{\partial u}\right|_{x=x_0,\, u=u_0} \tilde{u}(t) \quad + \quad \text{high-order terms}$$

$$\tilde{y}(t) = \left.\frac{\partial g}{\partial x}\right|_{x=x_0,\, u=u_0} \tilde{x}(t) + \left.\frac{\partial g}{\partial u}\right|_{x=x_0,\, u=u_0} \tilde{u}(t) \quad + \quad \text{high-order terms}$$

where

$$\frac{\partial f}{\partial x} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{\text{Jacobian matrix}}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}, \quad \frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}$$

For a small neighborhood of the operating point, i.e.,

$$\tilde{u}(t) = u(t) - u_0, \quad \tilde{x}(t) = x(t) - x_0, \quad \tilde{y}(t) = y(t) - y_0$$

are small, we can neglect the higher-order terms and approximate the original system by the following linear system:

$$\dot{\tilde{x}}(t) = A\,\tilde{x}(t) + B\,\tilde{u}(t)$$

$$\tilde{y}(t) = C\,\tilde{x}(t) + D\,\tilde{u}(t)$$

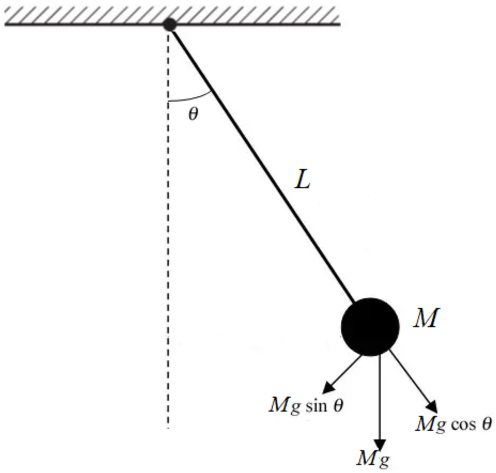
where

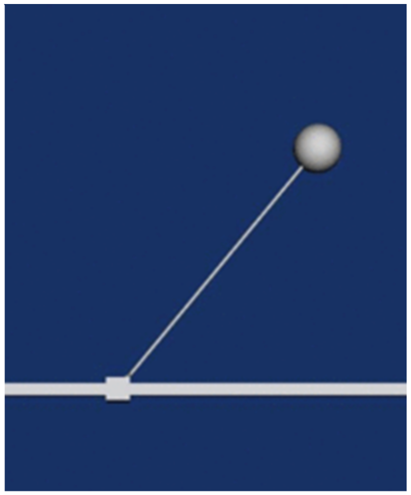
$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_0,\, u=u_0}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=x_0,\, u=u_0}$$

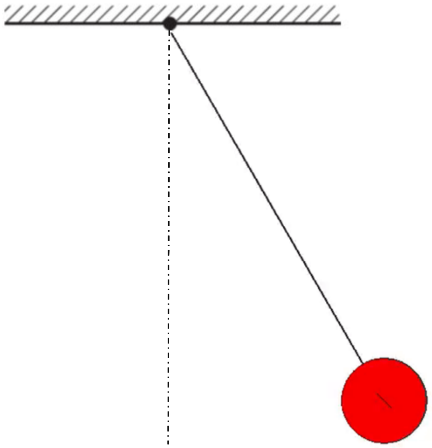
$$C = \left.\frac{\partial g}{\partial x}\right|_{x=x_0,\, u=u_0}, \quad D = \left.\frac{\partial g}{\partial u}\right|_{x=x_0,\, u=u_0}$$

This linear system is called the linearized state space model of the original nonlinear system.

Example: Revisit the (inverse) pendulum system studied earlier







We have obtained earlier a nonlinear dynamic equation governing the system

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ -\dfrac{g}{L}\sin\theta + \dfrac{1}{ML^2}u \end{pmatrix} = f(x,u), \quad x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

Let us define the system output to be θ. We have the output equation

$$y = \theta = g(x,u)$$

We note that there is an operating point of the system at

$$(u_0, x_0, y_0) = (0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0) \quad \Rightarrow \quad 0 = f(0,0) = \begin{pmatrix} \dot{\theta} \\ -\dfrac{g}{L}\sin\theta + \dfrac{1}{ML^2}u \end{pmatrix}_{\substack{u=0 \\ \theta=0, \dot{\theta}=0}}, \quad y_0 = 0$$

In the small neighborhood of the operating point, i.e., when θ is small, we have

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,\,u=0} = \begin{bmatrix} \dfrac{\partial f_1}{\partial \theta} & \dfrac{\partial f_1}{\partial \dot{\theta}} \\ \dfrac{\partial f_2}{\partial \theta} & \dfrac{\partial f_2}{\partial \dot{\theta}} \end{bmatrix}_{\theta=0,\,\dot{\theta}=0,\,u=0} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L}\cos\theta & 0 \end{bmatrix}_{\theta=0,\,\dot{\theta}=0,\,u=0} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L} & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x=0,\,u=0} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_n}{\partial u} \end{bmatrix}_{x=0,\,u=0} = \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix}, \qquad C = \left.\frac{\partial g}{\partial x}\right|_{x=0,\,u=0} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L} & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

Linearized model around $\theta = 0$

Another operating point of the system is at

$$(u_0, x_0, y_0) = \left(0, \begin{pmatrix} \pi \\ 0 \end{pmatrix}, \pi\right) \Rightarrow \; 0 = f\left(\begin{pmatrix} \pi \\ 0 \end{pmatrix}, 0\right) = \begin{pmatrix} \dot\theta \\ -\dfrac{g}{L}\sin\theta + \dfrac{1}{ML^2}u \end{pmatrix}_{\substack{u=0 \\ \theta=\pi,\dot\theta=0}}, \quad y_0 = \pi$$

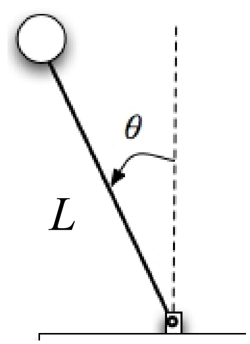
In the small neighborhood of the operating point, we have

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\begin{pmatrix} \pi \\ 0 \end{pmatrix}, u=0} = \begin{bmatrix} \dfrac{\partial f_1}{\partial \theta} & \dfrac{\partial f_1}{\partial \dot\theta} \\ \dfrac{\partial f_2}{\partial \theta} & \dfrac{\partial f_2}{\partial \dot\theta} \end{bmatrix}_{\theta=\pi,\,\dot\theta=0,\,u=0} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L}\cos\theta & 0 \end{bmatrix}_{\theta=\pi,\,\dot\theta=0,\,u=0} = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{L} & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x=\begin{pmatrix} \pi \\ 0 \end{pmatrix}, u=0} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_n}{\partial u} \end{bmatrix}_{x=\begin{pmatrix} \pi \\ 0 \end{pmatrix}, u=0} = \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix}, \qquad C = \left.\frac{\partial g}{\partial x}\right|_{x=\begin{pmatrix} \pi \\ 0 \end{pmatrix}, u=0} = [1 \quad 0]$$

$$\Rightarrow \begin{pmatrix} \dot{\tilde\theta} \\ \ddot{\tilde\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{L} & 0 \end{bmatrix} \begin{pmatrix} \tilde\theta \\ \dot{\tilde\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix} u, \quad \tilde y = [1 \quad 0] \begin{pmatrix} \tilde\theta \\ \dot{\tilde\theta} \end{pmatrix}, \quad \begin{aligned} \tilde\theta &= \theta - \pi \\ \tilde y &= y - \pi \end{aligned}$$

Inverse Pendulum



Linearized model around $\theta = \pi$!

Feedback linearization of nonlinear systems

There exist a class of nonlinear systems

$$\dot{x}(t) = f\big(x(t), u(t)\big), \quad y(t) = g\big(x(t), u(t)\big)$$

for which we can find a pre-feedback law of the following form

$$u(t) = h(y(t)) + \breve{u}(t)$$

such that when it is applied to the given nonlinear system, the resulting system is linear, i.e.,

$$\dot{x}(t) = A\, x(t) + B\, \breve{u}(t), \quad y(t) = C\, x(t) + D\, \breve{u}(t)$$

Such a technique is commonly called as feedback linearization.

Example: Let us consider the pendulum system once again, i.e.,

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ -\dfrac{g}{L}\sin\theta + \dfrac{1}{ML^2}u \end{pmatrix} = f(x,u), \quad x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}, \qquad y = \theta = g(x,u)$$

Let us apply a pre-feedback control law

$$u(t) = gML\sin\theta(t) + \breve{u}(t) = gML\sin y(t) + \breve{u}(t)$$

which implies

$$\begin{pmatrix} \dot\theta \\ \ddot\theta \end{pmatrix} = \begin{pmatrix} \dot\theta \\ -\frac{g}{L}\sin\theta + \frac{1}{ML^2}u \end{pmatrix} = \begin{pmatrix} \dot\theta \\ -\frac{g}{L}\sin\theta + \frac{1}{ML^2}\left(gML\sin\theta + \breve{u}\right) \end{pmatrix} = \begin{pmatrix} \dot\theta \\ \frac{1}{ML^2}\breve{u} \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot\theta \end{pmatrix} + \begin{bmatrix} 0 \\ 1/ML^2 \end{bmatrix} \breve{u}$$

$$\Rightarrow \begin{cases} \dot{x} = A\,x + B\,\breve{u}, \quad x = \begin{pmatrix} \theta \\ \dot\theta \end{pmatrix} \\ y = \theta = [1 \quad 0]\,x = C\,x \end{cases}$$

We indeed obtain a linear system in the **entire** state space through. Such a technique has been widely used in the nonlinear research community.

Transfer function of linear systems

Throughout the rest of this course, we will be dealing with linear systems in the state space form

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad y(t) = C\,x(t) + D\,u(t)$$

By Laplace transformation on both sides of the above equations, we obtain

$$L\left[\dot{x}(t)\right] = sX(s) - x(0^-) = L\left[A\,x(t) + B\,u(t)\right] = AX(s) + BU(s)$$

$$L\left[y(t)\right] = Y(s) = L\left[C\,x(t) + D\,u(t)\right] = CX(s) + DU(s)$$

ss2tf

which implies

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0^-)$$

$$Y(s) = CX(s) + DU(s) = \left[C(sI - A)^{-1}B + D\right]U(s) + C(sI - A)^{-1}x(0^-)$$

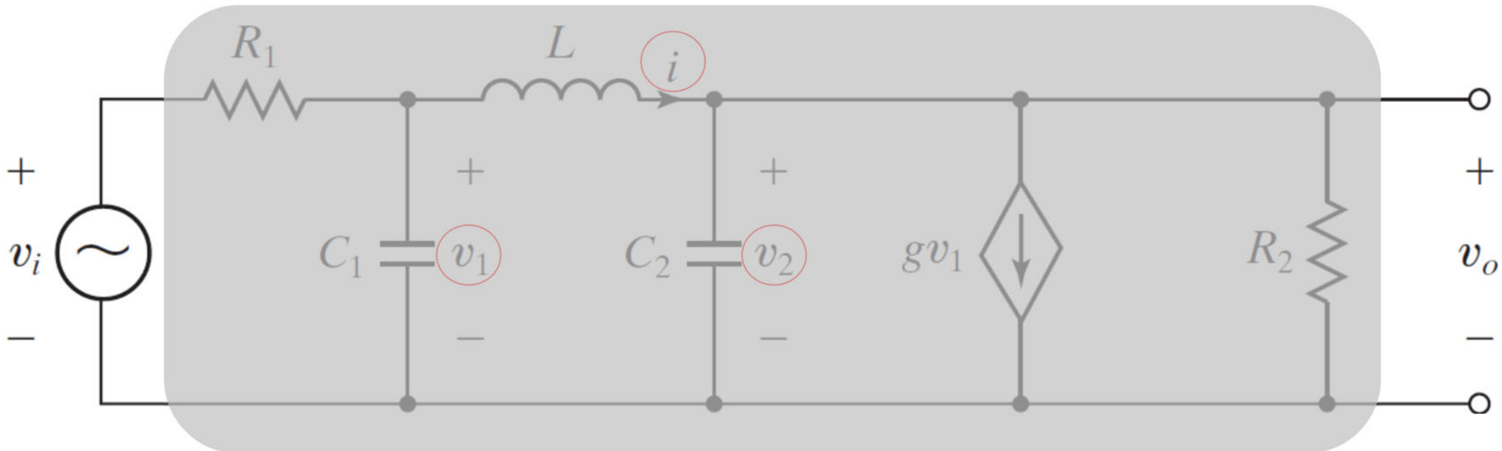
For the case when the initial condition is $x(0^-) = 0$,

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s) := G(s)U(s)$$

where $G(s) = C(sI - A)^{-1}B + D$ is called the system **transfer function matrix**.

Time domain *vs* frequency domain…

Let us revisit the electrical system studied earlier



For simplicity, assume $R_1 = R_2 = 1\,\Omega$, $L = 1\,\text{H}$, $C_1 = C_2 = 1\,\text{F}$, and $g = 100$. We then have

$$\dot{x} = \begin{pmatrix} -1 & 0 & -1 \\ -100 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_i, \quad v_o = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x, \quad x = \begin{pmatrix} v_1 \\ v_2 \\ i \end{pmatrix}$$

internal variables

which has an (input-output) transfer function

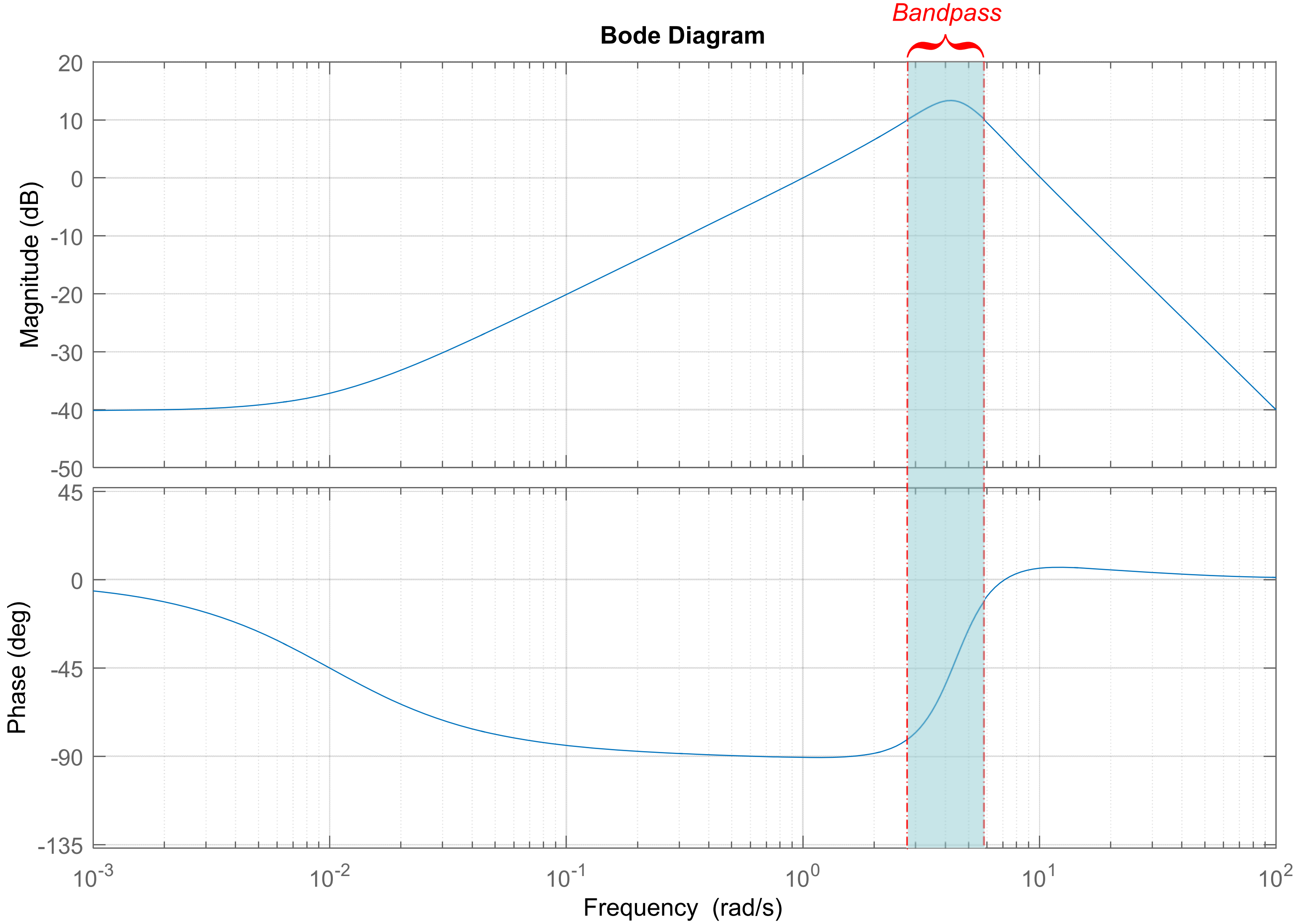
Ratio of input-output magnitudes at ω

$$G(s) = \frac{V_o}{V_i} = \frac{-100s + 1}{s^3 + 2s^2 + 3s + 102} \quad \Rightarrow \quad G(j\omega) = G(s)\big|_{s=j\omega} \rightarrow \begin{cases} |G(j\omega)| \\ \angle G(j\omega) \end{cases}$$

Phase shifting at ω

Frequency domain response…

Just an illustration…



State, input and output transformations

It is very often that we need to perform some transformations on the state, input and output variables, for the ease of systems analysis and control design. These transformations, so long as they are nonsingular, in fact, preserve all the structural properties of the given system.

Consider again the following system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad y(t) = C\,x(t) + D\,u(t)$$

We define a set of nonsingular state, input and output transformations, $\Gamma_{\rm s}$, $\Gamma_{\rm i}$ and $\Gamma_{\rm o}$, respectively, i.e.,

$$x = \Gamma_{\rm s}\tilde{x}, \quad u = \Gamma_{\rm i}\tilde{u}, \quad y = \Gamma_{\rm o}\tilde{y} \quad \Rightarrow \quad \left(\quad \tilde{x} = \Gamma_{\rm s}^{-1}x, \quad \tilde{u} = \Gamma_{\rm i}^{-1}u, \quad \tilde{y} = \Gamma_{\rm o}^{-1}y \quad \right)$$

which implies

$$\dot{\tilde{x}} = \Gamma_{\rm s}^{-1}\dot{x} = \Gamma_{\rm s}^{-1}(Ax + Bu) = \left(\Gamma_{\rm s}^{-1}A\Gamma_{\rm s}\right)\tilde{x} + \left(\Gamma_{\rm s}^{-1}B\Gamma_{\rm i}\right)\tilde{u} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$$

$$\tilde{y} = \Gamma_{\rm o}^{-1}y = \Gamma_{\rm o}^{-1}(Cx + Du) = \left(\Gamma_{\rm o}^{-1}C\Gamma_{\rm s}\right)\tilde{x} + \left(\Gamma_{\rm o}^{-1}D\Gamma_{\rm i}\right)\tilde{u} = \tilde{C}\tilde{x} + \tilde{D}\tilde{u}$$

We obtain a transformed system characterized by

$$\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,\tilde{u}(t), \quad \tilde{y}(t) = \tilde{C}\,\tilde{x}(t) + \tilde{D}\,\tilde{u}(t)$$

which has a transfer function

$$\begin{aligned}\tilde{G}(s) = \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D} &= \left(\Gamma_{\mathrm{o}}^{-1} C \Gamma_{\mathrm{s}}\right)\left(sI - \Gamma_{\mathrm{s}}^{-1} A \Gamma_{\mathrm{s}}\right)^{-1}\left(\Gamma_{\mathrm{s}}^{-1} B \Gamma_{\mathrm{i}}\right) + \left(\Gamma_{\mathrm{o}}^{-1} D \Gamma_{\mathrm{i}}\right) \\ &= \Gamma_{\mathrm{o}}^{-1}\left[C\left(sI - A\right)^{-1} B + D\right]\Gamma_{\mathrm{i}} \\ &= \Gamma_{\mathrm{o}}^{-1} G(s)\Gamma_{\mathrm{i}}\end{aligned}$$

For single-input and single-output (SISO) system,

$$\tilde{G}(s) = \alpha\, G(s), \quad \alpha \neq 0$$

We note that the nonsingular transformations of the system state, input and output have been proven to be a powerful tool for solving many systems and control problems. We will see very often this technique used in this course. Nevertheless, we first illustrate it by an example…

Example: Consider a system characterized by

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 3 & 1 & -3 \end{bmatrix} x$$

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With the state transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x = \Gamma_s \tilde{x} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 3 & 0 \\ -2 & -1 & 1 \end{bmatrix} \tilde{x}, \quad \tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}$$

we obtain a transformed system

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

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$$y = 8 \times \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tilde{x}$$

It has an identical transfer function as the original one: $G(s) = \tilde{G}(s) = \dfrac{8}{s^3 - 4s^2}$.

Realization of Transfer Functions

$$G(s) = \frac{Y(s)}{U(s)} \quad \Rightarrow \quad \xrightarrow{u(t)} \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array}} \xrightarrow{y(t)}$$

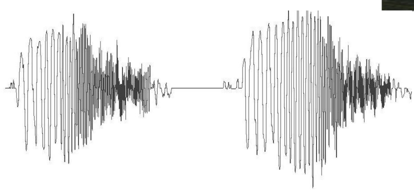
Black-box system identification

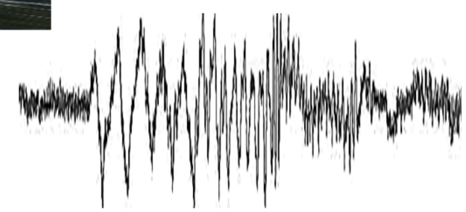
input

output

$u(t)$

$y(t)$





Laplace transforms

FFT is actually used instead…

$U(s)$

$$G(s) = Y(s) \big/ U(s)$$

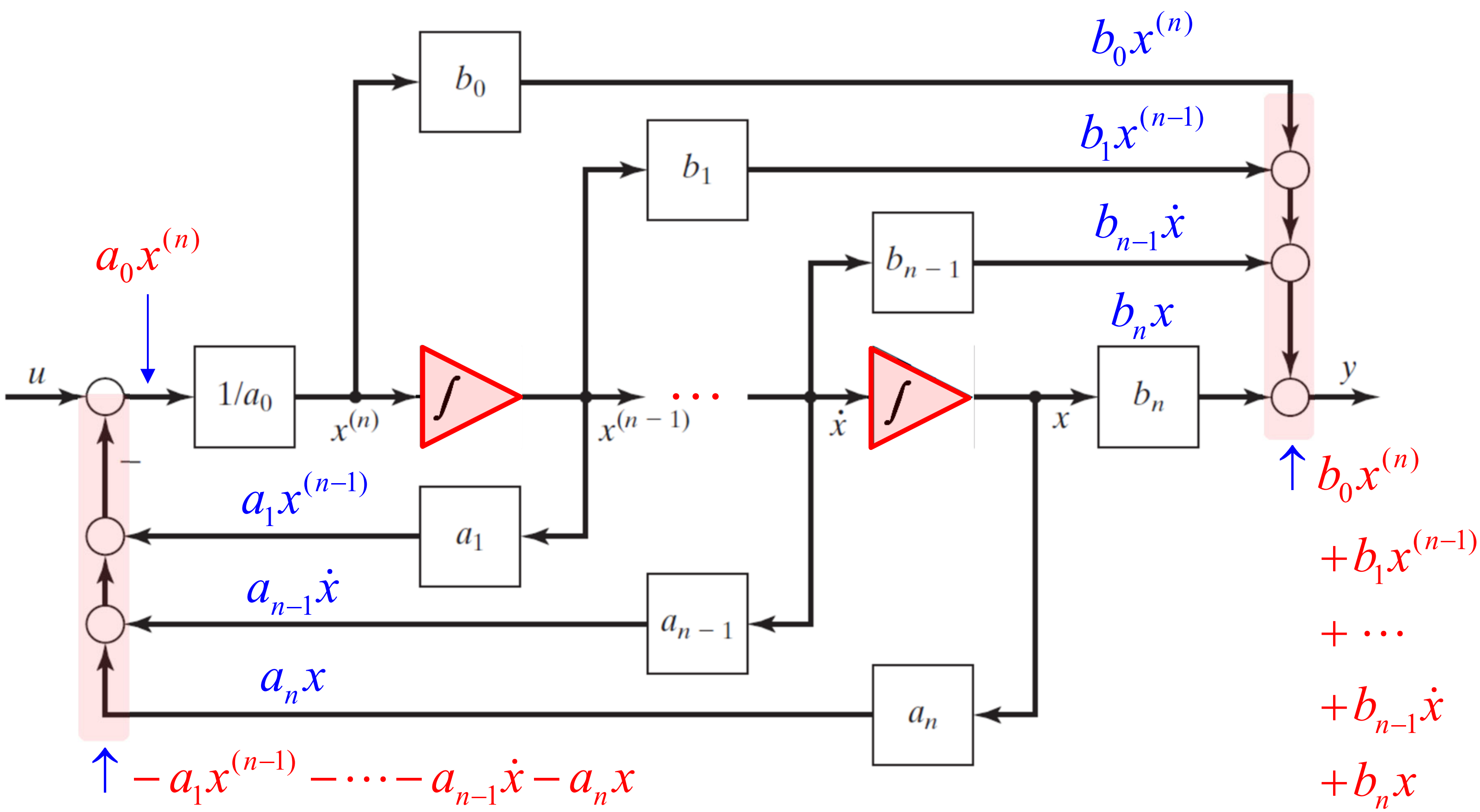
$Y(s)$

Transfer Function *– A linear model in frequency domain*

Let a SISO system be given by a proper n-th order transfer function

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0 s^n+b_1 s^{n-1}+\cdots+b_n}{a_0 s^n+a_1 s^{n-1}+\cdots+a_n}:=\frac{b(s)}{a(s)}, \quad a_0\neq 0$$

A physical realization of the above transfer function is shown in the figure below:



To show this, we note that

$$a_0 x^{(n)}(t) = -a_1 x^{(n-1)}(t) - \cdots - a_n x(t) + u(t)$$

Taking the Laplace transforms (with 0 initial conditions), we obtain

$$a_0 s^n X(s) = -a_1 s^{n-1} X(s) - \cdots - a_n X(s) + U(s)$$

$$\Downarrow$$

$$\left(a_0 s^n + a_1 s^{n-1} + \cdots + a_n\right) X(s) = U(s)$$

$$\Downarrow$$

$$a(s) X(s) = U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}}$$

Also note that

$$y(t) = b_0 x^{(n)}(t) + b_1 x^{(n-1)}(t) + \cdots + b_n x(t)$$

$$\Downarrow$$

$$Y(s) = b_0 s^n X(s) + b_1 s^{n-1} X(s) + \cdots + b_n X(s) = b(s) X(s)$$

Let us define a state variable vector

$$\mathbf{x} = \begin{pmatrix} x^{(n-1)} \\ \vdots \\ \dot{x} \\ x \end{pmatrix}$$

Controller Form Realization

Then the corresponding state space model is given as

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} -\dfrac{a_1}{a_0} & \cdots & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_n}{a_0} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} \dfrac{1}{a_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(t) = A\mathbf{x} + Bu$$

$$y(t) = \begin{pmatrix} b_1 - a_1 \dfrac{b_0}{a_0} & \cdots & b_{n-1} - a_{n-1} \dfrac{b_0}{a_0} & b_n - a_n \dfrac{b_0}{a_0} \end{pmatrix} \mathbf{x}(t) + \frac{b_0}{a_0} u(t) = C\mathbf{x} + Du$$

Exercise: Verify that $G(s) = C(sI - A)^{-1} B + D$.

Another realization of the same $G(s)$ is as follows:

$$\dot{\mathbf{x}}(t)=\begin{pmatrix} -\dfrac{a_1}{a_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{a_{n-1}}{a_0} & 0 & \cdots & 1 \\ -\dfrac{a_n}{a_0} & 0 & \cdots & 0 \end{pmatrix}\mathbf{x}(t)+\begin{pmatrix} b_1-a_1\dfrac{b_0}{a_0} \\ \vdots \\ b_{n-1}-a_{n-1}\dfrac{b_0}{a_0} \\ b_n-a_n\dfrac{b_0}{a_0} \end{pmatrix}u(t)=A\,\mathbf{x}+B\,u$$

$$y(t)=\begin{pmatrix} \dfrac{1}{a_0} & 0 & \cdots & 0 \end{pmatrix}\mathbf{x}(t)+\frac{b_0}{a_0}u(t)=C\,\mathbf{x}+D\,u$$

This realization is called the **observer form realization**. We note that the realization of the transfer function to the state space form is generally non-unique. **There are many forms of realization for any given transfer function!**

Exercise: Verify that $G(s)=C(sI-A)^{-1}B+D$.

Example: Find the controller and observer form realizations of

$$G(s)=\frac{1}{s(s+1)}=\frac{1}{s^2+s}=\frac{0\cdot s^2+0\cdot s+1}{s^2+s+0}$$

tf2ss

The controller form realization is given as

$$\dot{\mathbf{x}}_1=\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}\mathbf{x}_1+\begin{bmatrix}1\\ 0\end{bmatrix}u=A\,\mathbf{x}_1+B\,u,\quad y=\begin{bmatrix}0 & 1\end{bmatrix}\mathbf{x}_1+0\cdot u=C\,\mathbf{x}_1+D\,u$$

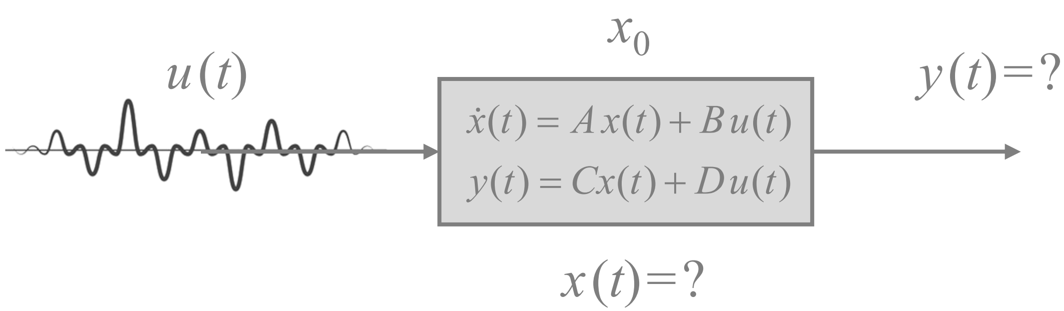
The observer form realization is given as

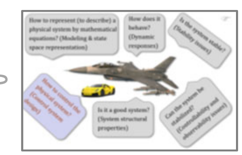
$$\dot{\mathbf{x}}_2=\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}\mathbf{x}_2+\begin{bmatrix}0\\ 1\end{bmatrix}u=A\,\mathbf{x}_2+B\,u,\quad y=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}_2+0\cdot u=C\,\mathbf{x}_2+D\,u$$

We note that $\mathbf{x}_1$ and $\mathbf{x}_2$ are related by the following nonsingular transformation:

$$\mathbf{x}_1=\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}\mathbf{x}_2$$

Dynamical Responses of Linear Systems





We will focus primarily on continuous-time linear time-invariant systems characterized by the following state and output equations:

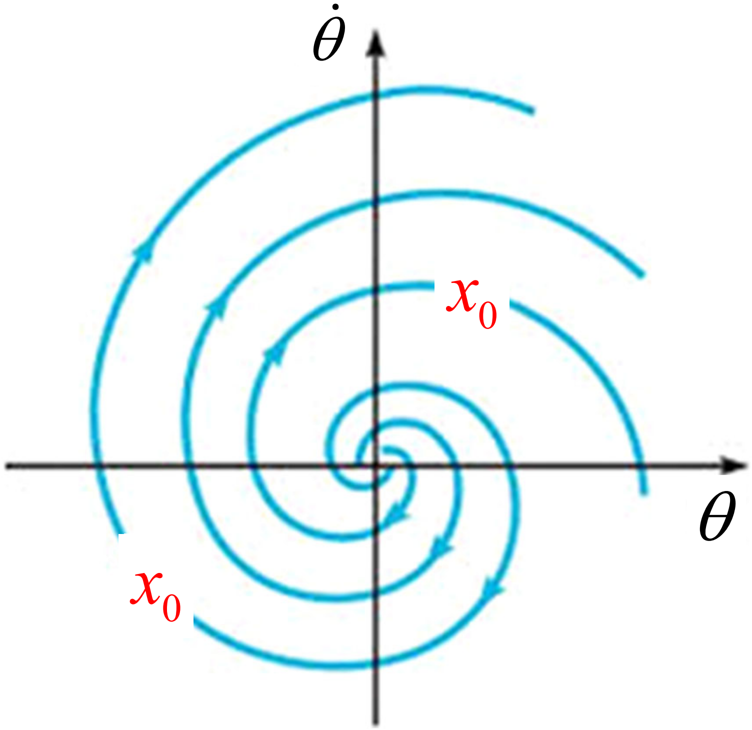
$$\Sigma \ : \ \begin{cases} \dot{x}(t) = A \ x(t) + B \ u(t), \\ y(t) = C \ x(t) + D \ u(t), \end{cases} \tag{3.1.1}$$

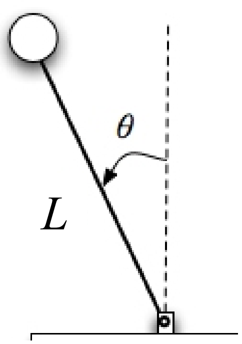
where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the system input, $y(t) \in \mathbb{R}^p$ is the system output, and A, B, C and D are constant matrices of appropriate dimensions. Also, n is referred to the order of the system in (3.1.1), which is used throughout this whole course unless otherwise specified.

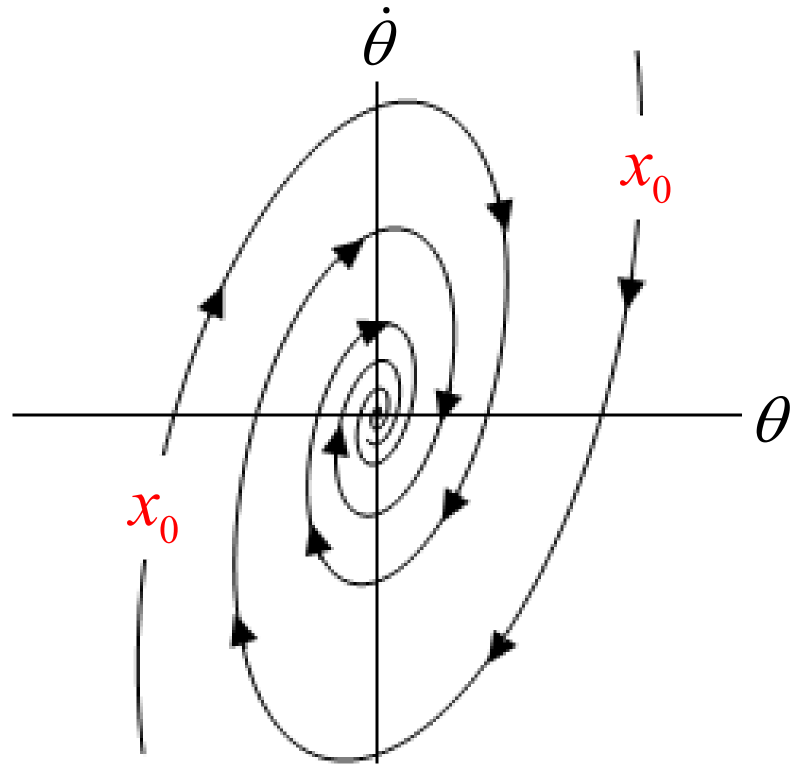
The solution of the state variable or the state response, $x(t)$, of Σ with an initial condition $x_0 = x(0)$ can be uniquely expressed as

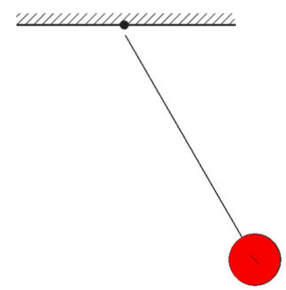
$$x(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-\tau)} B u(\tau) d\tau, \quad t \geq 0, \tag{3.2.1}$$

where the first term is the response due to the initial state, x_0, and the second term is the response excited by the external control force, $u(t)$.









$$x_0 = \begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \end{pmatrix}_{t=0}$$

Phase plane – Illustration of solutions to some 2nd order systems…

To introduce the definition of a matrix exponential function, we derive this result by separating it into the following two cases:

i) the system is free of external input, i.e., $u(t) = 0$; and

ii) the system has a zero initial state, i.e., $x_0 = 0$.

$$x(t) = \underbrace{\mathrm{e}^{At} x_0}_{\text{(i)}} + \underbrace{\int_0^t \mathrm{e}^{A(t-\tau)} B u(\tau) d\tau}_{\text{(ii)}}, \quad t \geq 0, \qquad (3.2.1)$$

(i) Due to initial condition with no external force

(ii) Due to external force with zero initial condition

1. For the case when the external force $u(t) = 0$, the state equation of (3.1.1) reduces to

$$\dot{x} = Ax, \quad x(0) = x_0. \tag{3.2.2}$$

Let the solution to the above autonomous system be expressed as

$$x(t) = \bar{\alpha}_0 + \bar{\alpha}_1 t + \bar{\alpha}_2 t^2 + \cdots = \sum_{k=0}^{\infty} \bar{\alpha}_k t^k, \quad t \geq 0, \tag{3.2.3}$$

where $\bar{\alpha}_k \in \mathbb{R}^n$, $k = 0, 1, \ldots$, are parameters to be determined. Substituting (3.2.3) into (3.2.2), we obtain

$$\begin{aligned} \dot{x}(t) &= \underline{\bar{\alpha}_1} + \underline{2\bar{\alpha}_2}t + \underline{3\bar{\alpha}_3}t^2 + \cdots = Ax \\ &= \underline{A\bar{\alpha}_0} + \underline{A\bar{\alpha}_1}t + \underline{A\bar{\alpha}_2}t^2 + \cdots. \end{aligned} \tag{3.2.4}$$

Since the equality in (3.2.4) has to be true for all $t \geq 0$, we have

$$\bar{\alpha}_1 = A\bar{\alpha}_0, \quad \bar{\alpha}_2 = \frac{1}{2}A\bar{\alpha}_1 = \frac{1}{2!}A^2\bar{\alpha}_0, \quad \bar{\alpha}_3 = \frac{1}{3}A\bar{\alpha}_2 = \frac{1}{3!}A^3\bar{\alpha}_0,$$

and in general,

$$\bar{\alpha}_k = \frac{1}{k!}A^k\bar{\alpha}_0, \quad k = 0, 1, 2, \ldots, \tag{3.2.5}$$

which together with the given initial condition imply

$$\underline{x(t)} = \left(\sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k\right)\bar{\alpha}_0 = \underline{\mathrm{e}^{At}x_0}, \quad t \geq 0, \tag{3.2.6}$$

where

$$\mathrm{e}^{At} := \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k. \tag{3.2.7}$$

Properties of matrix exponential

It is straightforward to verify that

expm

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \right)$$

$$= A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = A\left(I + At + \frac{1}{2!}A^2t^2 + \cdots \right) = Ae^{At}$$

$$= \left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \right)A = e^{At}A \qquad (3.2.8)$$

exchangeable

For every $t, \tau \in \mathbb{R}$,

$$e^{At}e^{A\tau} = e^{A(t+\tau)}.$$

For every $t \in \mathbb{R}$, e^{At} is nonsingular and

$$\left(e^{At}\right)^{-1} = e^{-At}.$$

♣ **Jordan Canonical Form:** For every $n \times n$ matrix, there exists a non-singular similarity transformation P such that

$$J = PAP^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_\ell \end{bmatrix},$$

where each J_i *is a* Jordan block *of the form*

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

where each λ_i is an eigenvalue of A, and the number ℓ of Jordan blocks is equal to the total number of independent eigenvectors of A.

♣ **Example:** Given a matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

Camille Jordan
1838–1922
French Mathematician

its Jordan canonical form is given by

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad J = PAP^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

jcf
ex1037

Matrix A has three eigenvalue at $\lambda = 2$, but with only one independent eigenvector.

Note: It can be computed using an m-function JCF in Linear Systems Toolkit.

♣ **Example:** Given a matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

> 0 (i.e., positive definite) as it is symmetric and has all its eigenvalues > 0 !

which is symmetric. Its eigenvalues are given by

$$\lambda_1 = 0.468, \quad \lambda_2 = 1.653, \quad \lambda_3 = 3.879$$

rjd
ex1040

We can find a non-singular transformation such that

$$P = \begin{bmatrix} -0.449 & -0.293 & -0.844 \\ 0.844 & -0.449 & -0.293 \\ -0.293 & -0.844 & 0.449 \end{bmatrix} \Rightarrow J = PAP^{-1} = \begin{bmatrix} 0.468 & 0 & 0 \\ 0 & 1.653 & 0 \\ 0 & 0 & 3.879 \end{bmatrix}$$

Note: It can be computed using an m-function JCF/RJD in Linear Systems Toolkit.

Properties of matrix exponential (cont.)

For the case when A is a **diagonal matrix**, i.e.,

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \quad \Rightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

When A is given by a **Jordan block**, i.e.,

$$A = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix} \quad \Rightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}/2! & \cdots & t^{n-1} e^{\lambda t}/(n-1)! \\ 0 & e^{\lambda t} & te^{\lambda t} & \cdots & t^{n-2} e^{\lambda t}/(n-2)! \\ & & e^{\lambda t} & \cdots & t^{n-3} e^{\lambda t}/(n-3)! \\ & 0 & & \ddots & \vdots \\ & & & & e^{\lambda t} \end{bmatrix}$$

2. For the case when the system (3.1.1) has a zero initial condition, *i.e.*, $x_0 = 0$, but with a nonzero external input, $u(t)$, we consider the following equality:

$$\frac{d}{dt}\left(\mathrm{e}^{-At}x\right) = \frac{d\mathrm{e}^{-At}}{dt}x + \mathrm{e}^{-At}\dot{x} = -e^{-At}Ax + e^{-At}\dot{x}$$

$$= \mathrm{e}^{-At}(\dot{x} - Ax) = \mathrm{e}^{-At}Bu(t). \quad (3.2.9)$$

Integrating both sides of (3.2.9), we obtain

$$\mathrm{e}^{-At}x(t) - x_0 = \mathrm{e}^{-At}x(t) = \int_0^t \mathrm{e}^{-A\tau}Bu(\tau)d\tau, \qquad (3.2.10)$$

which implies that

$$x(t) = \mathrm{e}^{At}\int_0^t \mathrm{e}^{-A\tau}Bu(\tau)d\tau = \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau. \qquad (3.2.11)$$

By the superposition property of linear systems, we obtain the solution of the state response of the given system in (3.1.1) as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad t \geq 0, \tag{3.2.1}$$

Moreover, the uniqueness of the above solution to (3.1.1) with an initial condition $x(0) = x_0$ can be shown as follows: Suppose x_1 and x_2 are the solutions to (3.1.1) with $x_1(0) = x_2(0) = x_0$. Let $\tilde{x}(t) = x_1(t) - x_2(t)$, and thus $\tilde{x}_0 = \tilde{x}(0) = 0$. We have

$$\dot{\tilde{x}} = \dot{x}_1 - \dot{x}_2 = Ax_1 + Bu - Ax_2 - Bu = A\tilde{x}. \tag{3.2.12}$$

It follows from (3.2.6) that $\tilde{x}(t) = e^{At}\tilde{x}_0 \equiv 0$, *i.e.*, $x_1(t) \equiv x_2(t)$ for all $t \geq 0$.

Lastly, it is simple to see that the corresponding output response of the system (3.1.1) is given as:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t), \quad t \geq 0. \tag{3.2.13}$$

The term *zero-input response* refers to output response due to the initial state and in the absence of an input signal. The terms *unit step response* and the *impulse response*, for the continuous-time system (3.1.1) respectively refer to the output responses of (3.2.13) with zero initial conditions to the input signals,

$$u(t) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad u(t) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \delta(t), \tag{3.2.14}$$

where $\delta(t)$ is a unit *impulse function*.

Recap:

Property of an impulse function $\delta(t)$ — for any continuous function $f(t)$,

$$f(t)\delta(t-a) = f(a)\delta(t) \quad \text{and} \quad \int_c^d f(t)\delta(t-a)dt = \begin{cases} f(a) & \text{if } c < a < d \\ 0 & \text{otherwise} \end{cases}$$

Example: Find the state and output responses of the following system

ss
ex1089

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x(0) = x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

with u being a unit step function, i.e., $u(t) = 1$, and a impulse function $u(t) = \delta(t)$.

Solution: By the property of matrix exponential, we have

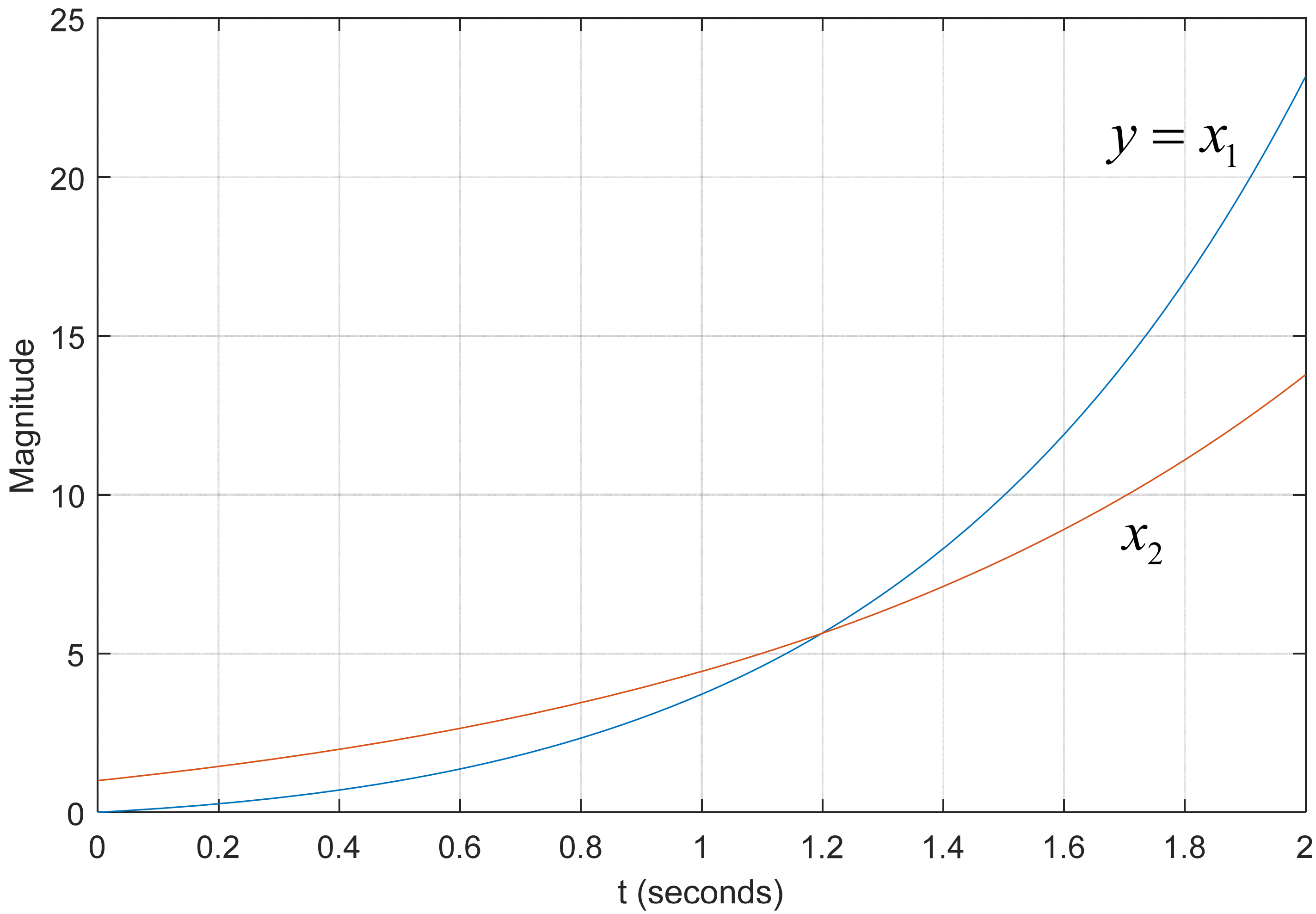
$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \quad \Rightarrow \quad e^{At} x_0 = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} te^t \\ e^t \end{pmatrix}$$

$$\Rightarrow \quad e^{A(t-\tau)} B u(\tau) = \begin{bmatrix} e^{t-\tau} & (t-\tau)e^{t-\tau} \\ 0 & e^{t-\tau} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 = \begin{pmatrix} (t-\tau)e^{t-\tau} \\ e^{t-\tau} \end{pmatrix}$$ for a unit step function.

and $$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \begin{pmatrix} (2t-1)e^t + 1 \\ 2e^t - 1 \end{pmatrix}, \quad y(t) = (2t-1)e^t + 1$$

…unit step state response… unit step output response…

State response due to a unit step input…



initial
step

State response due to a unit impulse input can be calculated as follows: Noting

$$e^{At}x_0 = \begin{pmatrix} te^t \\ e^t \end{pmatrix}$$

and

$$e^{A(t-\tau)}Bu(\tau) = \begin{bmatrix} e^{t-\tau} & (t-\tau)e^{t-\tau} \\ 0 & e^{t-\tau} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \delta(\tau) = \begin{pmatrix} (t-\tau)e^{t-\tau} \\ e^{t-\tau} \end{pmatrix} \delta(\tau)$$

we have

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{pmatrix} te^t \\ e^t \end{pmatrix} + \int_{0^-}^t \begin{pmatrix} (t-\tau)e^{t-\tau} \\ e^{t-\tau} \end{pmatrix} \delta(\tau)d\tau$$

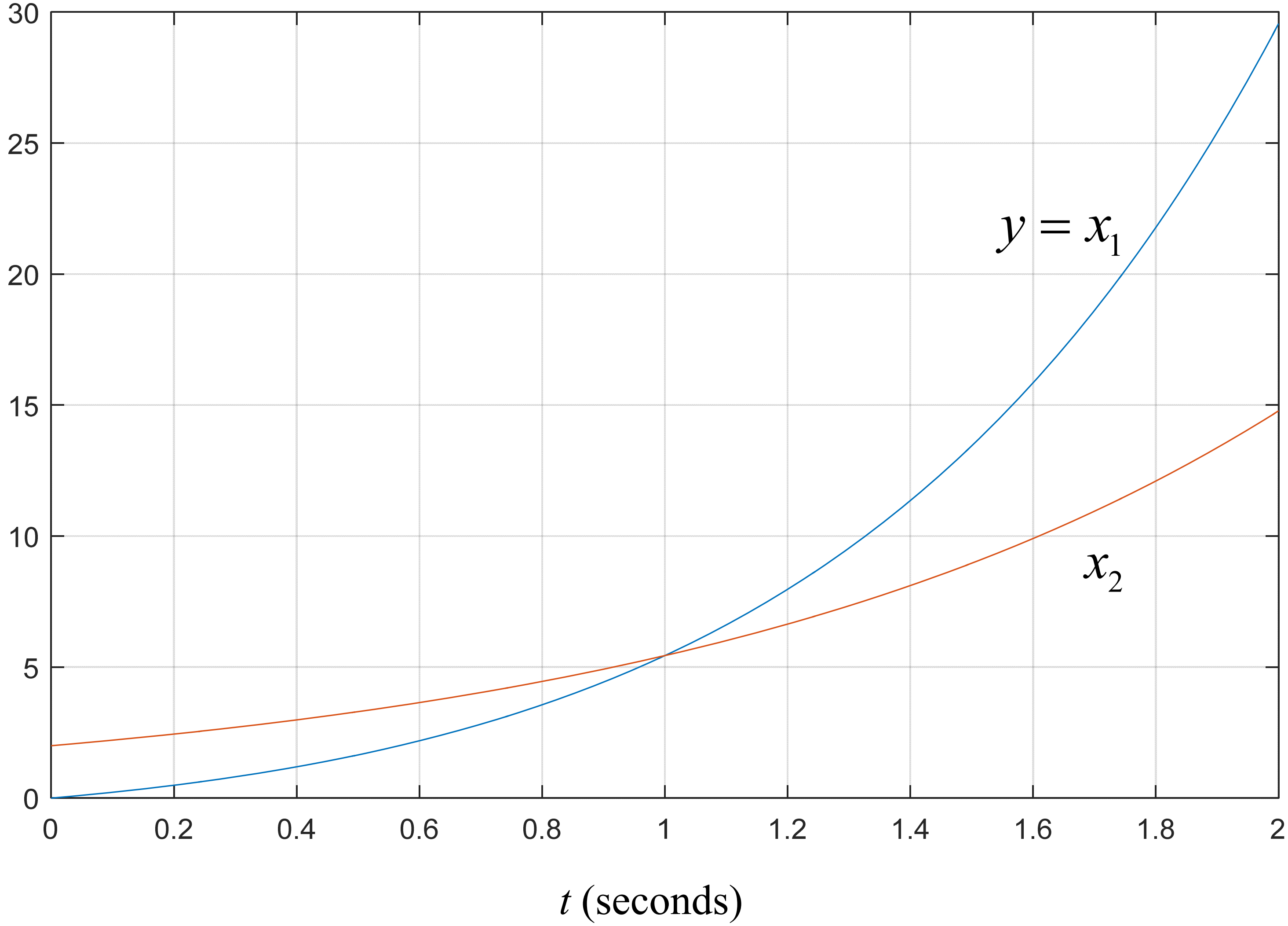
$$= \begin{pmatrix} te^t \\ e^t \end{pmatrix} + \begin{pmatrix} te^t \\ e^t \end{pmatrix} = \begin{pmatrix} 2te^t \\ 2e^t \end{pmatrix}$$ …unit impulse state response…

$$y(t) = 2te^t$$ …unit impulse output response…

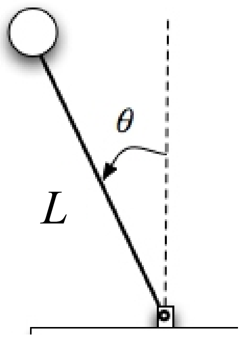
$$\int_c^d f(t)\delta(t-a)dt = f(a)$$

State response due to a unit impulse input…

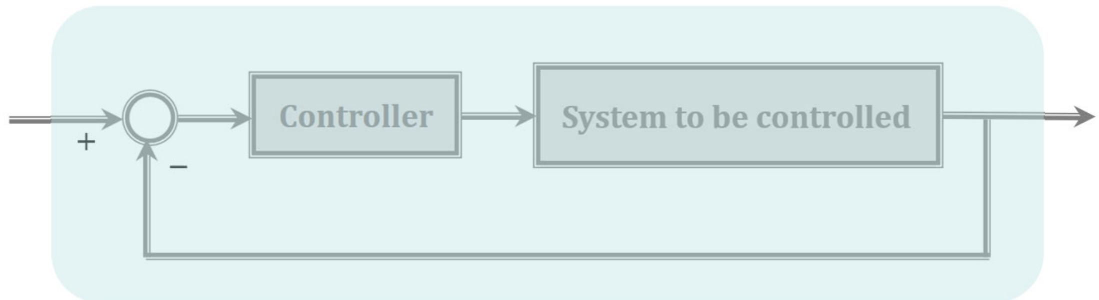
impulse

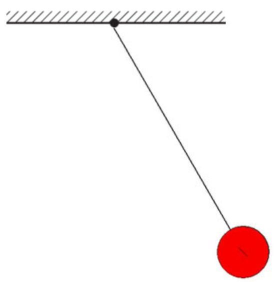


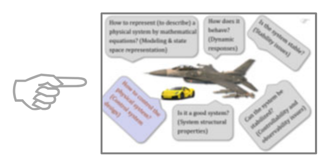
System Stability



$$\dot{x}(t) = A\,x(t)$$







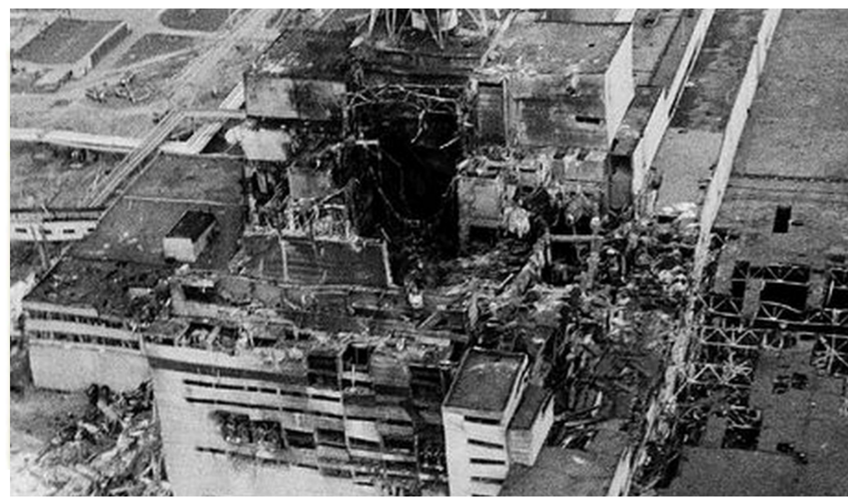
FEATURE 2003

Respect the Unstable

The practical, physical (and sometimes dangerous) consequences of control must be respected, and the underlying principles must be clearly and well taught.

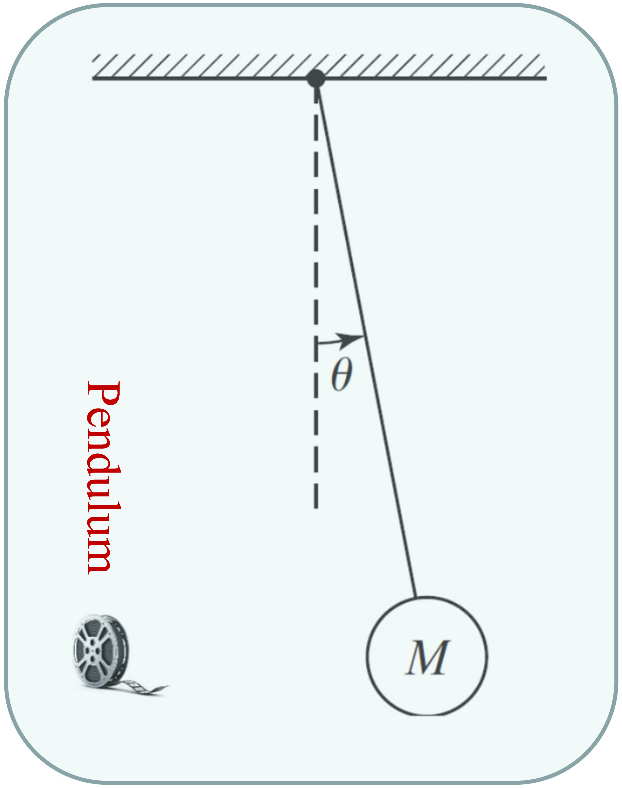
By Gunter Stein

Feedback contro all around us in mod ical life. They are at wo omes, our cars, our factories, our transportation systems, our defense systems—everywhere we look. Certainly, one of the great achievements of the international controls research community is that the

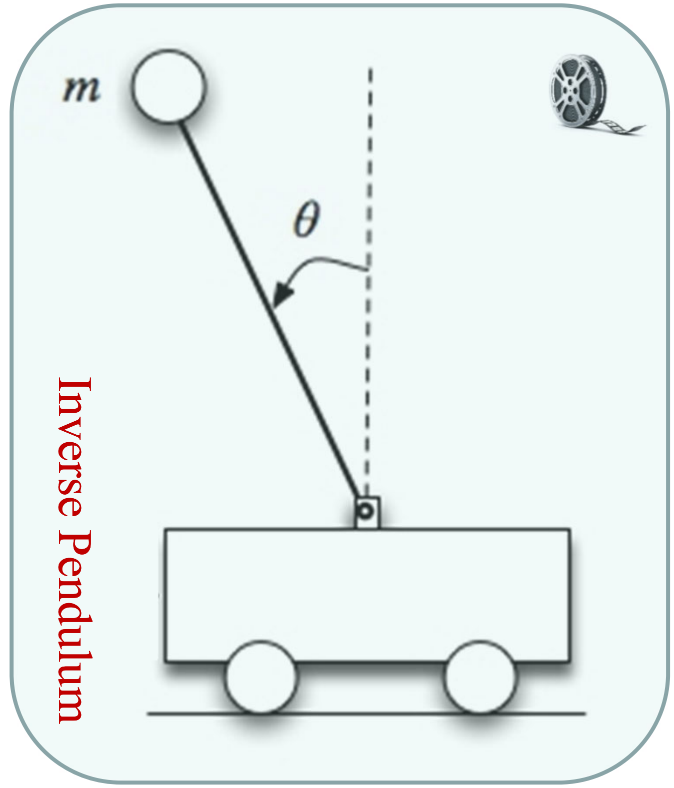


Chernobyl Nuclear Disaster

Examples of system stability…



Stable?



Stable?

Bounded-input bounded output (BIBO) stability

A system is said to be BIBO stable if for any bounded input $u(t)$, i.e.,

$$|u(t)| \leq u_m < \infty, \quad \text{for all } t \geq 0$$

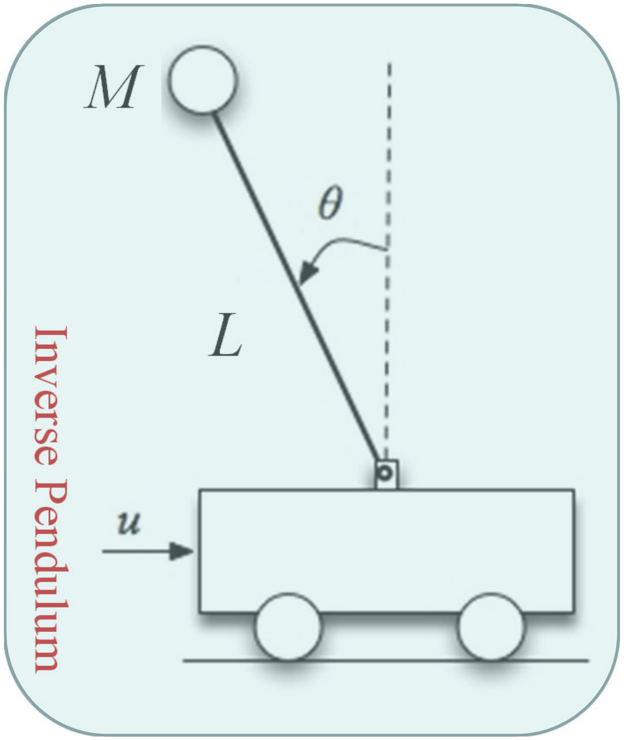
the corresponding output $y(t)$ is also bounded.

For a continuous-time linear time-invariant (LTI) system, the condition for BIBO stability is that its impulse response, $h(t)$, be absolutely integrable, i.e., its L_1 norm exists:

$$\int_0^\infty |h(t)|\, dt = \|h\|_1 < \infty$$

Note that BIBO stability is only applicable when the system is initially relaxed, i.e., with initial condition being 0.

Example: The linear model of the inverse pendulum system around $\theta_0 = \pi$ is



$$\begin{pmatrix} \dot{\tilde{\theta}} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ML^2} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix}$$

which has a transfer function

$$G(s) = \frac{1/ML^2}{s^2 - g/L} = \frac{2}{s^2 - 1}$$

Set $1/ML^2 = 2, \; g/L = 1$ for simplicity

The impulse response $h(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{2}{s^2-1}\right] = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = e^t - e^{-t}$

$$\Rightarrow \quad \int_0^\infty |h(t)|\,dt \ge \int_0^\infty e^t\,dt - \int_0^\infty e^{-t}\,dt = \infty \quad \Rightarrow \quad \text{the system is not BIBO stable.}$$

Internal stability

Stability, more specifically internal stability, is always a primary issue in designing a meaningful control system. For linear systems, either the continuous-time system (3.1.1), *e.g.*, $\dot{x}(t) = A\,x(t), \quad x(0) = x_0$, the notion of internal stability of the system is related to the behavior of its state trajectory in the absence of the external input, u. Thus, the internal stability is related to the trajectory of

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{3.3.1}$$

The system (3.1.1) is said to be *marginally stable* or *stable in the sense of Lyapunov* or simply *stable* if the state trajectory corresponding to every bounded initial condition x_0 is bounded. It is said to be *asymptotically stable* if it is stable and, in addition, for any initial condition, the corresponding state trajectory $x(t)$ of (3.3.1) satisfies,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \mathrm{e}^{At} x_0 = 0. \tag{3.3.3}$$

Note: For LTI systems, asymptotic stability and exponential stability are equivalent.

It is straightforward to verify that the continuous-time linear system (3.1.1) or (3.3.1) is stable if and only if all the eigenvalues of A are in the closed left-half complex plane with those on the $j\omega$ axis having Jordan blocks of size 1. It is asymptotically stable if and only if all the eigenvalues of A are in the open left-half complex plane, *i.e.*, $\lambda(A) \subset \mathbb{C}^-$. This can be shown by first transforming A into a Jordan canonical form, say

$$J = P^{-1}AP = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix}, \tag{3.3.4}$$

where $P \in \mathbb{C}^{n \times n}$ is a nonsingular matrix, and

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}, \quad i = 1, 2, \ldots, q. \tag{3.3.5}$$

We proceed to define a non-singular state transformation

$$x = P\tilde{x} \quad \Rightarrow \quad \dot{\tilde{x}} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}AP\tilde{x} = J\tilde{x}, \quad \tilde{x}_0 = P^{-1}x_0$$

It follows from (3.2.6) that the solution to the transformed system is given by

$$\tilde{x}(t) = e^{Jt}\tilde{x}_0 \quad \Rightarrow \quad x(t) = P\tilde{x}(t) = Pe^{Jt}\tilde{x}_0 = \left(Pe^{Jt}P^{-1}\right)x_0 = e^{At}x_0$$

Alternatively, we note that

$$A = PJP^{-1} \quad \Rightarrow \quad A^k = \left(PJP^{-1}\right)\left(PJP^{-1}\right)\cdots\left(PJP^{-1}\right) = PJ^kP^{-1}$$

which implies

$$e^{At} = \sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k = \sum_{k=0}^{\infty}\frac{1}{k!}PJ^kP^{-1}t^k = P\left(\sum_{k=0}^{\infty}\frac{1}{k!}J^k t^k\right)P^{-1} = Pe^{Jt}P^{-1}$$

Thus,

$$x(t) = e^{At}x_0 = \left(Pe^{Jt}P^{-1}\right)x_0$$

Then, we have

$$e^{At} = P e^{Jt} P^{-1} = P \begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_q t} \end{bmatrix} P^{-1}, \tag{3.3.6}$$

where

$$e^{J_i t} = \left[\begin{array}{cc|ccc} e^{\lambda_i t} & t e^{\lambda_i t} & \cdots & t^{n_i - 1} e^{\lambda_i t}/(n_i - 1)! \\ 0 & e^{\lambda_i t} & \cdots & t^{n_i - 2} e^{\lambda_i t}/(n_i - 2)! \\ \hline \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_i t} \end{array}\right], \tag{3.3.7}$$

$i = 1, 2, \ldots, q.$

We note that the result in (3.3.7) follows from the properties of matrix exponential given earlier.

Take note on the off-diagonal elements in (3.3.7), which are functions of *t* powers!

It is now clear that $e^{J_i t} \to 0$ as $t \to \infty$ if and only if $\lambda_i \in \mathbb{C}^-$, and thus

$$\lim_{t\to\infty} e^{At} x_0 = P \begin{bmatrix} \lim\limits_{t\to\infty} e^{J_1 t} & & & \\ & \lim\limits_{t\to\infty} e^{J_2 t} & & \\ & & \ddots & \\ & & & \lim\limits_{t\to\infty} e^{J_q t} \end{bmatrix} P^{-1} x_0 = 0, \quad (3.3.8)$$

for any $x_0 \in \mathbb{R}^n$, if and only if $\lambda_i \in \mathbb{C}^-$, $i = 1, 2, \ldots, q$, or $\lambda(A) \subset \mathbb{C}^-$. On the other hand, the solutions remain bounded for all initial conditions if and only if $\lambda(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$ and $n_i = 1$ for $\lambda_i(A) \in \mathbb{C}^0$.

In other words, the given system is asymptotically stable, i.e., the state trajectory converts to zero as time progresses, if and only if all the eigenvalues of A (which are also called the poles of the given system or A) are on the open left-half complex plane. The given is marginally stable if and only if all the eigenvalues or poles of A are on the closed left-half plane with those on the imaginary axis being simple (why?).

Remark: The BIBO stability does not imply internal stability as it can be seen from the following simple example:

$$\dot{x} = 1 \cdot x + 0 \cdot u, \quad y = x$$

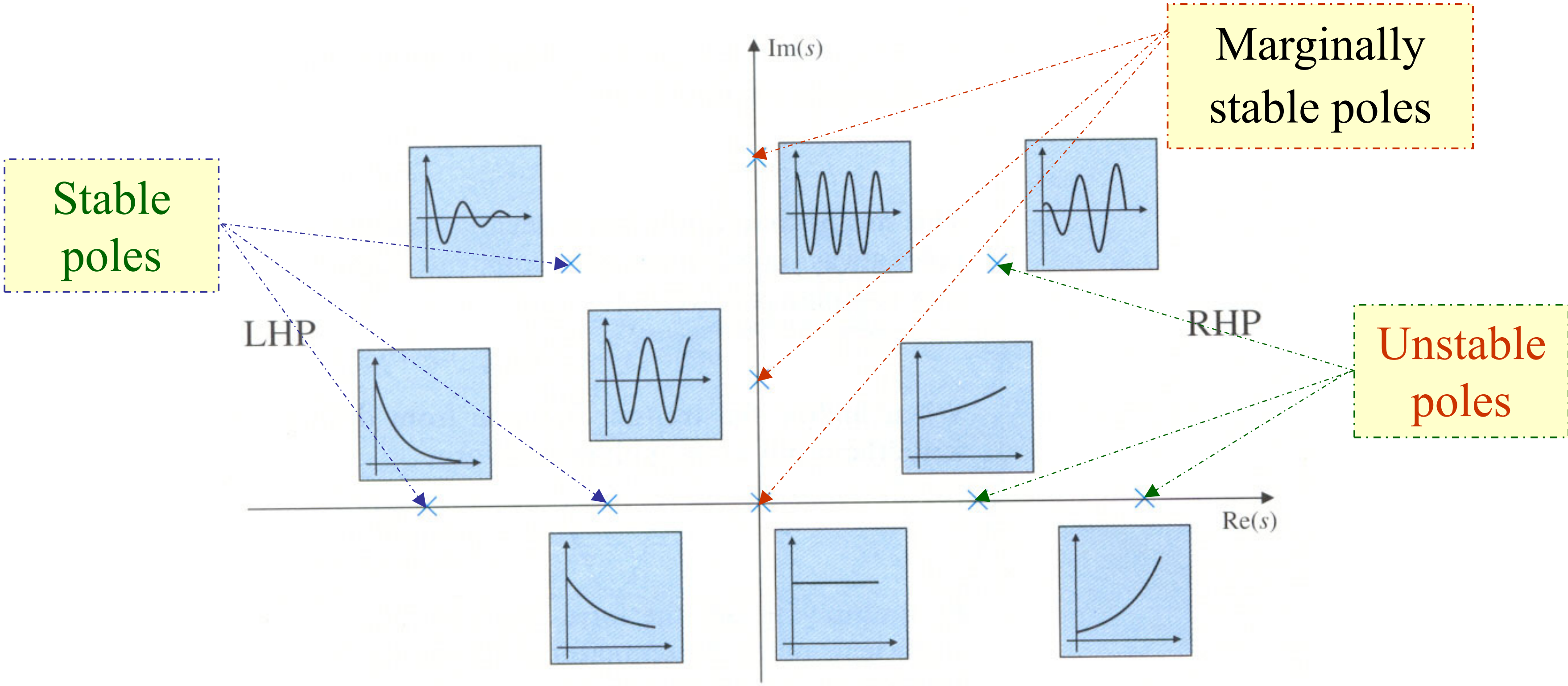
which is a BIBO stable system, but not internally stable as it has an unstable pole at $s = 1$. Any non-zero initial condition will cause the state (and output) variable blowing up to infinity.

On the other hand, the internal (asymptotic or exponential) stability of an LTI system does imply its BIBO stability. This can be shown by finding its impulse response of the system and showing that the L_1 norm of the impulse output response is bounded and hence the system is BIBO stable.

However, it will be shown by a counterexample (Q.6 in Homework Assignment 1) that the marginally internal stability does not guarantee the given system is BIBO stable. In fact, we can show that a marginally internally stable system is always BIBO unstable.

Summary of internal stability

A linear time-invariant system is said to be asymptotically stable if all its poles are located on the left-half complex plane (LHP), marginally stable if all its poles are in closed LHP with those on imaginary axis being simple, and unstable otherwise…



$$e^{\lambda t} = e^{(\sigma + j\omega)t} = e^{\sigma t}\left(\cos \omega t + j \sin \omega t\right)$$

Lyapunov stability of dynamical systems

Consider a general dynamic system, $\dot{x} = f(x)$ with $f(0) = 0$.

If there exists a so-called Lyapunov function $V(x)$, which satisfies the following conditions:

Aleksandr Lyapunov
1857–1918

1. $V(x)$ is continuous in x and $V(0) = 0$;
2. $V(x) > 0$ (positive definite);
3. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$ (negative definite),

then we can say that the system is asymptotically stable at $x = 0$. If in addition,

$$V(x) \to \infty, \quad \text{as} \quad \|x\| \to \infty$$

then we can say that the system is globally asymptotically stable at $x = 0$. In this case, the stability is independent of the initial condition $x(0)$.

Lyapunov stability for LTI systems

The following result is particularly useful for stability analysis when numerical values of system matrix are unknown. It will be used in coming lectures when we deal with control systems design.

Theorem 3.3.1. *The continuous-time system of (3.3.1) is asymptotically stable if and only if for any given positive definite matrix $Q = Q' \in \mathbb{R}^{n\times n}$, the Lyapunov equation*

$$A'P + PA = -Q \tag{3.3.9}$$

has a unique and positive definite solution $P = P' \in \mathbb{R}^{n\times n}$.

We note that unlike Lyapunov stability theory for general dynamical systems on the previous page, Theorem 3.3.1 gives a necessary and sufficient condition for the stability of LTI systems.

The result of Theorem 3.3.1 also holds for $Q \geq 0$ and (A, Q) being observable (the concept of observability is to be studied in the next section).

Proof. The asymptotic stability of the system implies that all eigenvalues of A have negative real parts. Thus, the following matrix is well defined,

$$P = \int_0^\infty \mathrm{e}^{A't} Q \mathrm{e}^{At} dt. \tag{3.3.10}$$

In what follows, we will show that such a P is the unique solution to the Lyapunov equation (3.3.9) and is positive definite.

First, substitution of (3.3.10) in (3.3.9) yields

$$\begin{aligned} A'P + PA &= \int_0^\infty A'\mathrm{e}^{A't} Q \mathrm{e}^{At} dt + \int_0^\infty \mathrm{e}^{A't} Q \mathrm{e}^{At} A dt \\ &= \int_0^\infty \frac{d}{dt}\left(\mathrm{e}^{A't} Q \mathrm{e}^{At}\right) dt \\ &= \mathrm{e}^{A't} Q \mathrm{e}^{At}\Big|_{t=0}^{\infty} \\ &= -Q, \end{aligned}$$

$$\frac{d}{dt}\mathrm{e}^{At} = A\mathrm{e}^{At} = \mathrm{e}^{At}A$$

where we have used the fact that $\mathrm{e}^{At} \to 0$ as $t \to \infty$. This shows that P as defined in (3.3.10) is indeed a solution to (3.3.9).

To show that the solution (3.3.9) is unique, let P_1 and P_2 both be a solution, *i.e.*,

$$A'P_1 + P_1A = -Q, \tag{3.3.12}$$

and

$$A'P_2 + P_2A = -Q. \tag{3.3.13}$$

Subtracting (3.3.13) from (3.3.12) yields

$$A'(P_1 - P_2) + (P_1 - P_2)A = 0, \tag{3.3.14}$$

which implies that

$$\mathrm{e}^{A't}A'(P_1-P_2)\mathrm{e}^{At} + \mathrm{e}^{A't}(P_1-P_2)A\mathrm{e}^{At} = \frac{d}{dt}\mathrm{e}^{A't}(P_1-P_2)\mathrm{e}^{At} = 0. \tag{3.3.15}$$

Integration of (3.3.15) from $t=0$ to ∞ yields

$$\mathrm{e}^{A't}(P_1 - P_2)\mathrm{e}^{At}\Big|_{t=0}^{\infty} = P_1 - P_2 = 0. \tag{3.3.16}$$

This shows that P as defined in (3.3.10) is the unique solution to the Lyapunov equation (3.3.9).

It is clear that this P is symmetric since Q is. The positive definiteness of P follows from the fact that, for any nonzero $x \in \mathbb{R}^n$,

$$x'Px = \int_0^\infty x' \mathrm{e}^{A't} Q \mathrm{e}^{At} x dt > 0, \tag{3.3.17}$$

which in turn follows from the facts that Q is positive definite and that e^{At} is nonsingular for any t.

Conversely, for any $Q > 0$, if the Lyapunov equation has a solution $P > 0$, we define a Lyapunov function

$$V(x) = x' P x$$

which obviously a continuous function in x and positive definite, and

$$\dot{V}(x) = \dot{x}' P x + x' P \dot{x} = (Ax)' P x + x' P A x = x'(A'P + PA)x = -x'Qx < 0$$

Furthermore,

$$V(x) \to \infty, \quad \text{as} \quad \|x\| \to \infty$$

By the Lyapunov stability theorem, $\dot{x} = Ax$ is asymptotically stable.

On the other hand, we can prove the result by directly determining the locations of the eigenvalues of matrix A. If there are positive definite P and Q that satisfy the Lyapunov equation (3.3.9), i.e.,

$$A'P + PA = -Q \tag{3.3.9}$$

then all eigenvalues of matrix A have negative real parts, and thus it is stable. We let λ be an eigenvalue of A with an associated eigenvector $v \neq 0$, *i.e.*,

$$Av = \lambda v,$$

which also implies that

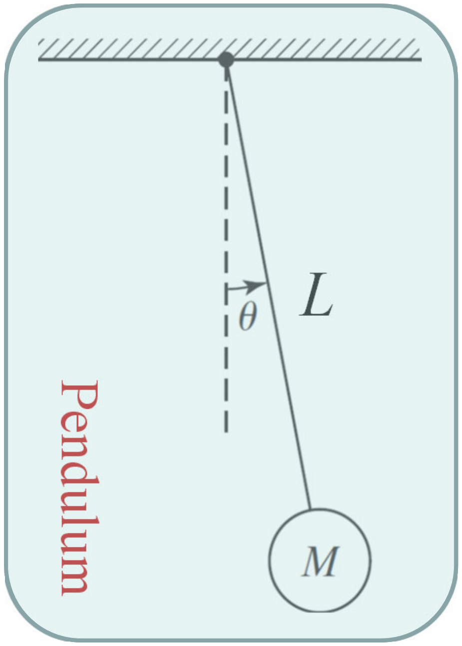
$$v^* A' = \lambda^* v^*.$$

Pre-multiplying and post-multiplying (3.3.9) by v^* and v respectively yields

$$-v^* Q v = v^* A' P v + v^* P A v = (\lambda^* + \lambda) v^* P v = 2\mathrm{Re}(\lambda) v^* P v,$$

which implies that $\mathrm{Re}(\lambda) < 0$, as both P and Q are positive definite. ■

Example: The linear model of the pendulum system around $\theta_0 = 0$ is



$$\begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L} & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

The eigenvalues of the system matrix are

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ \dfrac{g}{L} & \lambda \end{vmatrix} = \lambda^2 + \frac{g}{L} \quad \Rightarrow \quad \lambda_{1,2} = \pm j\sqrt{\frac{g}{L}}$$

The above pendulum system has two simple poles on the imaginary axis of the complex plane. It is thus a marginally stable system.

> Recall that it was showed earlier that the inverse pendulum system is BIBO unstable. It is easy to verify that the system matrix of the inverse pendulum has two poles at $\pm\sqrt{g/L}$. Clearly, it is an internally unstable system.

Example: Consider an LTI system

$$\dot{x} = A\,x = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -3 \end{bmatrix} x$$

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eig
ex1112

whose system matrix A has eigenvalues at

$$\lambda_1 = -3.7321, \quad \lambda_2 = -3, \quad \lambda_3 = -0.2679,$$

respectively. The system is clearly stable. Let

$$Q = I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

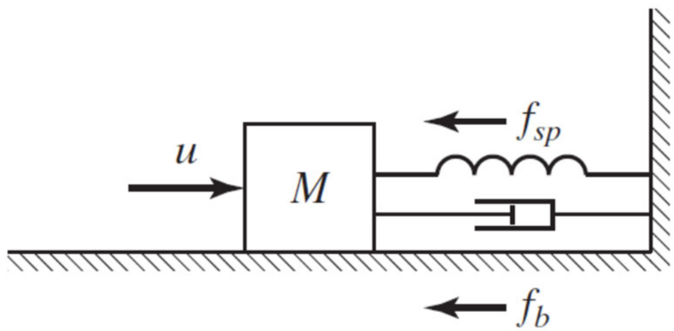
The solution to the corresponding Lyapunov equation $A'P + PA = -Q$ is given by

$$P = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 3 \end{bmatrix} > 0 \quad \Rightarrow \quad \text{the given system is stable!}$$

Homework Assignment 1 (due in two weeks)

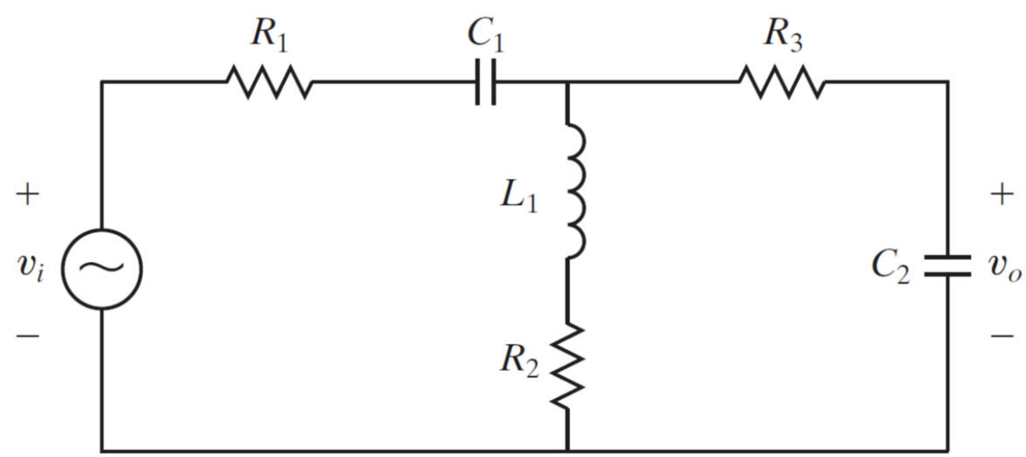
Q.1. Consider the mechanical system shown in the figure below. Here $u(t)$ is an external force applied to the mass M, $y(t)$ is the displacement of the mass with respect to the position when the spring is relaxed. The spring force and friction force are given respectively by

$$f_{sp}(t) = k(1 + ay^2(t))y(t), \qquad f_b(t) = b\dot{y}(t).$$



1. Write the differential equation model of this system.
2. Write a state space description of the system.
3. Is the system linear? If it is not linear, linearize it around the operating point with $u_0 = 0$.
4. Find the transfer function of the linearized system.

Q.2. Consider the electric circuit network in the figure below. Let the input be $v_i(t)$ and output be $v_o(t)$.

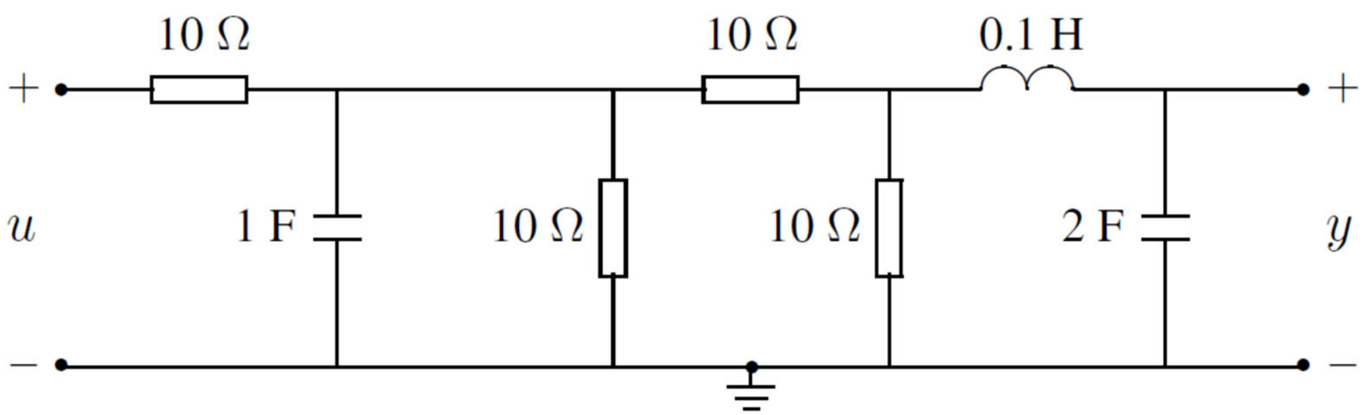


1. Derive the state and output equation of the network.
2. Find the transfer function of the network.

Assuming that $R_1 = R_2 = R_3 = 1\,\Omega$, $C_1 = C_2 = 1\,\text{F}$ and $L_1 = 1\,\text{H}$,

3. Find the unit step response of the network.
4. Find the unit impulse response of the network.

Q.3. Consider an electric network shown in the circuit below with its input, u, being a voltage source, and output, y, being the voltage across the 2 F capacitor. Assume that the initial voltages across the 1 F and 2 F capacitors are 1 V and 2 V, respectively, and that the inductor is initially uncharged.



(a) Derive the state and output equations of the network.

(b) Find the unit step response of the network.

(c) Find the unit impulse response of the network.

(d) Determine the stability of the network.

Q.4. Given a linear system, $\dot{x} = Ax + Bu$, with $x(t_1) = x_1$ and $x(t_2) = x_2$ for some $t_1 > 0$ and $t_2 > 0$, show that

$$\int_{t_1}^{t_2} \mathrm{e}^{-A\tau} Bu(\tau) d\tau = \mathrm{e}^{-At_2} x_2 - \mathrm{e}^{-At_1} x_1.$$

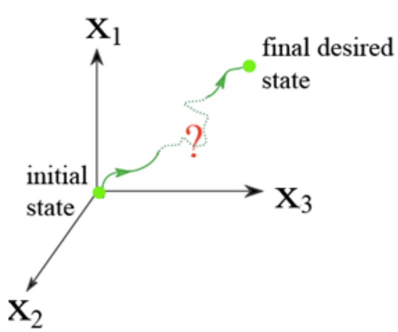
Q.5. Given

$$\mathrm{e}^{At} = \begin{bmatrix} -\mathrm{e}^{-t} + \alpha \mathrm{e}^{-2t} & -\mathrm{e}^{-t} + \beta \mathrm{e}^{-2t} \\ 2\mathrm{e}^{-t} - 2\mathrm{e}^{-2t} & 2\mathrm{e}^{-t} - \mathrm{e}^{-2t} \end{bmatrix},$$

determine the values of the scalars α and β, and the matrices A and A^{100}.

Q.6. Show that the pendulum system is a BIBO unstable system even though it was proved to be internally marginally stable. Identify a bounded input signal such that when it is applied to the pendulum, the resulting output response will go unbounded.

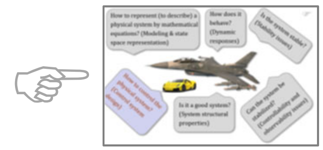
For simplicity, you can assume that $ML^2 = 1$ and $g = L$.



$u(t)$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$y(t)$



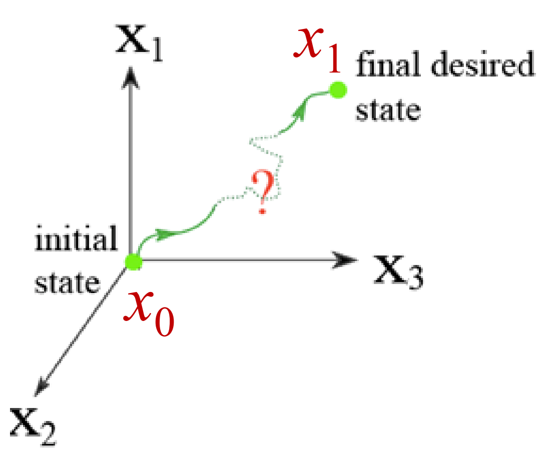
Controllability and Observability

Controllability and stabilizability

Let us first focus on the issue of controllability. The concept of controllability is about controlling the state trajectory of a given system through its input. Simply stated, a system is said to be controllable if its state can be controlled in the state space from any point to any other point through an appropriate control input within a finite time interval. For a linear time-invariant system, it is equivalent to controlling the state trajectory from an arbitrary point to the origin of the state space. To be more precise, we consider the following continuous-time system:

$$\Sigma \ : \ \dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{3.4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.



The system (3.4.1) is said to be controllable if for every $x_1 \in \mathbb{R}^n$ and every finite $t_1 > 0$, there exists a control signal $u(t)$, $t \in [0, t_1]$, such that the resulting state trajectory goes from a given initial condition $x(0) = x_0$ to $x(t_1) = x_1$. Otherwise, it is said to be uncontrollable.

Theorem 3.4.1. *The given system* Σ *of (3.4.1) is controllable if and only if the matrix*

$$W_{\rm c}(t) := \int_0^t {\rm e}^{-A\tau} B B' {\rm e}^{-A'\tau} d\tau \tag{3.4.2}$$

is nonsingular for all $t > 0$. $W_{\rm c}(t)$ *is called the* controllability grammian *of* Σ.

Proof. If $W_{\rm c}(t)$ is nonsingular for all $t > 0$, for a fixed $t_1 > 0$, we let

$$u(t) = -B'{\rm e}^{-A't} W_{\rm c}^{-1}(t_1)(x_0 - {\rm e}^{-At_1} x_1), \quad t \in [0, t_1]. \tag{3.4.3}$$

Then, by (3.2.1), i.e., $x(t) = {\rm e}^{At} x_0 + \int_0^t {\rm e}^{A(t-\tau)} B u(\tau) d\tau$, we have

$$\begin{aligned}
x(t_1) &= {\rm e}^{At_1} x_0 + \int_0^{t_1} {\rm e}^{A(t_1-t)} B u(t) dt \\
&= {\rm e}^{At_1} x_0 - \left(\int_0^{t_1} {\rm e}^{A(t_1-t)} B B' {\rm e}^{-A't} dt \right) W_{\rm c}^{-1}(t_1)(x_0 - {\rm e}^{-At_1} x_1) \\
&= {\rm e}^{At_1} x_0 - {\rm e}^{At_1} \left(\int_0^{t_1} {\rm e}^{-At} B B' {\rm e}^{-A't} dt \right) W_{\rm c}^{-1}(t_1)(x_0 - {\rm e}^{-At_1} x_1) \\
&= {\rm e}^{At_1} x_0 - {\rm e}^{At_1} x_0 + x_1 = x_1.
\end{aligned}$$

By definition, Σ is controllable.

We prove the converse by contradiction. Suppose Σ is controllable, but $W_c(t)$ is singular for some $t_1 > 0$. Then, there exists a nonzero $x_0 \in \mathbb{R}^n$ such that

$$x_0' W_c(t_1) x_0 = x_0' \left(\int_0^{t_1} e^{-At} BB' e^{-A't} dt \right) x_0 = 0 \tag{3.4.5}$$

Thus, we have

$$\begin{aligned} 0 &= \int_0^{t_1} x_0' e^{-At} BB' e^{-A't} x_0 dt \\ &= \int_0^{t_1} \left(B' e^{-A't} x_0 \right)' \left(B' e^{-A't} x_0 \right) dt \\ &= \int_0^{t_1} \left| B' e^{-A't} x_0 \right|^2 dt, \end{aligned} \tag{3.4.6}$$

which implies the $m \times 1$ vector function

$$\underline{B' e^{-A't} x_0 = 0}, \quad \forall t \in [0, t_1]. \tag{3.4.7}$$

Since Σ is controllable, by definition, for any x_1, there exists a control $u(t)$ such that

$$x_1 = \mathrm{e}^{At_1} x_0 + \int_0^{t_1} \mathrm{e}^{At_1} \mathrm{e}^{-At} Bu(t) dt. \tag{3.4.8}$$

In particular, for $x_1 = 0$, we have

$$0 = \mathrm{e}^{At_1} x_0 + \mathrm{e}^{At_1} \int_0^{t_1} \mathrm{e}^{-At} Bu(t) dt, \tag{3.4.9}$$

or

$$x_0 = -\int_0^{t_1} \mathrm{e}^{-At} Bu(t) dt, \tag{3.4.10}$$

which together with (3.4.7) imply that

$$|x_0|^2 = x_0' x_0 = \left[-\int_0^{t_1} \mathrm{e}^{-At} Bu(t) dt \right]' x_0 = -\int_0^{t_1} u'(t) \overset{=0}{\boxed{B' \mathrm{e}^{-A't} x_0}} dt = 0.$$

This is a contradiction as $x_0 \neq 0$. Hence, $W_\mathrm{c}(t)$ is nonsingular for all $t > 0$. ■

$$x(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau, \quad t \geq 0, \tag{3.2.1}$$

Theorem 3.4.2. *The given system Σ of (3.4.1) is controllable if and only if*

$$\operatorname{rank}(Q_{\mathrm{c}}) = n, \tag{3.4.11}$$

where

$$Q_{\mathrm{c}} := [\, B \quad AB \quad \cdots \quad A^{n-1}B \,] \tag{3.4.12}$$

is called the controllability matrix *of Σ.*

This is the most commonly used result to determine the controllability of an LTI system. It only involves checking the rank of a constant matrix generated from the given system, rather than time domain functions.

However, one should note that the determination of the rank of the controllability matrix sometimes can be ill-conditioned when the system order is high. Nonetheless, it is still much easier than checking the condition in Theorem 3.4.1.

Note: CTRB in MATLAB Control Toolbox calculates the controllability matrix.

Proof. We again prove this theorem by contradiction. Suppose $\text{rank}\,(Q_c) = n$, but Σ is uncontrollable. Then, it follows from Theorem 3.4.1 that

$$W_c(t_1) = \int_0^{t_1} e^{-At} B B' e^{-A't} dt, \quad \forall t_1 > 0 \tag{3.4.13}$$

is singular for some $t_1 > 0$. Also, it follows from the proof of Theorem 3.4.1, *i.e.*, equation (3.4.7), that there exists a nonzero $x_0 \in \mathbb{R}^n$ such that

$$x_0' e^{-At} B = 0, \quad \forall t \in [0, t_1]. \tag{3.4.14}$$

Differentiating (3.4.14) with respect to t and letting $t = 0$, we obtain

$$x_0' B = 0, \;\; x_0' A B = 0, \;\; \dots, \;\; x_0' A^{n-1} B = 0, \tag{3.4.15}$$

or

$$x_0' [\, B \quad AB \quad \cdots \quad A^{n-1} B \,] = x_0' Q_c = 0, \tag{3.4.16}$$

which together with the fact that $x_0 \neq 0$ imply $\text{rank}\,(Q_c) < n$. Obviously, this is a contradiction, and hence, Σ is controllable.

Conversely, we will show that if Σ is controllable, then rank $(Q_c) = n$. If Σ is controllable, but rank $(Q_c) \neq n$, *i.e.*, rank $(Q_c) < n$, then, there exists a nonzero $x_0 \in \mathbb{R}^n$ such that $x_0' Q_c = 0$, *i.e.*,

$$x_0' B = 0, \quad x_0' A B = 0, \quad \ldots, \quad x_0 A^{n-1} B = 0. \tag{3.4.17}$$

It follows from the Cayley–Hamilton Theorem, *i.e.*, (2.3.33), that

$$x_0' A^k B = 0, \quad k = n, n+1, \ldots \tag{3.4.18}$$

Thus, we have

$$x_0' \mathrm{e}^{-At} B = 0 \tag{3.4.19}$$

and

$$x_0' \left(\int_0^t \mathrm{e}^{-A\tau} B B' \mathrm{e}^{-A'\tau} d\tau \right) x_0 = x_0' W_c(t) x_0 = 0 \tag{3.4.20}$$

which implies that $W_c(t)$ is singular for all $t > 0$, and hence, by Theorem 3.4.1, the given system Σ is uncontrollable. This is a contradiction. Thus, Q_c has to be of full rank. ■

Example: Consider an LTI system

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rank
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$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

Calculate the controllability matrix, we obtain

$$Q_\mathrm{c} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 1 & 2 & 5 \end{bmatrix}, \quad \mathrm{rank}(Q_\mathrm{c}) = 2 < 3$$

The given system is uncontrollable.

Example: Consider an LTI system

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

Calculate the controllability matrix, we obtain

$$Q_{\mathrm{c}} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 1 & 1 & 3 \end{bmatrix}, \quad \mathrm{rank}(Q_{\mathrm{c}}) = 3$$

The given system is controllable.

Theorem 3.4.3. *The given system Σ of (3.4.1) is controllable if and only if, for every eigenvalue of A, λ_i, $i = 1, 2, \ldots, n$,*

$$\operatorname{rank} \begin{bmatrix} \lambda_i I - A & B \end{bmatrix} = n. \tag{3.4.21}$$

This theorem is known as the PBH (Popov–Belevitch–Hautus) test, *developed by Popov* [109], *Belevitch* [11] *and Hautus* [63].

The proof of the above result can be found in Chen, Lin and Shamash (2004). The significance of the PBH test is that it leads to the introduction of another important concept in control theory – the system stabilizability, which turns out to be a necessary and sufficient condition to stabilize a system to be controlled.

P

Vasile M. Popov
Romanian American
1928–

B

Vitold Belevitch
Belgian Mathematician
1921–1999

H

Malo Hautus
Eindhoven University of Technology
1940–

We note that Theorem 3.4.3 builds an interconnection between the system controllability and the eigenstructure of the system matrix, *i.e.*, A. The system is controllable if all the eigenvalues of A satisfy the condition given in (3.4.21). On the other hand, the system is not controllable if one or more eigenvalues of A do not satisfy the condition given in (3.4.21). As such, we call an eigenvalue of A a *controllable mode* if it satisfies (3.4.21). Otherwise, it is said to be an *uncontrollable mode*. In many control system design methods, it is not necessary to require the given system to be controllable. The system can be properly controlled if all its uncontrollable modes are stable. Such a system is said to be *stabilizable* as it can still be made stable through a proper state feedback control. For easy reference, in what follows, we highlight the concept of stabilizability.

The system (3.4.1) is said to be stabilizable if all its uncontrollable modes are asymptotically stable. Otherwise, the system is said to be unstabilizable.

Example: Consider an LTI system

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

ex1125

It is verified earlier that the given system is uncontrollable as its controllability matrix Q_c has a rank of $2 < 3$.

The eigenvalues of A are respectively at $-1,\ 1-\sqrt{2},\ 1+\sqrt{2}$.

Using the PBH test,

$$\text{rank}\begin{bmatrix} -1 \cdot I - A & B \end{bmatrix} = \text{rank}\begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -2 & -1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = 2 < 3$$

Thus, $\lambda_1 = -1$ is an uncontrollable mode. Without any further calculation, one can conclude that the other two modes are controllable as Q_c has a rank of 2.

Nonetheless, let us proceed the PBH test for the other two modes…

For $\lambda_2 = 1-\sqrt{2}$,

$$\text{rank}\begin{bmatrix}\left(1-\sqrt{2}\right)\cdot I - A & B\end{bmatrix} = \text{rank}\begin{bmatrix} 1-\sqrt{2} & -1 & -1 & 1 \\ -1 & -\sqrt{2} & -1 & 1 \\ -1 & -1 & 1-\sqrt{2} & 1 \end{bmatrix} = 3$$

Thus, λ_2 is a controllable mode. For $\lambda_3 = 1+\sqrt{2}$,

$$\text{rank}\begin{bmatrix}\left(1+\sqrt{2}\right)\cdot I - A & B\end{bmatrix} = \text{rank}\begin{bmatrix} 1+\sqrt{2} & -1 & -1 & 1 \\ -1 & \sqrt{2} & -1 & 1 \\ -1 & -1 & 1+\sqrt{2} & 1 \end{bmatrix} = 3$$

which implies that λ_3 is also a controllable mode. As the only uncontrollable mode is stable, the given system is stabilizable.

Theorem 3.4.4. *For the given system Σ of (3.1.1), the following two statements are equivalent:*

1. *The pair (A, B) is stabilizable.*
2. *There exists an $F \in \mathbb{R}^{m \times n}$ such that, under the state feedback law*

$$u = Fx, \tag{3.4.23}$$

the resulting closed-loop system is asymptotically stable, i.e., $A + BF$ has all its eigenvalues in $\mathbb{C}^-$.

The above result is heavily used in control systems design. It shows that the stabilizability of a given system is necessary for any control problem if one wishes to make a controlled system stable.

One should not proceed to carry out a control system design any further if the given system is not stabilizable. Instead of designing a controller, the designer should try to re-design the system to be controlled.

Observability and detectability

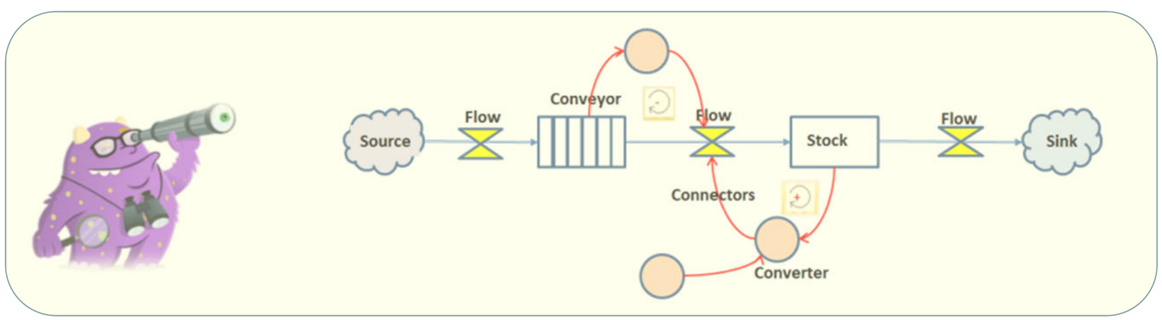
Similarly, we can introduce the concept of observability and detectability for the following unforced system Σ:

$$\dot{x} = Ax, \quad y = Cx, \tag{3.4.24}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and A and C are constant matrices of appropriate dimensions. Basically, the system of (3.4.24) is said to be observable if we are able to reconstruct (or observe) the state variable, x, using only the measurement output y. More precisely, we have the following definition.

Definition 3.4.3. The given system Σ of (3.4.24) is said to be **observable** if for any $t_1 > 0$, the initial state $x(0) = x_0$ can be uniquely determined from the measurement output $y(t)$, $t \in [0, t_1]$. Otherwise, Σ is said to be **unobservable**.

able to see what is going on inside the system



Theorem 3.4.6. *The given system Σ of (3.1.1) is observable if and only if either one of the following statements is true:*

1. *The* observability matrix *of* Σ,

$$Q_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{3.4.27}$$

is of full rank, i.e., $\operatorname{rank}(Q_o) = n$.

2. *For every eigenvalue of* A, $\lambda_i, i = 1, 2, \ldots, n$,

$$\operatorname{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n. \tag{3.4.28}$$

Definition 3.4.4. *The given system Σ of (3.1.1) is said to be* detectable *if all its unobservable modes are asymptotically stable. Otherwise, Σ is said to be* undetectable.

Theorem 3.4.7. *For the given system Σ of (3.1.1), the following two statements are equivalent:*

1. *The pair (A, C) is detectable.*
2. *There exists a $K \in \mathbb{R}^{n\times p}$ such that $A + KC$ has all its eigenvalues in $\mathbb{C}^-$.*

Furthermore, the following dynamical equation utilizing only the system output and control input is capable of asymptotically estimating the system state trajectory, $x(t)$, without knowing its initial value x_0:

$$\dot{\hat{x}} = A\hat{x} + Bu - K(y - C\hat{x} - Du), \quad \hat{x}_0 \in \mathbb{R}^n, \tag{3.4.29}$$

i.e., $e(t) := x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. The dynamical equation of (3.4.29) is commonly called the state observer *or* estimator *of Σ.*

We note that all modern control techniques with measurement feedback using the above observer framework or its variant form!

Example: Consider an LTI system

$$\dot{x} = A\,x = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x, \quad y = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} x$$

Calculate the observability matrix (m-function OBSV), we obtain

$$Q_\text{o} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 3 \\ 6 & 5 & 5 \end{bmatrix}, \quad \text{rank}(Q_\text{o}) = 2 < 3$$

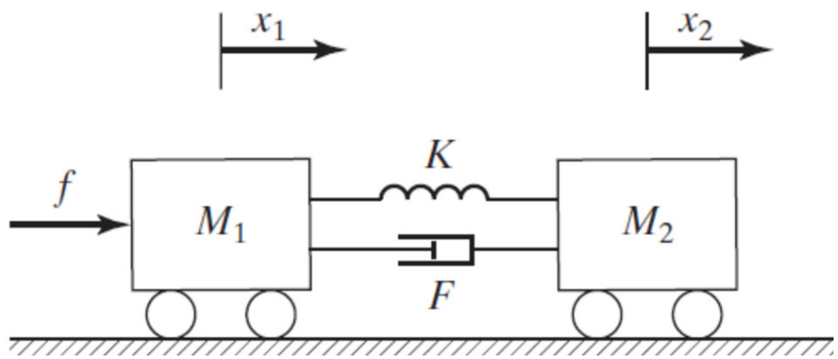
The given system is unobservable. The unobservable mode is –1 as

$$\text{rank}\begin{bmatrix} C \\ -1 \cdot I - A \end{bmatrix} = \text{rank}\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = 2 < 3$$

The system is detectable. In fact, the given system has two modes at –1 with one being unobservable and one not.

Homework Assignment 2 (due in one week)

Q.1. It was shown in the section of dynamic modeling that the two-cart system can be described by the following state space model:



$$\begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K & -F & K & F \\ 0 & 0 & 0 & 1 \\ K & F & -K & -F \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} f, \quad y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix}$$

(a) Determine the stability of the network.

(b) Determine the controllability and observability of the network.

Q.2. Given a linear time-invariant system, $\dot{x} = Ax + Bu$, let

$$\tilde{A} := \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix}.$$

(a) Verify that $\mathrm{e}^{\tilde{A}t}$ has the form

$$\mathrm{e}^{\tilde{A}t} = \begin{bmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{bmatrix}.$$

(b) Show that the controllability grammian of the system is given by

$$W_{\mathrm{c}}(t) = \int_0^t \mathrm{e}^{-A\tau} BB' \mathrm{e}^{-A'\tau} d\tau = E_3'(t) E_2(t).$$

Q.3. Show that if (A, B) is uncontrollable, then $(A + \alpha I, B)$ is also uncontrollable for any $\alpha \in \mathbb{R}$.

Q.4. Consider an uncontrollable system, $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume that

$$\operatorname{rank}(Q_\mathrm{c}) = \operatorname{rank}([\,B \quad AB \quad \cdots \quad A^{n-1}B\,]) = r < n.$$

Let $\{q_1, q_2, \ldots, q_r\}$ be a basis for the range space of the controllability matrix, Q_c, and let $\{q_{r+1}, \ldots, q_n\}$ be any vectors such that

$$T = [\,q_1 \quad q_1 \quad \cdots \quad q_r \quad q_{r+1} \quad \cdots \quad q_n\,]$$

is nonsingular. Show that the state transformation

$$x = T\tilde{x} = T\begin{pmatrix} \tilde{x}_\mathrm{c} \\ \tilde{x}_{\bar{\mathrm{c}}} \end{pmatrix}, \quad \tilde{x}_\mathrm{c} \in \mathbb{R}^r, \quad \tilde{x}_{\bar{\mathrm{c}}} \in \mathbb{R}^{n-r},$$

transforms the given system into the form

$$\begin{pmatrix} \dot{\tilde{x}}_\mathrm{c} \\ \dot{\tilde{x}}_{\bar{\mathrm{c}}} \end{pmatrix} = \begin{bmatrix} A_{\mathrm{cc}} & A_{\mathrm{c}\bar{\mathrm{c}}} \\ 0 & A_{\bar{\mathrm{c}}\bar{\mathrm{c}}} \end{bmatrix} \begin{pmatrix} \tilde{x}_\mathrm{c} \\ \tilde{x}_{\bar{\mathrm{c}}} \end{pmatrix} + \begin{bmatrix} B_\mathrm{c} \\ 0 \end{bmatrix} u,$$

where $(A_{\mathrm{cc}}, B_\mathrm{c})$ is controllable. Show that the uncontrollable modes of the system are given by $\lambda(A_{\bar{\mathrm{c}}\bar{\mathrm{c}}})$.

Q.5. Verify the result in Q.4 for the following systems:

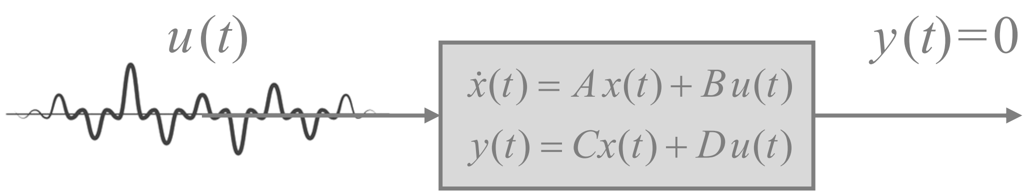
$$\dot{x} = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 0 & 2 & -2 \\ -1 & -1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} u,$$

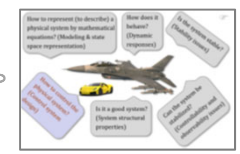
and

$$\dot{x} = \begin{bmatrix} -3 & -3 & 1 & 0 \\ 26 & 36 & -3 & -25 \\ 30 & 39 & -2 & -27 \\ 30 & 43 & -3 & -32 \end{bmatrix} x + \begin{bmatrix} 3 & 3 \\ -2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} u.$$

Q.6. Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, show than if the pair (A, B) is controllable (detectable) if and only if $(A^{\mathrm{T}}, B^{\mathrm{T}})$ is observable (stabilizable).

System Invertibility and Invariant Zeros





Good systems *vs*. bad systems…

It is meaningless to talk about a good system or a bad system without controller design in the picture. When controlling a given system, its (structural) properties do play crucial roles. Up to now, we have learned that…

1. An unstable system is bad as it blows up everything inside out.

   Solution: To employ a control law to stabilize it, if possible. How to work out a stabilizing controller for an unstable system is the story of Part 2.

2. An unstabilizable system is bad as it cannot be stabilized and thus cannot be controlled.

   Solution: No solution besides redesigning the system itself.

3. An undetectable system is bad as it cannot be stabilized and controlled.

   Solution: No solution besides redesigning the system itself.

There are more to be added to the above list as we progress. There are systems that can be controlled but would generally yield bad control performance.

System invertibility

The topic of system invertibilities has been left out in many popular texts in linear systems (for example, in almost all the references listed for this course), although it is important and crucial in almost every control problem.

By definition, it is clear that an invertible system has to be a square system, i.e., the number of the system inputs, m, and the number of the system outputs, p, are identical. A square system is, however, not necessarily invertible. Unfortunately, confusion between invertibility and square systems is common in the literature. Many people take it for granted that a square system is invertible. We illustrate this in the following example.

ex1351

Example 3.5.1. Consider a system Σ of (3.1.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.5.4}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.5.5}$$

Note that both matrices B and C are of full rank. It is controllable and observable, and has a transfer function:

$$G(s) = \frac{1}{s^3 - 3s^2 + s} \begin{bmatrix} (s-1)^2 & s-1 \\ s-1 & 1 \end{bmatrix}. \tag{3.5.6}$$

Clearly, although square, it is a degenerate system as the determinant of $G(s)$ is identical to zero.

Recall the given system (3.1.1), which has a transfer function

$$G(s) = C(sI - A)^{-1}B + D. \tag{3.5.1}$$

Definition 3.5.1. *Consider the linear time-invariant system Σ of (3.1.1). Then,*

1. *Σ is said to be* left invertible *if there exists a rational function matrix of s, say $L(s)$, such that*
$$L(s)G(s) = I_m. \tag{3.5.2}$$

2. *Σ is said to be* right invertible *if there exists a rational function matrix of s, say $R(s)$, such that*
$$G(s)R(s) = I_p. \tag{3.5.3}$$

3. *Σ is said to be* invertible *if it is both left and right invertible.*

4. *Σ is said to be* degenerate *if it is neither left nor right invertible.*

A rational function is a ratio of two polynomials…

Example (left invertibility): Consider an LTI system

ss2tf
ex1144

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = C\,x + D\,u = \begin{bmatrix} 2 & 1 & 1 \\ 9 & 8 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

which has a transfer function

$$G(s) = \frac{\begin{pmatrix} 4s^2 + 5s + 1 \\ 22s^2 + 30s + 8 \end{pmatrix}}{s^3 - s^2 - 3s - 1}$$

It is easy to see that

$$L(s) = \begin{pmatrix} \dfrac{s^3 - s^2 - 3s - 1}{2\left(4s^2 + 5s + 1\right)} & \dfrac{s^3 - s^2 - 3s - 1}{2\left(22s^2 + 30s + 8\right)} \end{pmatrix} \quad \Rightarrow \quad L(s)G(s) = 1$$

Example (right invertibility): Consider an LTI system

ss2tf
ex1145

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} u$$

$$y = C\,x + D\,u = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \end{bmatrix} u$$

which has a transfer function

$$G(s) = \frac{\begin{pmatrix} 4s^2 + s5 + 1 & 9s^2 + 8s + 1 \end{pmatrix}}{s^3 - s^2 - 3s - 1}$$

It is easy to see that

$$R(s) = \begin{pmatrix} \dfrac{s^3 - s^2 - 3s - 1}{2\left(4s^2 + 5s + 1\right)} \\[2ex] \dfrac{s^3 - s^2 - 3s - 1}{2\left(9s^2 + 8s + 1\right)} \end{pmatrix} \quad \Rightarrow \quad G(s)R(s) = 1$$

Example (invertible system): Consider an LTI system

ss2tf
ex1146

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} u$$

$$y = C\,x + D\,u = \begin{bmatrix} 2 & 1 & 1 \\ 9 & 8 & 5 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

which has a transfer function

$$G(s) = \frac{\begin{bmatrix} s^3 + 3s^2 + 2s & 9s^2 + 8s + 1 \\ 22s^2 + 30s + 8 & s^3 + 47s^2 + 49s + 11 \end{bmatrix}}{s^3 - s^2 - 3s - 1}$$

Exercise: Find the inverse of the above system, i.e., find

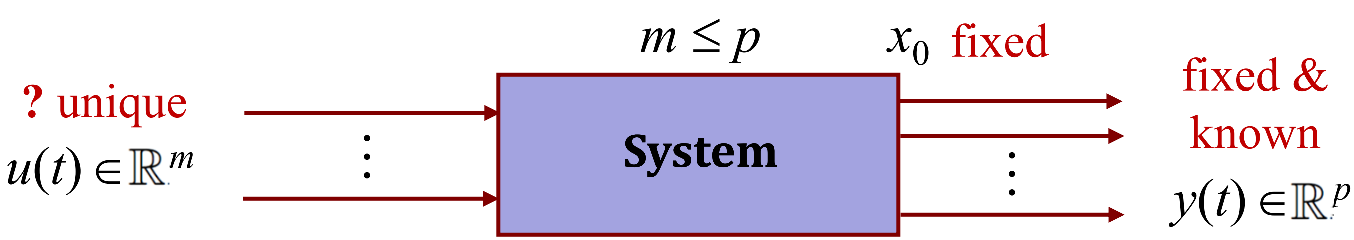
$$G^{-1}(s) = \cdots\cdots$$

For $D=I$, it follows from (2.3.15) on p. 33 that $G^{-1}(s) = I - C(sI - A + BC)^{-1}B$.

Interpretation of system invertibility

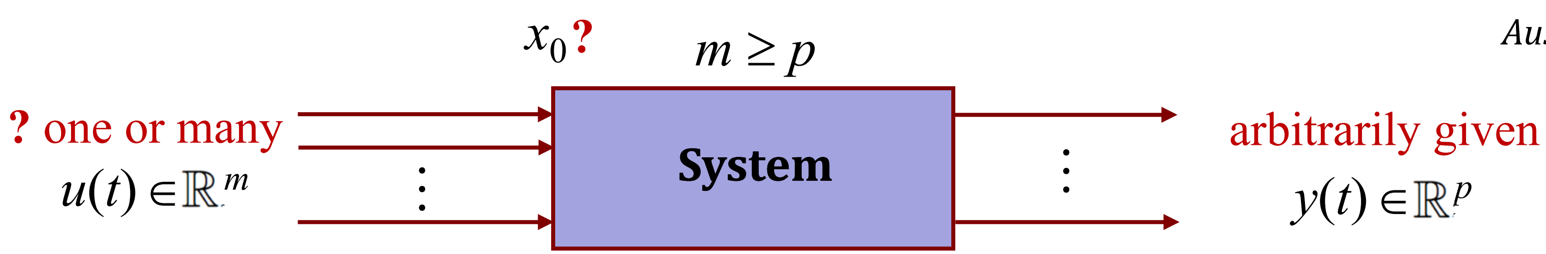
The system left and right invertibilities can be interpreted in the time domain as follows.

- For a left invertible system, given an output $y(t)$ produced by the system with an initial condition x_0, one is able to identify a unique control signal $u(t)$ that generates the given output $y(t)$.



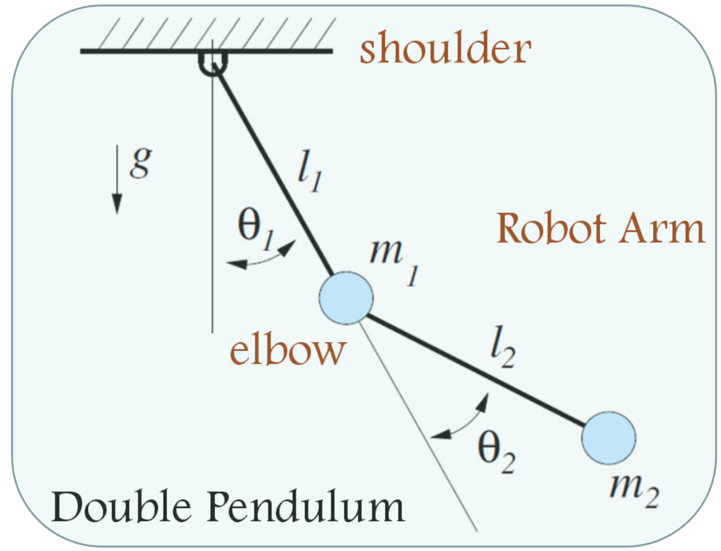
Peter Moylan
University of Newcastle
Australia

- For a right invertible system, for any given signal $y_{\rm ref}(t) \in \mathbb{R}^p$, one is able to determine a (or many) control input $u(t)$ and an (or many) initial condition x_0 for the system, which would produce an output $y(t) = y_{\rm ref}(t)$.



A good example that illustrates a left invertible system is **underactuated robot manipulators** or a double pendulum…

Consider the double pendulum system on the right, where the output variables are θ_1 and θ_2. If we have torque to control both the elbow and the shoulder, the double pendulum system is fully actuated and the resulting dynamical system is invertible. If there is only one actuator providing torque to the elbow, the pendulum is underactuated and the resulting system is left invertible. In such a case, the system does not enough control authorities to drive all the output variables to desired values as illustrated on the previous page.



A left invertible system would cause problems in output tracking. Dually, a right invertible system (over-actuated) is good for output tracking but would degrade the performance of the overall system with output feedback controllers where an observer is used. The concepts of left and right invertibility are dual. This will be clear in Part 2 when we study advanced control design techniques.

Normal rank and invariant zeros

Definition 3.6.1. *Consider the given system Σ of (3.1.1). The* normal rank *of its transfer function $G(s) = C(sI - A)^{-1}B + D$, or in short,* $\text{normrank}\{G(s)\}$, *is defined as*

$$\text{normrank}\{G(s)\} = \max\left\{\text{rank}\,[G(\lambda)] \,\middle|\, \lambda \in \mathbb{C}\right\}. \tag{3.6.2}$$

We note that Example 3.5.1 given earlier has a 2x2 transfer function matrix

$$G(s) = \frac{1}{s^3 - 3s^2 + s}\begin{bmatrix} (s-1)^2 & s-1 \\ s-1 & 1 \end{bmatrix}$$

The **normal rank** of this function matrix is 1.

Historically, many researchers had made lots of **mistakes** in defining **system zeros**. Normal rank was introduced to give a correct and precise definition of zeros, more specifically the invariant zeros, for multivariable systems.

Invariant zeros

Definition 3.6.2. *Consider the given system Σ of (3.1.1). A scalar $\beta \in \mathbb{C}$ is said to be an* invariant zero *of Σ if*

$$\operatorname{rank}\{P_\Sigma(\beta)\} < n + \operatorname{normrank}\{G(s)\}. \tag{3.6.4}$$

Here

$$P_\Sigma(s) := \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

which is known as the so-called Rosenbrock system matrix.

Howard H. Rosenbrock
1920–2010

We note that

- Invariant zeros play a crucial role in designing sensible control systems.
- For a SISO system, invariant zeros are identical to the zeros or transmission zeros, i.e., the roots of the numerator of its transfer function.

Other but incorrect definition of transmission zeros has been used in the literature. The same mistake has been spread over all the places including our textbook by C.T. Chen…

Definition

Given the system (1), the *transmission zeros* of (1) are defined to be the set of complex numbers λ which satisfy the following inequality

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} < n + \underline{\min(p,m)} \qquad (2)$$

incorrect…

♣ E. J. Davidson and S. H. Wang, "Properties and calculation of transmission zeros of linear multivariable systems," *Automatica*, pp. 643–658, 1974.

♣♣ E. J. Davidson and S. H. Wang, "Remark on multiple transmission zeros of a system," *Automatica*, p. 195, 1976.

Edward Davison
University of Toronto

Chi-Tsong Chen
Stony Brook University

Example: Consider an LTI system

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We will demonstrate using MATLAB that for any scalar λ on the complex plane,

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = 4 < n + \min(p,m) = 5$$

ex1351

Clearly, by definition, if β is an invariant zero of Σ, then there exist a nonzero vector $x_{\mathrm{R}} \in \mathbb{C}^n$ and a vector $w_{\mathrm{R}} \in \mathbb{C}^m$ such that

$$P_\Sigma(\beta)\begin{pmatrix} x_{\mathrm{R}} \\ w_{\mathrm{R}} \end{pmatrix} = \begin{bmatrix} \beta I - A & -B \\ C & D \end{bmatrix}\begin{pmatrix} x_{\mathrm{R}} \\ w_{\mathrm{R}} \end{pmatrix} = 0. \tag{3.6.5}$$

Here, x_{R} and w_{R} are respectively called the right *state zero direction* and right *input zero direction* associated with the invariant zero β of Σ.

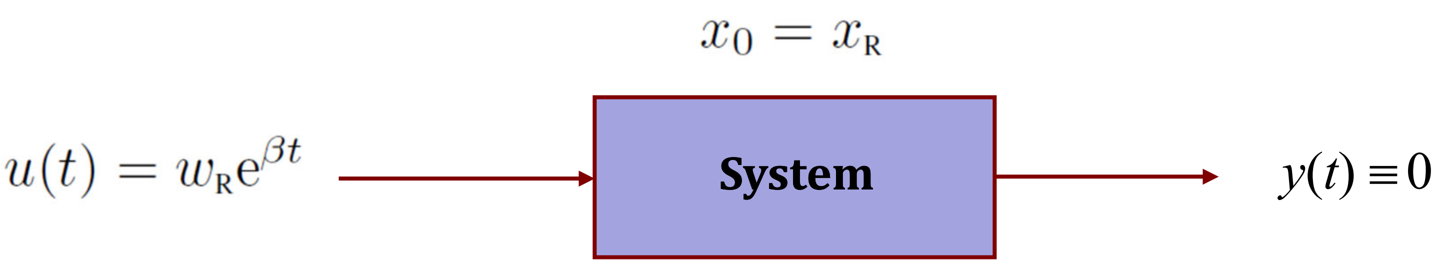
Proposition 3.6.1. *Let β be an invariant zero of Σ with a corresponding right state zero direction x_{R} and a right input zero direction w_{R}. Let the initial state of Σ be $x_0 = x_{\mathrm{R}}$ and the system input be*

$$u(t) = w_{\mathrm{R}} \mathrm{e}^{\beta t}, \quad t \geq 0. \tag{3.6.6}$$

Then, the output of Σ is identically zero, i.e., $y(t) = 0$, $t \geq 0$, and

$$x(t) = x_{\mathrm{R}} \mathrm{e}^{\beta t}, \quad t \geq 0. \tag{3.6.7}$$

This implies that with an appropriate initial state, the system input signal at an appropriate direction and frequency is totally blocked from the system output.



Interpretation of invariant zeros (transmission zeros)

We note that physically

- An invariant zero β with a state zero direction x_R and input zero direction w_R means that the input signal at frequency $e^{\beta t}$ entering the system at the direction w_R will be totally blocked by the system provided that the initial condition of the given system is x_R.
- There are cases that a certain complex frequency, say β, might be totally blocked in all input directions. Such a β is called a **blocking zero** of the given system.

Proof. First, it is simple to verify that (3.6.5), i.e.,

$$P_\Sigma(\beta)\begin{pmatrix} x_{\rm R} \\ w_{\rm R} \end{pmatrix} = \begin{bmatrix} \beta I - A & -B \\ C & D \end{bmatrix}\begin{pmatrix} x_{\rm R} \\ w_{\rm R} \end{pmatrix} = 0$$

$$\Downarrow$$

$$Ax_{\rm R} + Bw_{\rm R} = \beta x_{\rm R}, \quad \underline{Cx_{\rm R} + Dw_{\rm R} = 0}. \tag{3.6.8}$$

We first show that $x(t) = x_{\rm R}\mathrm{e}^{\beta t}$ is a solution to the system Σ of (3.1.1) with the initial condition $x_0 = x_{\rm R}$ and the input $u(t)$ given in (3.6.6). Indeed, with $u(t)$ of (3.6.6) and $x(t)$ of (3.6.7), we have

$$Ax + Bu = Ax_{\rm R}\mathrm{e}^{\beta t} + Bw_{\rm R}\mathrm{e}^{\beta t} = (Ax_{\rm R} + Bw_{\rm R})\mathrm{e}^{\beta t} = \beta x_{\rm R}\mathrm{e}^{\beta t} = \dot{x}. \tag{3.6.9}$$

Thus, $x(t)$ is indeed a solution to the state equation of Σ and it satisfies the initial condition $x(0) = x_{\rm R}$. In fact, $x(t)$ as given in (3.6.7) is the unique solution (see, e.g., Section 3.2). Next, we have

$$y(t) = Cx(t) + Du(t) = \underline{(Cx_{\rm R} + Dw_{\rm R})}\mathrm{e}^{\beta t} \equiv 0, \quad t \geq 0. \tag{3.6.10}$$

This concludes the proof of Proposition 3.6.1. ■

Example: Consider an LTI system

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 9 & 8 & 5 \end{bmatrix} x$$

tzero
invz
ex1155

which has a transfer function $G(s) = \dfrac{31s^2 + 34s + 7}{s^3 - s^2 - 3s - 1}$ with a normal rank of 1.

Since it is a SISO system, its invariant zeros are the zeros or roots of the numerator of its transfer function

$$z_1 = \frac{-17 + 6\sqrt{2}}{31} = -0.2747, \quad z_2 = \frac{-17 - 6\sqrt{2}}{31} = -0.8221$$

It is easy to check for each of them, the rank of the corresponding Rosenbrock system matrix drops.

For MIMO systems, the computation of invariant zeros are rather complicated! The m-function TZERO in MATLAB and INVZ in Linear Systems Toolkit can do the job.

Remarks:

- In this course, we define an LTI system to be of **minimum phase** if all its invariant zeros are in the LHP (note that we don't need the system to be stable). Otherwise, it is called to be of **nonminimum phase**.
- Invariant zeros are **invariant** under state feedback and output injection, i.e., we cannot re-place the locations of invariant zeros through a feedback control law. On the other hand, we can freely assign a closed-loop pole so long as its corresponding mode is controllable.
- A nonminimum phase zero would cause a lot of problems in designing a control system. The overall control performance would be bad.
  - In particular, the time-domain response of a nonminimum phase system to a step input might have an undershoot.
  - The frequency-domain performance will be limited as to be seen in the results given in Part 2.

Why are invariant zeros invariant?

Consider the LTI system characterized by

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

with a state feedback law $u = F\,x + v$, the resulting closed-loop system is given by

$$\Sigma_{\mathrm{F}} : \begin{cases} \dot{x} = \left(A + BF\right)x + B\,v \\ y = \left(C + DF\right)x + D\,v \end{cases}$$

and the corresponding Rosenbrock system matrix is

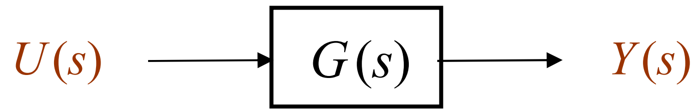
$$\begin{bmatrix} sI - A - BF & -B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}$$

Obviously, s is an invariant zero of Σ if and only if it is an invariant zero of Σ_{F}, i.e., invariant zeros are invariant under state feedback. Similarly, we can show that the invariant zeros are invariant under output injection, i.e.,

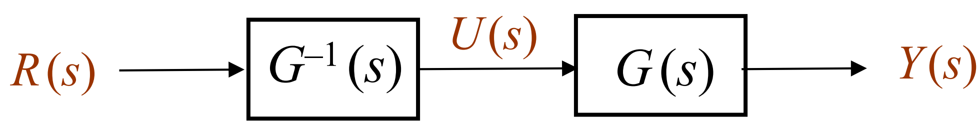
$$\begin{bmatrix} sI - A - KC & -B - KD \\ C & D \end{bmatrix} = \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

Why are bad (nonminimum phase) invariant zeros bad?

For simplicity, we consider a SISO with a transfer function $G(s)$, i.e.,



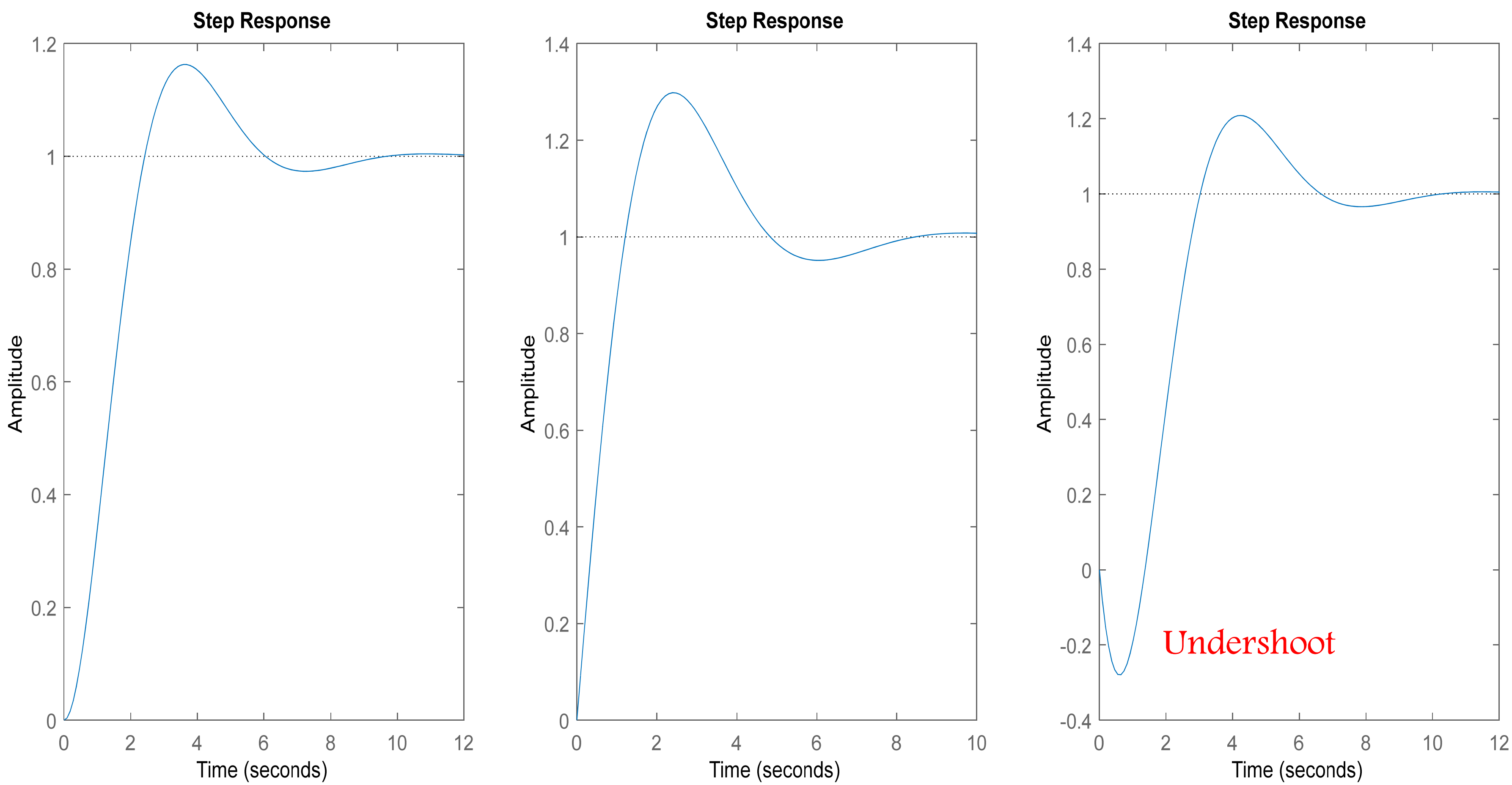
If we want the output to track a reference r, the simplest way is to design a control law of the following form



which results in pole-zero cancellations. Actually, almost all the control techniques to be studied in Part 2 possess inherent pole-zero cancellations whenever the zeros of the given systems are stable. Unfortunately, **unstable pole-zero cancellations are not allowed in control system design** (to be explained in the class). As such, the unstable phase zeros would limited the performance of the closed-loop system. For instance, the unstable zeros would cause an undershoot in its step response…

Example: The step responses of the systems with stable and unstable zeros…

$$G(s)=\frac{Y(s)}{R(s)}=\frac{1}{s^2+s+1}\qquad\qquad G(s)=\frac{1+s}{s^2+s+1}\qquad\qquad G(s)=\frac{1-s}{s^2+s+1}$$



Good systems *vs.* bad systems (cont.)…

1. An unstable system is bad as it blows up everything inside out.

   Solution: To employ a control law to stabilize it, if possible. How to work out a stabilizing controller for an unstable system is the story of Part 2.

2. An unstabilizable system is bad as it cannot be stabilized and controlled.

   Solution: To redesign the system itself.

3. An undetectable system is bad as it cannot be stabilized and controlled.

   Solution: To redesign the system itself.

4. A degenerate system is bad as it would yield bad performance in the overall control system.
   - In state feedback control, left invertible (underactuated) systems would generally yield bad performance.
   - In observer-based feedback control, right invertible systems would cause troubles.

   Solution: To redesign the system if better performance is wanted.

5. A nonminimum phase system is bad as it would yield bad control performance.

   Solution: To redesign the system if better performance is wanted.

Finally, note that some good systems could also be improved to be better ones…

How to get rid of bad invariant zeros? ⋆

Even though the unstable invariant zeros (or nonminimum phase or bad systems in general) cannot be changed by feedback control laws, they can be relocated by…

- **Reselection** of the system **actuators** (matrix B) and/or
- **Replacement** of the measurement **sensors** (matrix C)

ex1160

Example: Consider a system characterized by

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x \;\Rightarrow\; G(s) = \frac{s-1}{s^3 - s^2 - 2s + 1}$$

nonminimum phase

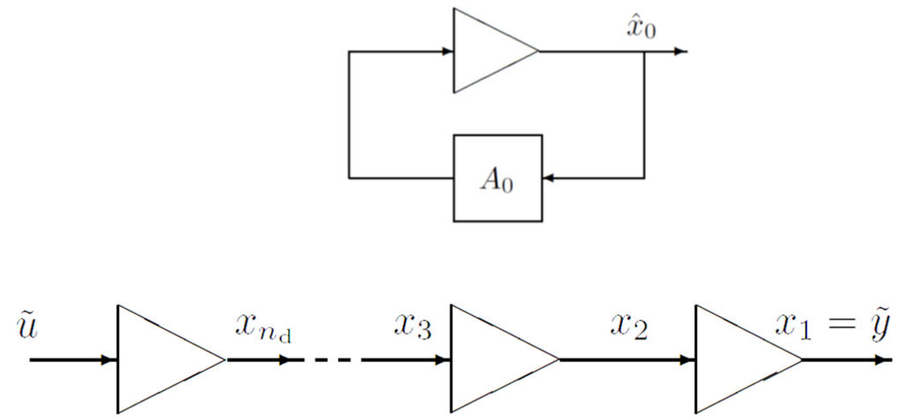
If we replace the measurement sensor to measure the first state variable instead, i.e.,

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \;\Rightarrow\; G(s) = \frac{1}{s^3 - s^2 - 2s + 1} \;\Rightarrow\; \text{minimum phase!}$$

Note: There are techniques that can also be used to solve all the problems highlighted on the previous slide.


Assignment of Complete Structural Properties of Linear Systems via Sensor Selection

Some Structural Decompositions



The performance of a control system is primarily determined by the structural properties of the system to be controlled, rather than the control law controlling it…

A good system can be controlled by a simple controller.

A bad system cannot perform well no matter what control law is used.

Why and what?

Structural properties play an important role in our understanding of linear systems in the state space representation. The structural canonical form representation of linear systems not only reveals the structural properties but also facilitates the design of feedback laws that meet various control objectives. In particular, it decomposes the system into various subsystems. These subsystems, along with the interconnections that exist among them, clearly show the structural properties of the system. The simplicity of the subsystems and their explicit interconnections with each other lead us to a deeper insight into how feedback control would take effect on the system, and thus to the explicit construction of feedback laws that meet our design specifications. The discovery of structural canonical forms and their applications in feedback design for various performance specifications has been an active area of research for a long time. The effectiveness of the structural decomposition approach has also been extensively explored in nonlinear systems and control theory in the recent past.

Unsensed systems

We now proceed to introduce the controllability structural decomposition (CSD) for the unsensed system characterized by

$$\dot{x} = Ax + Bu, \tag{4.4.1}$$

where as usual $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input.

We note that the CSD is also commonly known as the Brunovsky canonical form (1970). But the same result was reported by Luenberger earlier in 1967.

Pavel Brunovský
1934–2018
Slovakian Mathematician

David Luenberger
Stanford University
USA

Theorem 4.4.1 (CSD). *Consider the unsensed system of (4.4.1) with B being of full rank. Then, there exist nonsingular state and input transformations $T_{\rm s} \in \mathbb{R}^{n\times n}$ and $T_{\rm i} \in \mathbb{R}^{m\times m}$ such that, in the transformed input and state,*

$$x = T_{\rm s}\tilde{x}, \quad u = T_{\rm i}\tilde{u}, \tag{4.4.2}$$

such that the transformed system $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$ has the following form:

uncontrollable modes

$$\tilde{A} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \tag{4.4.7}$$

controllable pairs

where $\{k_1, k_2, \cdots, k_m\}$ are called the controllability index of (A, B).

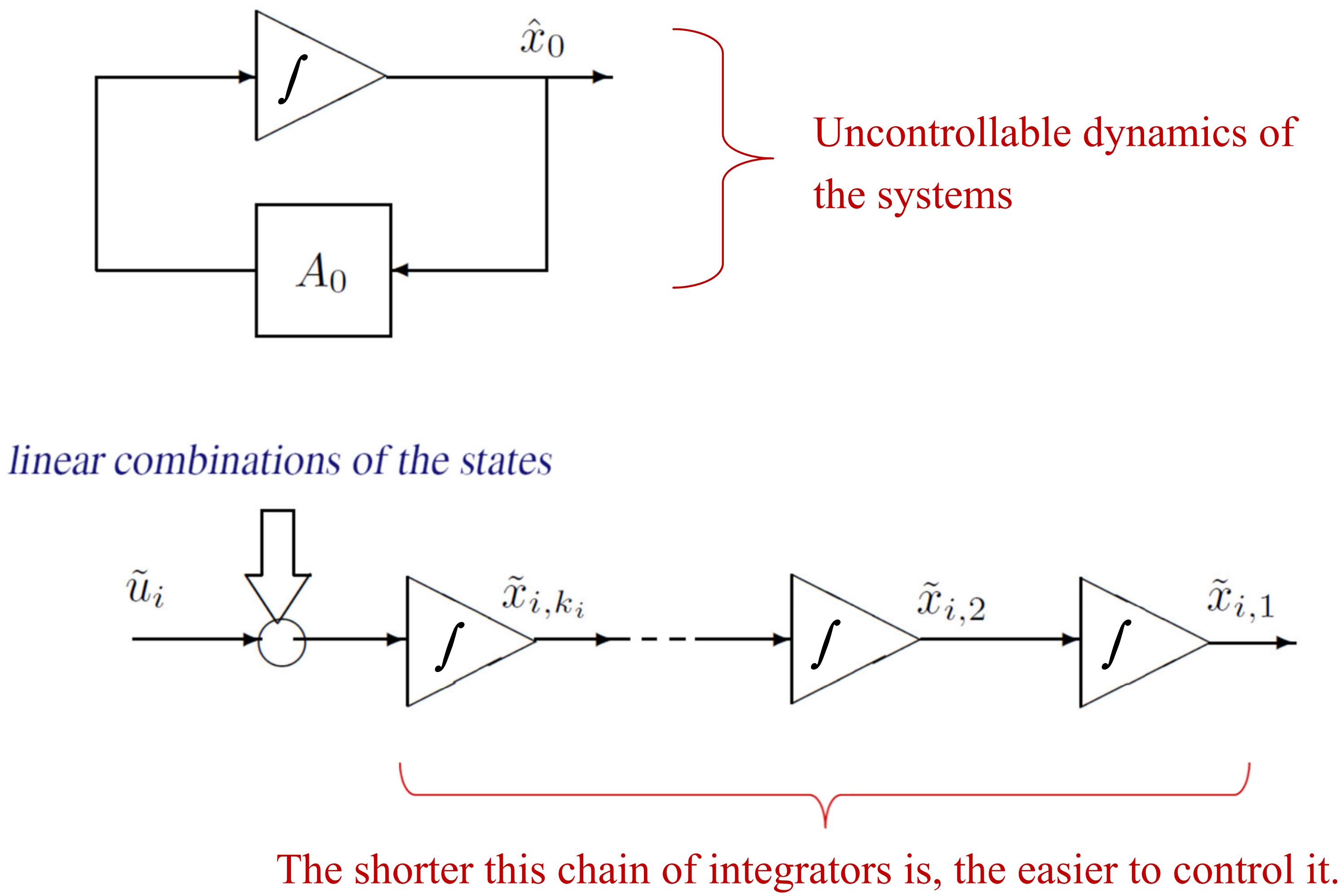


Figure 4.4.1: Interpretation of the controllability structural decomposition.

Example: Consider an LTI system $\dot{x} = A\,x + B\,u$ with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \\ 4 & 3 \\ 5 & 2 \\ 6 & 1 \end{bmatrix}$$

Using the CSD function in Linear System Toolkit, we obtain a state transformation

$$T_s = \frac{1}{15} \begin{bmatrix} -63 & -292 & 15 & 126 & -86 & 90 \\ -38 & -226 & 30 & 101 & -68 & 75 \\ 25 & -70 & 45 & -25 & -35 & 60 \\ 49 & -109 & 60 & 77 & -122 & 45 \\ -62 & -13 & 75 & -1 & -29 & 30 \\ 37 & -127 & 90 & 26 & -41 & 15 \end{bmatrix}$$

csd
ex1166

which transforms the given system into the CSD form, i.e.,

$$T_s^{-1} A T_s = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.8286 & 0.1619 & 0.0095 & -0.2286 & 0.4381 & -0.2095 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.8381 & -2.5714 & 3.4095 & 1.4381 & -2.4286 & 1.9905 \end{bmatrix}, \quad T_s^{-1} B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This controllability structural decomposition form is particularly useful if we want to design a state feedback control law to place the closed-loop system poles to any desired locations. By using a proper pre-feedback gain, we can simplify the above pair to the following form.

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

This special form is particular useful in designing state feedback control law as illustrated on the next page…

Pole placement is trivial in the CSD form. For simplicity, we consider a 3rd order matrix pair in the CSD or Brunovsky canonical form, i.e.,

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \Delta_1 & \Delta_2 & \Delta_3 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and a set of desired closed-loop system poles at $\lambda_1, \lambda_2, \lambda_3$, respectively. The desired characteristic polynomial is then given as

$$\chi(s)=(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)=s^3+a_1 s^2+a_2 s+a_3$$

It is straightforward to show that the following state feedback gain F would place the closed-loop poles at the desired locations

☞ $$F=-\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \end{bmatrix}-\begin{bmatrix} a_3 & a_2 & a_1 \end{bmatrix}=\begin{bmatrix} -\Delta_1-a_3 & -\Delta_2-a_2 & -\Delta_3-a_1 \end{bmatrix}$$

$$\Rightarrow \quad A+BF=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad \Rightarrow \quad \lambda(A+BF)=\lambda_1, \lambda_2, \lambda_3$$ ☞

Unforced systems

We consider an unforced system Σ characterized by

$$\dot{x} = Ax, \quad y = Cx, \tag{4.3.1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, and A and C are constant matrices of appropriate dimensions. We note that there are quite a number of canonical forms associated with such a system, *e.g.*, the observable canonical form and the observability canonical form (see, *e.g.*, Chen [33] and Kailath [70]). These canonical forms are effective in studying the observability of the given system. However, they are not adequate to show the more intrinsic system structural properties (see, for example, Q.4 in Homework Assignment No. 2).

We proceed to present next an observability structural decomposition (OSD), which is dual of the CSD introduced earlier, i.e.,

$$\rightarrow\rightarrow\rightarrow \quad \text{OSD of } (A, C) \quad \Leftrightarrow \quad \text{CSD of } (A', C') \quad \leftarrow\leftarrow\leftarrow$$

Theorem 4.3.1 (OSD). *Consider the unforced system of (4.3.1) with C being of full rank. Then, there exist nonsingular state transformation $T_\mathrm{s} \in \mathbb{R}^{n\times n}$ and nonsingular output transformation $T_\mathrm{o} \in \mathbb{R}^{p\times p}$ such that, in the transformed state and output,*

$$x = T_\mathrm{s}\tilde{x}, \quad y = T_\mathrm{o}\tilde{y}, \tag{4.3.2}$$

such that the transformed system $\dot{\tilde{x}} = \tilde{A}\,\tilde{x},\;\; y = \tilde{C}\,\tilde{x}$ has the following form:

unobservable modes

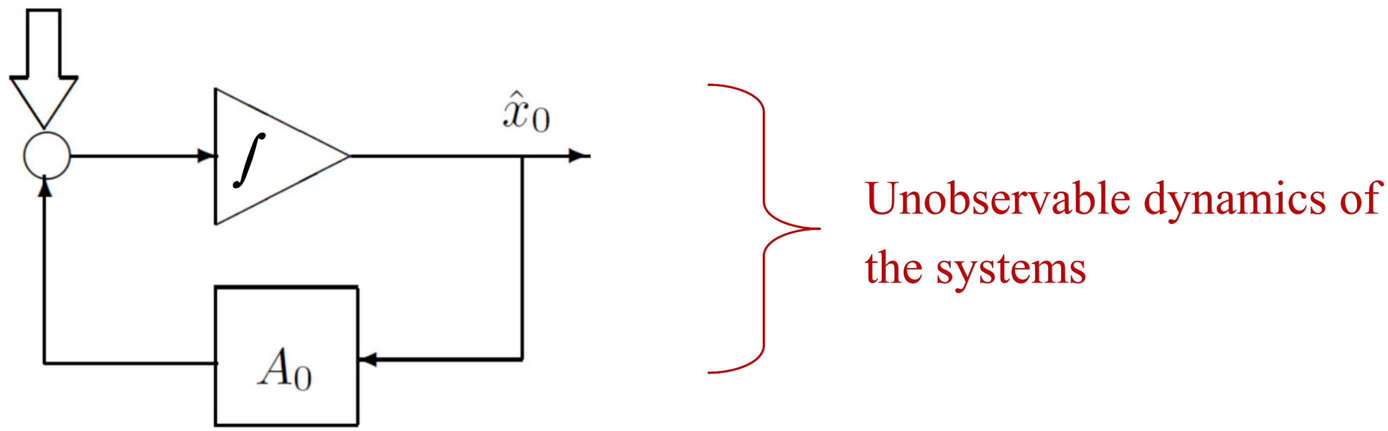
$$\tilde{A} = T_\mathrm{s}^{-1} A T_\mathrm{s} = \begin{bmatrix} A_0 & \star & 0 & \cdots & \star & 0 \\ 0 & \star & I_{k_1-1} & \cdots & \star & 0 \\ 0 & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{k_p-1} \\ 0 & \star & 0 & \cdots & \star & 0 \end{bmatrix},$$

$$\tilde{C} = T_\mathrm{o}^{-1} C T_\mathrm{s} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

observable pairs

where $\star$ represents a matrix of less interest.

Also, $\{k_1, k_2, \cdots, k_p\}$ are called the observability index of (A, C).



Note: the signals indicated by double-edged arrows are some linear combinations of $\tilde{y}_i$.

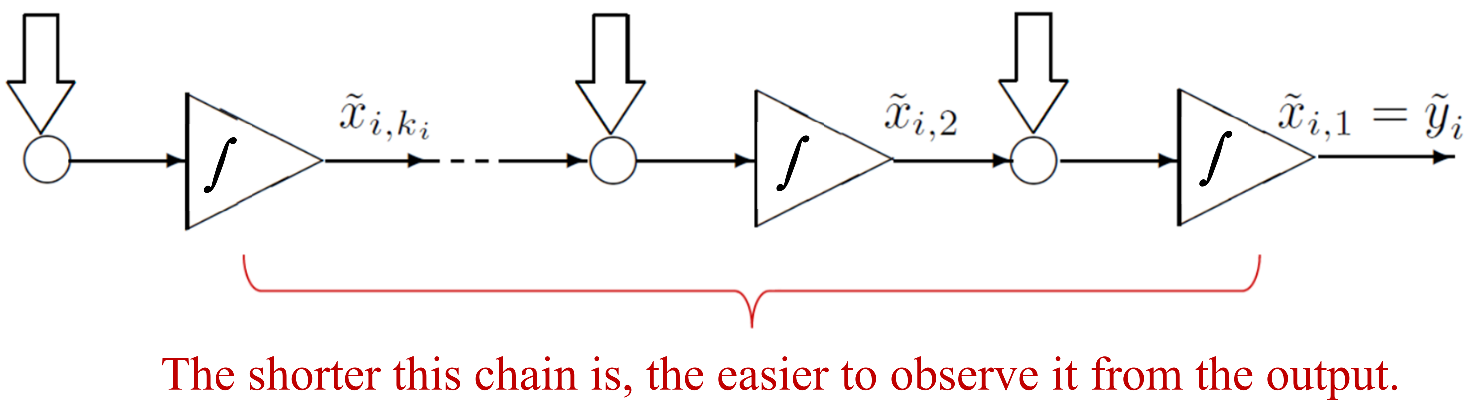


Figure 4.3.1: Interpretation of the observability structural decomposition.

osd
ex1431

Example 4.3.1. Consider an unforced system (4.3.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -2 & -1 & 4 & -2 & 3 & 0 \\ -2 & -1 & 3 & -1 & 3 & 0 \\ 1 & 1 & -2 & 3 & -2 & 0 \\ 2 & 1 & -2 & 2 & -3 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}, \tag{4.3.46}$$

and

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}. \tag{4.3.47}$$

The following transformations will bring the system into the OSD form:

$$T_s = \begin{bmatrix} 0 & 2 & 2 & -1 & -0.6667 & -0.5556 \\ 0 & 0 & -2 & 2 & 0.3333 & 0.4444 \\ 0 & -2 & -1 & 3 & 1 & 0.3333 \\ 0 & -7 & -3 & 3 & 2 & 1 \\ 0 & -2 & 0 & 0 & 0.3333 & 0.1111 \\ 1 & -2 & 0 & 0.3333 & 0.6667 & 0 \end{bmatrix}, \quad T_o = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

such that the transformed system $\dot{\tilde{x}} = \tilde{A}\,\tilde{x}, \;\; y = \tilde{C}\,\tilde{x}$ has the following form:

unobservable modes

$$\tilde{A} = T_{\rm s}^{-1} A T_{\rm s} = \left[\begin{array}{c|cc|ccc} -1 & 2.3333 & 0 & 4.3333 & 0 & 0 \\ \hline 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & -0 & 0 & 5 & 1 & 0 \\ 0 & -14 & 0 & -14 & 0 & 1 \\ 0 & 6 & 0 & 6 & 0 & 0 \end{array}\right],$$

and

$$\tilde{C} = T_{\rm o}^{-1} C T_{s} = \left[\begin{array}{c|cc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

From the OSD form, it is simple to see that the given system is unobservable, but detectable as the unobservable mode is –1. There are two observable pairs associated with the system.

Note: It can be computed using an m-function OSD in Linear Systems Toolkit.

Illustrative Example: Consider a linear system characterized by

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x, \quad y = Cx = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

We define a new state variable

$$\bar{x}_1 = y = Cx = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x \quad \Rightarrow \quad \dot{\bar{x}}_1 = \dot{y} = C\dot{x} = CAx = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} x$$

which is independent of $\bar{x}_1 = y$. We proceed to define

$$\bar{x}_2 = \dot{\bar{x}}_1 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} x \quad \Rightarrow \quad \dot{\bar{x}}_2 = C\ddot{x} = CA^2 x = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix} x$$

which is independent of $\bar{x}_1$ and $\bar{x}_2$. We proceed to define

$$\bar{x}_3 = \dot{\bar{x}}_2 = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix} x$$

which implies

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 4 & 5 & 3 \end{bmatrix} x = S^{-1} x \quad \Rightarrow \quad S = \begin{bmatrix} -4 & -3 & 2 \\ 5 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}$$

ex1174

We obtain a transformed system

$$\dot{\bar{x}} = \bar{A}\,\bar{x} = \left(S^{-1}AS\right)\bar{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}, \quad y = \bar{C}\,\bar{x} = \left(C\,S\right)\bar{x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\bar{x}$$

$$\dot{\bar{x}}_1 = \bar{x}_2, \quad y = \bar{x}_1$$

$$\dot{\bar{x}}_2 = \bar{x}_3$$

$$\dot{\bar{x}}_3 = 1 \cdot \bar{x}_1 + 3 \cdot \bar{x}_2 + \left(1 \times 1\right) \cdot \bar{x}_3$$

We define another set of new state variables…

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 - 1 \cdot \bar{x}_1 \\ \bar{x}_3 - 3 \cdot \bar{x}_1 - 1 \cdot \bar{x}_2 \end{pmatrix} = T^{-1}\,\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \bar{x}$$

$$\Rightarrow \quad \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \bar{x} = T\,\tilde{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_1 + \tilde{x}_2 \\ 4\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_1 + \tilde{x}_2 \\ 4\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \end{pmatrix}$$

We obtain a final transformed system

$$\dot{\bar{x}} = \begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{pmatrix} = \begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 - \dot{\bar{x}}_1 \\ \dot{\bar{x}}_3 - 3\dot{\bar{x}}_1 - \dot{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 - \bar{x}_2 \\ \bar{x}_1 + 3\bar{x}_2 + \bar{x}_3 - 3\bar{x}_2 - \bar{x}_3 \end{pmatrix}$$

$$= \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 - \bar{x}_2 \\ \bar{x}_1 \end{pmatrix} = \begin{pmatrix} \bar{x}_2 \\ 4\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - \tilde{x}_1 - \tilde{x}_2 \\ \tilde{x}_1 \end{pmatrix} = \begin{pmatrix} \bar{x}_2 \\ 3\tilde{x}_1 + \tilde{x}_3 \\ \tilde{x}_1 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{x}_1 + \tilde{x}_2 \\ 3\tilde{x}_1 + \tilde{x}_3 \\ \tilde{x}_1 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}, \qquad y = \bar{x}_1 = \tilde{x}_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\tilde{C} = C\Gamma_s} \tilde{x}$$

The required state transformation

$$x = \Gamma_s \tilde{x} = (S\,T)\tilde{x} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tilde{x} \quad \Rightarrow \quad \tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Homework Assignment 3 (due in one week)

Q.1. It was showed earlier that the invariant zeros of linear systems are invariant under state feedback. More specifically, for a system characterized by

$$\dot{x} = A\,x + B\,u$$
$$y = C\,x + D\,u$$

with a state feedback $u = F x + v$, it gives a closed-loop system

$$\dot{x} = (A + BF)\,x + B\,v$$
$$y = (C + DF)\,x + D\,v$$

We have showed that if a scalar β is an invariant zero of the original system, it is also an invariant zero of the new one as well.

(a) Show that the state feedback law does not change the controllability property of the given system either.

(b) Show by a simple example that the state feedback law, however, may change the observability property of the given system.

Q.2. Verify that the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x,$$

is left invertible. Given an output

$$y(t) = \begin{pmatrix} \cos\omega t + \omega \sin\omega t \\ \mathrm{e}^t - \cos\omega t \end{pmatrix}, \quad t \geq 0,$$

which is produced by the given system with an initial condition,

$$x(0) = \begin{pmatrix} 0 \\ 1 \\ \omega^2 \end{pmatrix},$$

determine the corresponding control input, $u(t)$, which generates the above output, $y(t)$. Also, show that such a control input is unique.

Q.3. Verify that the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u, \quad y = [\,0 \quad 1 \quad 0\,]\,x,$$

is right invertible. Find an initial condition, $x(0)$, and a control input, $u(t)$, which together produce an output

$$y(t) = \alpha \cos \omega t, \quad t \geq 0.$$

Show that the solutions are nonunique.

Q.4. Given an unforced system

$$\dot{x} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} x, \quad y = [\,\alpha \quad \star \quad \cdots \quad \star\,]\,x,$$

where $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, show that the system is observable if and only if $\alpha \neq 0$.

Q.5. Given an unsensed system characterized by a matrix pair in the CSD form

$$\dot{x} = Ax + Bu, \quad \text{with } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let the output equation be $y = Cx$. Verify that the resulting system has

(a) No invariant zero if $C = [\ 1 \quad 0 \quad 0\ ]$;

(b) One invariant zero if $C = [\ 0 \quad 1 \quad 0\ ]$; and

(c) Two invariant zero if $C = [\ 0 \quad 0 \quad 1\ ]$.

Q.6. Given the matrix pair (A, B) as that in Q.5, determine an appropriate state feedback gain matrix F such that $A + BF$ has its eigenvalues at -1, $-1 \pm j$, respectively. Show that such an F is unique.

Show by an example that solutions to the pole placement problem for a multiple input system is non-unique. **Hint:** put the pair in the CSD form.

Advanced Concepts in Linear Systems★

System invariant structural indices (infinite zeros, etc…)

In what follows, however, we will introduce the well-known *Kronecker canonical form* for the system matrix $P_\Sigma(s)$, which is able to display the invariant zero structure, invertibility structures and infinite zero structure of Σ altogether. Although it is not a simple task (it is actually a pretty difficult task for systems with a high dynamical order), it can be shown (see Gantmacher [56]) that there exist nonsingular transformations U and V such that $P_\Sigma(s)$ can be transformed into the following form:

$$U P_\Sigma(s) V = \begin{bmatrix} \text{blkdiag}\{ sI-J, L_{l_1}, \dots, L_{l_{p_b}}, R_{r_1}, \dots, R_{r_{m_c}}, I-sH, I_{m_0} \} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.6.11}$$

where 0 is a zero matrix corresponding to the redundant system inputs and outputs.

Kronecker canonical form characterizes all the structure properties of linear time-invariant systems, i.e., it contains almost everything one needs to know about linear systems. More detailed illustrations on are given on the next…

Kronecker form of linear time-invariant system…

Invariant zeros

Left invertibility structure

$$
U P_\Sigma(s) V = \begin{bmatrix} \boxed{sI - J} & & & & & \\ & \begin{pmatrix} L_{l_1} & & \\ & \ddots & \\ & & L_{l_{p_b}} \end{pmatrix} & & & & 0 \\ & & \begin{pmatrix} R_{r_1} & & \\ & \ddots & \\ & & R_{r_{m_c}} \end{pmatrix} & & & \\ & 0 & & \boxed{I - sH} & & \\ & & & & I_{m_0} & \\ & & & & & 0 \end{bmatrix}
$$

Infinite zero structure

Right invertibility structure

Redundant inputs or outputs

rank (D)

J is in Jordan canonical form, and $sI - J$ has the following $\sum_{i=1}^{\delta} \tau_i$ pencils as its diagonal blocks,

$$sI_{n_{\beta_i,j}} - J_{n_{\beta_i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}, \tag{3.6.12}$$

$j = 1, 2, \ldots, \tau_i$ and $i = 1, 2, \ldots, \delta$; and L_{l_i}, $i = 1, 2, \ldots, p_{\mathrm{b}}$, is an $(l_i + 1) \times l_i$ bidiagonal pencil given by

$$L_{l_i} := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}, \tag{3.6.13}$$

R_{r_i}, $i = 1, 2, \ldots, m_c$, is an $r_i \times (r_i + 1)$ bidiagonal pencil given by

$$R_{r_i} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \tag{3.6.14}$$

H is nilpotent and in Jordan form, and $I - sH$ has the following m_d pencils as its diagonal blocks,

$$I_{q_i+1} - sJ_{q_i+1}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \quad q_i > 0, \; i = 1, 2, \ldots, m_\mathrm{d}, \tag{3.6.15}$$

and finally m_0 in I_{m_0} is the rank of D, *i.e.*, $m_0 = \mathrm{rank}\,(D)$.

Definition 3.6.3. *Consider the given system Σ of (3.1.1) whose system matrix $P_\Sigma(s)$ has a Kronecker form as in (3.6.11) to (3.6.15). Then,*

1. *β_i is said to be an* invariant zero *of Σ with a* geometric multiplicity *of τ_i* and an algebraic multiplicity *of $\sum_{j=1}^{\tau_i} n_{\beta_i,j}$. It has a zero structure*

$$S^\star_{\beta_i}(\Sigma) := \{n_{\beta_i,1}, n_{\beta_i,2}, \ldots, n_{\beta_i,\tau_i}\}. \tag{3.6.16}$$

β_i is said to be a simple invariant zero *if $n_{\beta_i,1} = \cdots = n_{\beta_i,\tau_i} = 1$.*

2. *The* left invertibility structure *of Σ is defined as*

$$S^\star_{\mathrm{L}}(\Sigma) := \{l_1, l_2, \ldots, l_{p_\mathrm{b}}\}. \tag{3.6.17}$$

3. *The* right invertibility structure *of Σ is defined as*

$$S^\star_{\mathrm{R}}(\Sigma) := \{r_1, r_2, \ldots, r_{m_\mathrm{c}}\}. \tag{3.6.18}$$

4. *Finally, m_0 is the number of the infinite zeros of Σ of order 0. The* infinite zero structure *of Σ of order higher than 0 is defined as:*

$$S^\star_\infty(\Sigma) := \{q_1, q_2, \ldots, q_{m_\mathrm{d}}\}. \tag{3.6.19}$$

We say that Σ has m_d infinite zeros of order $q_1, q_2, \ldots, q_{m_\mathrm{d}}$, respectively. If $q_1 = \cdots = q_{m_\mathrm{d}}$ and $m_0 = 0$, then Σ is said to be of uniform rank *q_1. If Σ is a SISO system, i.e., $m = 1$, q_1 is also called a* relative degree.

Everything about a linear system is characterized by these indices. Control performance is fully determined by these structural properties!

Example 3.6.1. Consider a system Σ of (3.1.1) characterized by

kcf
ex1361

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.6.20}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.6.21}$$

It can be shown (using the technique to be given later in Section 5.6 of Chapter 5) that with the following transformations

$U = \cdots \qquad V = \cdots$ (to be demonstrated using MATLAB in class)

Note: *U* and *V* can be obtained using m-function KCF in Linear Systems Toolkit.

the Kronecker canonical form of Σ is given as follows:

$$
UP_\Sigma(s)V = \left[\begin{array}{cc|cc|cc|ccccc}
s-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].
$$

Thus, we have $S_1^\star(\Sigma) = \{2\}$, $S_{\rm L}^\star(\Sigma) = \{2\}$, $S_{\rm R}^\star(\Sigma) = \{1\}$, $S_\infty^\star(\Sigma) = \{1, 2\}$, *i.e.*, Σ has a nonsimple invariant zero at $s = 1$, and two infinite zeros of order 1 and 2, respectively. Σ is degenerate as both $S_{\rm L}^\star(\Sigma)$ and $S_{\rm R}^\star(\Sigma)$ are nonempty.

Leopold Kronecker
1823–1891
German Mathematician

Felix Gantmacher
1908–1964
Soviet Mathematician

Structural Decompositions of LTI Systems★

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Theorem 5.2.1. *Consider the SISO system of (5.2.1). There exist nonsingular state, input and output transformations* $\Gamma_s \in \mathbb{R}^{n\times n}$, $\Gamma_i \in \mathbb{R}$ *and* $\Gamma_o \in \mathbb{R}$, *which decompose the state space of* Σ *into two subspaces,* x_a *and* x_d. *These two subspaces correspond to the finite zero and infinite zero structures of* Σ, *respectively. The new state space, input and output space of the decomposed system are described by the following set of equations:*

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \tag{5.2.2}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_d \end{pmatrix}, \quad x_a \in \mathbb{R}^{n_a}, \quad x_d \in \mathbb{R}^{n_d}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_d} \end{pmatrix}, \tag{5.2.3}$$

and

$$\dot{x}_a = A_{aa} x_a + L_{ad} \tilde{y}, \tag{5.2.4}$$

$$\dot{x}_1 = x_2, \quad \tilde{y} = x_1, \tag{5.2.5}$$

$$\dot{x}_2 = x_3, \tag{5.2.6}$$

$$\vdots$$

$$\dot{x}_{n_d - 1} = x_{n_d}, \tag{5.2.7}$$

$$\dot{x}_{n_d} = E_{da} x_a + E_1 x_1 + E_2 x_2 + \cdots + E_{n_d} x_{n_d} + \tilde{u}. \tag{5.2.8}$$

Furthermore, $\lambda(A_{aa})$ *contains all the system invariant zeros and* n_d *is the relative degree of* Σ.

$$\Sigma \ : \ \dot{x} = A\,x + B\,u, \qquad y = C\,x, \quad (5.2.1)$$

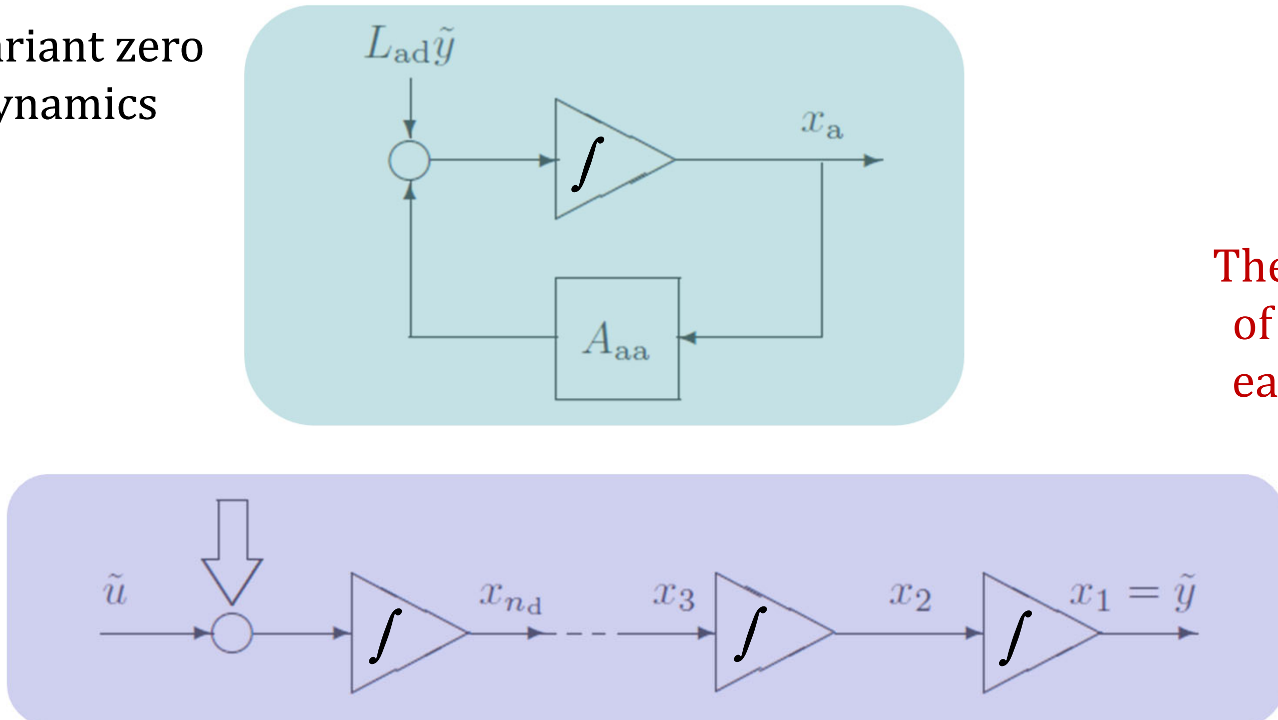
Transformed System:

$$\tilde{A} = \begin{bmatrix} A_{aa} & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ * & * & * & * & \cdots & * & * \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\tilde{A} + \tilde{B}\tilde{F} = \begin{bmatrix} A_{aa} & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Invariant zero dynamics



The shorter this chain of integrators is, the easier to control it…

Infinite zero structure

Note: the signal given by the double-edged arrow is a linear combination of the states.

Figure 5.2.1: Interpretation of structural decomposition of a SISO system.

5.3 Strictly Proper Systems

Next, we consider a general strictly proper linear system Σ characterized by

$$\begin{cases} \dot{x} = A\ x + B\ u, \\ y = C\ x, \end{cases} \tag{5.3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output. Without loss of generality, we assume that both B and C are of full rank. We have the following structural or special coordinate basis decomposition of Σ.

Theorem 5.3.1. *Consider the strictly proper system Σ characterized by (5.3.1). There exist a nonsingular state transformation, $\Gamma_{\mathrm{s}} \in \mathbb{R}^{n \times n}$, a nonsingular output transformation, $\Gamma_{\mathrm{o}} \in \mathbb{R}^{p \times p}$, and a nonsingular input transformation, $\Gamma_{\mathrm{i}} \in \mathbb{R}^{m \times m}$, that will reveal all the structural properties of Σ. More specifically, we have*

$$x = \Gamma_{\mathrm{s}} \tilde{x}, \quad y = \Gamma_{\mathrm{o}} \tilde{y}, \quad u = \Gamma_{\mathrm{i}} \tilde{u}, \tag{5.3.2}$$

with the new state variables

$$\tilde{x} = \begin{pmatrix} x_{\mathrm{a}} \\ x_{\mathrm{b}} \\ x_{\mathrm{c}} \\ x_{\mathrm{d}} \end{pmatrix}, \quad x_{\mathrm{a}} \in \mathbb{R}^{n_{\mathrm{a}}}, \quad x_{\mathrm{b}} \in \mathbb{R}^{n_{\mathrm{b}}}, \quad x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}, \quad x_{\mathrm{d}} \in \mathbb{R}^{n_{\mathrm{d}}}, \tag{5.3.3}$$

the new output variables

$$\tilde{y} = \begin{pmatrix} y_{\rm d} \\ y_{\rm b} \end{pmatrix}, \quad y_{\rm d} \in \mathbb{R}^{m_{\rm d}}, \quad y_{\rm b} \in \mathbb{R}^{p_{\rm b}}, \tag{5.3.4}$$

and the new input variables

$$\tilde{u} = \begin{pmatrix} u_{\rm d} \\ u_{\rm c} \end{pmatrix}, \quad u_{\rm d} \in \mathbb{R}^{m_{\rm d}}, \quad u_{\rm c} \in \mathbb{R}^{m_{\rm c}}. \tag{5.3.5}$$

Further, the state variable $x_{\rm d}$ *can be decomposed as:*

$$x_{\rm d} = \begin{pmatrix} x_{{\rm d},1} \\ x_{{\rm d},2} \\ \vdots \\ x_{{\rm d},m_{\rm d}} \end{pmatrix}, \quad y_{\rm d} = \begin{pmatrix} y_{{\rm d},1} \\ y_{{\rm d},2} \\ \vdots \\ y_{{\rm d},m_{\rm d}} \end{pmatrix}, \quad u_{\rm d} = \begin{pmatrix} u_{{\rm d},1} \\ u_{{\rm d},2} \\ \vdots \\ u_{{\rm d},m_{\rm d}} \end{pmatrix}, \tag{5.3.6}$$

$$x_{{\rm d},i} \in \mathbb{R}^{q_i}, \quad x_{{\rm d},i} = \begin{pmatrix} x_{{\rm d},i,1} \\ x_{{\rm d},i,1} \\ \vdots \\ x_{{\rm d},i,q_i} \end{pmatrix}, \quad i = 1, 2, \dots, m_{\rm d}, \tag{5.3.7}$$

with $q_1 \le q_2 \le \cdots \le q_{m_{\rm d}}$. The state variable $x_{\rm b}$ can be decomposed as

$$x_{\rm b} = \begin{pmatrix} x_{{\rm b},1} \\ x_{{\rm b},2} \\ \vdots \\ x_{{\rm b},p_{\rm b}} \end{pmatrix}, \quad y_{\rm b} = \begin{pmatrix} y_{{\rm b},1} \\ y_{{\rm b},2} \\ \vdots \\ y_{{\rm b},p_{\rm b}} \end{pmatrix}, \tag{5.3.8}$$

$$x_{{\rm b},i} \in \mathbb{R}^{l_i}, \quad x_{{\rm b},i} = \begin{pmatrix} x_{{\rm b},i,1} \\ x_{{\rm b},i,2} \\ \vdots \\ x_{{\rm b},i,l_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p_{\rm b}, \tag{5.3.9}$$

with $l_1 \le l_2 \le \cdots \le l_{p_{\rm b}}$. And finally, the state variable $x_{\rm c}$ can be decomposed as

$$x_{\rm c} = \begin{pmatrix} x_{{\rm c},1} \\ x_{{\rm c},2} \\ \vdots \\ x_{{\rm c},m_{\rm c}} \end{pmatrix}, \quad u_{\rm c} = \begin{pmatrix} u_{{\rm c},1} \\ u_{{\rm c},2} \\ \vdots \\ u_{{\rm c},m_{\rm c}} \end{pmatrix}, \tag{5.3.10}$$

$$x_{{\rm c},i} \in \mathbb{R}^{r_i}, \quad x_{{\rm c},i} = \begin{pmatrix} x_{{\rm c},i,1} \\ x_{{\rm c},i,2} \\ \vdots \\ x_{{\rm c},i,r_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_{\rm c}, \tag{5.3.11}$$

with $r_1 \le r_2 \le \cdots \le r_{m_{\rm c}}$.

The decomposed system can be expressed in the following dynamical equations:

$$\dot{x}_{\rm a} = A_{\rm aa} x_{\rm a} + L_{\rm ab} y_{\rm b} + L_{\rm ad} y_{\rm d}, \qquad (5.3.12)$$

invariant zero (JCF)

for each subsystem $x_{{\rm b},i}$, $i = 1, 2, \ldots, p_{\rm b}$,

$$\dot{x}_{{\rm b},i,1} = x_{{\rm b},i,2} + L_{{\rm bd},i,1} y_{\rm b} + L_{{\rm b},i,1} y_{\rm d}, \quad y_{{\rm b},i} = x_{{\rm b},i,1}, \qquad (5.3.13)$$

$$\dot{x}_{{\rm b},i,2} = x_{{\rm b},i,3} + L_{{\rm bd},i,2} y_{\rm b} + L_{{\rm b},i,2} y_{\rm d}, \qquad (5.3.14)$$

$$\vdots$$

$$\dot{x}_{{\rm b},i,l_i} = L_{{\rm bd},i,l_i} y_{\rm b} + L_{{\rm bd},i,l_i} y_{\rm d}, \qquad (5.3.15)$$

left invertibility structure (OSD)

for each subsystem $x_{{\rm c},i}$, $i = 1, 2, \ldots, m_{\rm c}$,

$$\dot{x}_{{\rm c},i,1} = x_{{\rm c},i,2} + L_{{\rm cb},i,1} y_{\rm b} + L_{{\rm cd},i,1} y_{\rm d}, \qquad (5.3.16)$$

$$\vdots$$

$$\dot{x}_{{\rm c},i,r_i-1} = x_{{\rm c},i,r_i} + L_{{\rm cb},i,r_i-1} y_{\rm b} + L_{{\rm cd},i,r_i-1} y_{\rm d}, \qquad (5.3.17)$$

$$\dot{x}_{{\rm c},i,r_i} = A_{{\rm c},i,{\rm a}} x_{\rm a} + A_{{\rm c},i,{\rm c}} x_{\rm c} + L_{{\rm cb},i,r_i} y_{\rm b} + L_{{\rm cd},i,r_i} y_{\rm d} + u_{{\rm c},i}, \qquad (5.3.18)$$

right invertibility structure (CSD)

and finally, for each subsystem $x_{d,i}$, $i = 1, 2, \ldots, m_d$,

$$\dot{x}_{d,i,1} = x_{d,i,2} + L_{d,i,1} y_d, \quad y_{d,i} = x_{d,i,1}, \tag{5.3.19}$$

$$\dot{x}_{d,i,2} = x_{d,i,3} + L_{d,i,2} y_d, \tag{5.3.20}$$

$$\vdots$$

$$\dot{x}_{d,i,q_i} = A_{d,i,a} x_a + A_{d,i,c} x_c + A_{d,i,b} x_b + A_{d,i,d} x_d + u_{d,i} \tag{5.3.21}$$

where $A_{aa}, L_{ab}, \ldots, A_{d,i,d}$ are constant matrices of appropriate dimensions.

infinite zero structure. why?

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Washington State University

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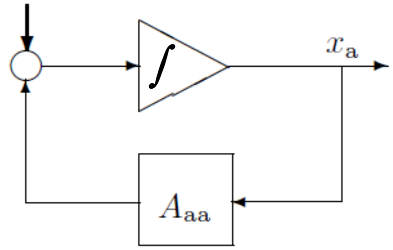
USA

We note that for each of these SISO subsystem, the corresponding transfer function from its input $u_{d,i}$ to its output $y_{d,i}$ can be expressed as

$$H_i(s) = \frac{1}{s^{q_i} + \cdots} \quad \Rightarrow \quad H_i(s)\big|_{s=\infty} = 0$$

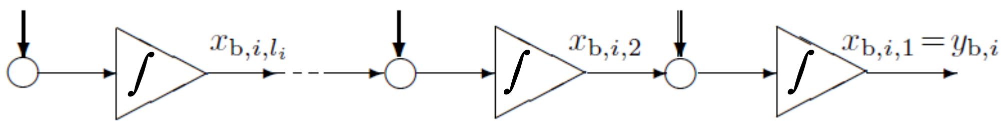
It has a **zero** at ∞ with an order of q_i.

x_a – *the subsystem without direct input and output:*



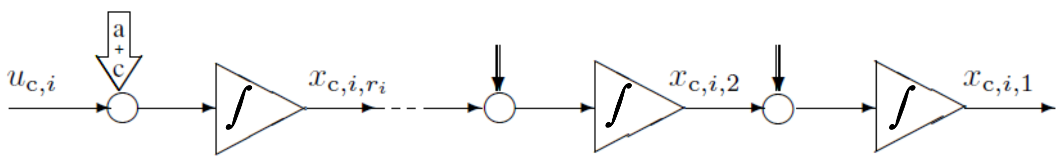
invariant zero

$x_{b,i}$ – *the chain of integrators without a direct input:*



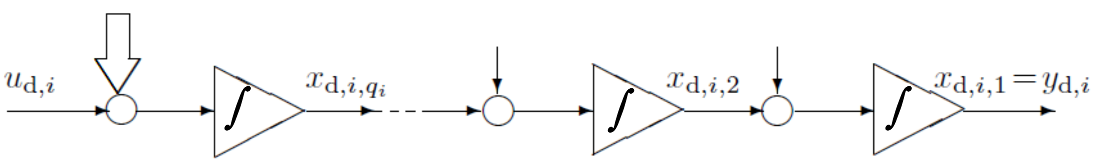
left invertibility structure

$x_{c,i}$ – *the chain of integrators without a direct output:*



right invertibility structure

$x_{d,i}$ – *the chain of integrators with direct input and output:*



infinite zero structure

scb
ex1531

Example 5.3.1. Consider a strictly proper system Σ characterized by (5.3.1) with

$$A = \begin{bmatrix} 0 & 0 & 2 & -1 & 2 & 0 & -1 & 2 & 0 & -1 & 0 & 2 & 2 \\ 0 & 2 & 4 & -5 & 3 & 2 & -4 & 3 & 2 & -4 & 0 & 5 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 0 & 3 & -3 & 0 & 3 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & -2 & 2 & 1 & -3 & 2 & 1 & 1 & 1 & -2 \\ 0 & 3 & 3 & -1 & -2 & 3 & 0 & -2 & 3 & 0 & 2 & 2 & -3 \\ 0 & 3 & 3 & -1 & -2 & 3 & -1 & -1 & 3 & 0 & 1 & 3 & -3 \\ 0 & 3 & 3 & -1 & -2 & 3 & -1 & 0 & 3 & 0 & 1 & 4 & -3 \\ 0 & 2 & 2 & 1 & -1 & 2 & 0 & 0 & 2 & 1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 & 2 & -2 & 0 & 2 & 1 & -2 & 0 \\ 0 & -1 & -3 & 7 & -3 & -1 & 4 & -3 & -1 & 4 & 2 & -4 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

The required state, input and output transformations…

$$
\Gamma_{\rm s} = \begin{bmatrix}
-1 & -1 & 1 & -1 & 3 & -3 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
-1 & 0 & 5 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\
0 & -1 & -2 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 2 & 0 & 4 & 0 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\
0 & -2 & 2 & 0 & 5 & 0 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\
1 & -3 & 7 & 0 & 14 & 1 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\
0 & -2 & 4 & 0 & 9 & 0 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\
1 & -1 & 5 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\
1 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1
\end{bmatrix}
$$

$$
\Gamma_{\rm i} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
\Gamma_{\rm o} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}
$$

These transformations are non-unique!

Note: It can also be done using an m-function `SCB` in Linear Systems Toolkit.

The transformed system $(\tilde{A},\tilde{B},\tilde{C})=(\Gamma_{\rm s}^{-1}A\Gamma_{\rm s},\Gamma_{\rm s}^{-1}B\Gamma_{\rm i},\Gamma_{\rm o}^{-1}C\Gamma_{\rm s})$

$$
\tilde{A}=\left[\begin{array}{cc|cccc|ccc|cccc}
-2 & 0 & 6 & 0 & 11 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
1 & 1 & -2 & 0 & -5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
\hline
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0\\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0\\
\hline
0 & 0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 10 & 0 & 16 & 0 & 0 & 0 & 1 & 4 & 8 & 0 & 0\\
5 & -1 & -1 & 0 & -3 & 0 & 1 & 1 & 2 & 5 & 13 & 0 & 0\\
\hline
2 & -2 & 12 & 6 & 23 & 8 & 1 & 1 & 2 & 2 & 16 & 6 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
2 & -2 & 8 & 3 & 15 & 4 & 1 & 1 & 2 & 1 & 14 & 5 & 1
\end{array}\right],\quad
\tilde{B}=\left[\begin{array}{cc|cc}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
\hline
0 & 0 & 1 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 1\\
\hline
1 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 1 & 0 & 0
\end{array}\right]
$$

$[\,F\,]$

$[\,K\,]$

$$
\tilde{C}=\left[\begin{array}{cc|cccc|ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$\bar{A}=\tilde{A}+\tilde{B}F+K\tilde{C}$ in an essential form shown on the next page…

The essential structures of the system…

$$\bar{A} = \left[\begin{array}{cc|cccc|ccc|cccc}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad \tilde{B} = \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{cc|cccc|ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

the essentials

two invariant zeros	left invertibility structure	right invertibility structure	infinite zero structure
$\lambda(A_{\mathrm{aa}}) = \{-2, 1\}$	$S_{\mathrm{L}}^{\star}(\Sigma) = \{2, 2\}$	$S_{\mathrm{R}}^{\star}(\Sigma) = \{1, 2\}$	$S_{\infty}^{\star}(\Sigma) = \{1, 3\}$

Just for fun…

The link between Taoism and chain of integrators...

论道法自然与全状态反馈控制系统之关系

山野蛮人

《系统与控制纵横》
第6卷第1期第81–84页, 2019

图 1：老子

【摘要】本文尝试用全状态反馈系统来诠释圣人老聃（见图 1）在他不朽的传世之作『道德经』[1]第二十五章中所阐述的『人法地，地法天，天法道，道法自然』的宇宙定律。

山野蛮人在他最近所谓的游记『阿拉斯加』[2]之结尾中写到：“大至天地万物，细如芸芸众生，究其宗者，在圣人老聃看来本质上也不过是一条短短的四阶积分链：『人法地，地法天，天法道』，始于道、终于人。而主宰这条积分链的控制系统也颇为简单，即老子所说的『道法自然』，一个全信息反馈的控制器。所谓的『大道至简』！”本文尝试揭示老子这宇宙定律与全状态反馈控制系统的关系。

human earth haven natural order of the universe

为了数学表达方便起见，我们令 x_{ren} = 人， x_{di} = 地， x_{tian} = 天， x_{dao} = 道

那么，『人法地，地法天，天法道』意指 $\dot{x}_{\text{ren}} = x_{\text{di}}$， $\dot{x}_{\text{di}} = x_{\text{tian}}$， $\dot{x}_{\text{tian}} = x_{\text{dao}}$

或可用以下矩阵状态方程式来表示：

$$\begin{pmatrix} \dot{x}_{\text{ren}} \\ \dot{x}_{\text{di}} \\ \dot{x}_{\text{tian}} \\ \dot{x}_{\text{dao}} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_{\text{ren}} \\ x_{\text{di}} \\ x_{\text{tian}} \\ x_{\text{dao}} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (3)$$

Brunovsky canonical form...

方程式（3）式是一个典型的四阶积分链系统。为了方便起见，我们令

$$x = \begin{pmatrix} x_{\text{ren}} \\ x_{\text{di}} \\ x_{\text{tian}} \\ x_{\text{dao}} \end{pmatrix} \qquad (4)$$

State variables...

为动态系统（3）的状态变量，其中 x 包含人、地、天和道，即自然界中的天地万物或所谓的自然。由此可见，老子所说的『道法自然』即是一个全状态反馈的控制器：

$$\dot{x}_{\text{dao}} = u = f(x) = F\,x \qquad (5)$$

State feedback...

A brief introduction of geometric approach to linear systems……

3.7 Geometric Subspaces

The geometric approach to linear systems and control theory has attracted much attention over the past few decades. It was started in the 1970s and quickly matured in the 1980s when researchers attempted to solve disturbance decoupling and almost disturbance decoupling problems, which require the design of appropriate control laws to make the influence of the exogenous disturbances to the controlled outputs equal to zero or almost zero (see, *e.g.*, Basile and Marro [9], Schumacher [126], Willems [151,152], Wonham [154], and Wonham and Morse [155]). In fact, most of the concepts in linear systems can be tackled and studied nicely within the geometric framework (see, *e.g.*, the classical text by Wonham [154] and a recent text by Trentelman *et al.* [141]). The geometric approach is mathematically elegant in expressing abstract concepts in linear systems. It is, however, hard to compute explicitly various subspaces defined in the framework.

The following are the definition and properties of the weakly unobservable subspace adopted from Trentelman *et al.* [141].

Definition 3.7.1. *Consider the continuous-time system* Σ *of (3.1.1). An initial state of* Σ, $x_0 \in \mathbb{R}^n$, *is called* weakly unobservable *if there exists an input signal* $u(t)$ *such that the corresponding system output* $y(t) = 0$ *for all* $t \geq 0$. *The subspace formed by the set of all weakly unobservable points of* Σ *is called the* weakly unobservable subspace *of* Σ *and is denoted by* $\mathcal{V}^*(\Sigma)$.

The following lemma shows that any state trajectory of Σ starting from an initial condition in $\mathcal{V}^*(\Sigma)$ with a control input that produces an output $y(t) = 0$, $t \geq 0$, will always stay inside the weakly unobservable subspace, $\mathcal{V}^*(\Sigma)$.

Lemma 3.7.1. *Let* x_0 *be an initial state of* Σ *with* $x_0 \in \mathcal{V}^*(\Sigma)$ *and* u *be an input such that the corresponding system output* $y(t) = 0$ *for all* $t \geq 0$. *Then the resulting state trajectory* $x(t) \in \mathcal{V}^*(\Sigma)$ *for all* $t \geq 0$.

Definitions of other geometric subspaces can be found in Chapter 3 of Chen et al…

Links between the special coordinate basis and geometric subspaces…

The structural decomposition decomposes the state space of Σ into several distinct parts. In fact, the state space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm a}^0 \oplus \mathcal{X}_{\rm a}^+ \oplus \mathcal{X}_{\rm b} \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d}. \tag{5.4.37}$$

Here $\mathcal{X}_{\rm a}^-$ is related to the stable invariant zeros, *i.e.*, the eigenvalues of $A_{\rm aa}^-$ are the stable invariant zeros of Σ. Similarly, $\mathcal{X}_{\rm a}^0$ and $\mathcal{X}_{\rm a}^+$ are respectively related to the invariant zeros of Σ located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_{\rm b}$ is related to the right invertibility, *i.e.*, the system is right invertible if and only if $\mathcal{X}_{\rm b} = \{0\}$, while $\mathcal{X}_{\rm c}$ is related to left invertibility, *i.e.*, the system is left invertible if and only if $\mathcal{X}_{\rm c} = \{0\}$. Finally, $\mathcal{X}_{\rm d}$ is related to zeros of Σ at infinity.

There are interconnections between the subsystems generated by the structural decomposition and various invariant geometric subspaces. The following properties show these interconnections.

Property 5.4.6. *The geometric subspaces defined in Definitions 3.7.2 and 3.7.4 are given by:*

1. $\boxed{\mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm a}^0 \oplus \mathcal{X}_{\rm c} \;\; \textit{spans} \;\; \mathcal{V}^-(\Sigma).}$
2. $\mathcal{X}_{\rm a}^+ \oplus \mathcal{X}_{\rm c} \;\; \textit{spans} \;\; \mathcal{V}^+(\Sigma).$
3. $\boxed{\mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm a}^0 \oplus \mathcal{X}_{\rm a}^+ \oplus \mathcal{X}_{\rm c} \;\; \textit{spans} \;\; \mathcal{V}^*(\Sigma).}$
4. $\mathcal{X}_{\rm a}^+ \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d} \;\; \textit{spans} \;\; \mathcal{S}^-(\Sigma).$
5. $\boxed{\mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm a}^0 \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d} \;\; \textit{spans} \;\; \mathcal{S}^+(\Sigma).}$
6. $\mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d} \;\; \textit{spans} \;\; \mathcal{S}^*(\Sigma).$
7. $\mathcal{X}_{\rm c} \;\; \textit{spans} \;\; \mathcal{R}^*(\Sigma).$
8. $\mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm a}^0 \oplus \mathcal{X}_{\rm a}^+ \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d} \;\; \textit{spans} \;\; \mathcal{N}^*(\Sigma).$

$$\tilde{x} = \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix} \begin{array}{l} \Leftarrow \text{invariant zeros} \\ \Leftarrow \text{left invertibility} \\ \Leftarrow \text{right invertibility} \\ \Leftarrow \text{infinite zeros} \end{array}$$

$$x_{\rm a} = \begin{pmatrix} x_{\rm a}^- \\ x_{\rm a}^0 \\ x_{\rm a}^+ \end{pmatrix} \begin{array}{l} \Leftarrow \text{stable zeros} \\ \Leftarrow \text{zeros on } j\omega \text{ axis} \\ \Leftarrow \text{unstable zeros} \end{array}$$

Partition of the state space in the special coordinate basis…

What are these geometric subspaces for?

Let us consider the following linear system

$$\dot{x} = A\,x + B\,u + E\,w, \quad z = C\,x + D\,u$$

where x is the state, u the control input, z the output and w is disturbance entering the system as an additional input.

We can show that there exists a state feedback control law $u = F\,x$ such that when it is applied to the given system, the resulting closed-loop system transfer matrix from w to z can be made perfectly zero (**disturbance decoupling**), i.e.,

$$H_{zw}(s) = \left(C + DF\right)\left(sI - A - BF\right)^{-1} E \equiv 0$$

if and only if $\mathrm{Im}(E) \subset \mathcal{V}^*(\Sigma)$. In the special coordinate basis,

$$\mathcal{X} = \mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_c \oplus \mathcal{X}_d$$

and $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c$ *spans* $\mathcal{V}^*(\Sigma)$. It means the disturbance input can only allow to enter in the subsystem spanned by $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c$.

If in addition it requires $A+BF$ to be asymptotically stable, the disturbance decoupling problem is solvable if and only if the disturbance enters the system through $\mathcal{X}_a^- \oplus \mathcal{X}_c$, which spans a geometric subspace $\mathcal{V}^-(\Sigma)$ if (A,B,C,D) has no invariant zeros on the imaginary axis.

Note that if (A,B,C,D) is right invertible and is of **minimum phase** with no infinite zeros, then $\mathcal{X}_a^- \oplus \mathcal{X}_c$ spans the entire state space $\mathcal{X}$ of the given system, which means the disturbance decoupling problem is solvable for any disturbance entering the system.

Such a system is **super good** for disturbance rejection under state feedback.

We will examine this issue further in the second part of this course when we are studying topics related to H_2 and H_∞ control.

We can also show that if (A,B,C,D) has no invariant zeros on the imaginary axis and if the disturbance enter the system through the subspace $\mathcal{S}^+(\Sigma)$, which is spanned by $\mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d}$, then there exists a stabilizing state feedback law such that when it is applied to the given system, the resulting closed-loop system is asymptotically stable and the resulting closed-loop transfer function matrix from w to z can be made arbitrarily small (**almost disturbance decoupling**).

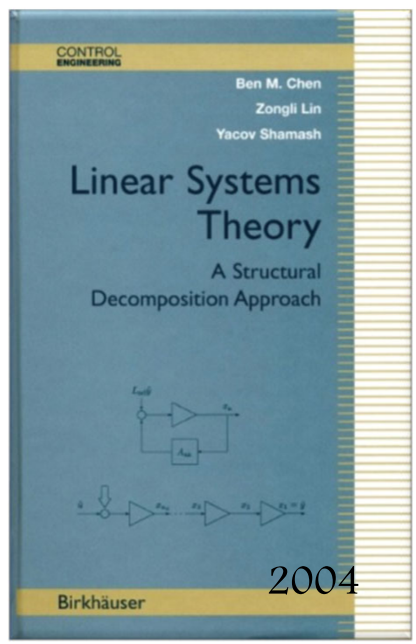
Note that if (A,B,C,D) is right invertible and is of **minimum phase**, then $\mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d}$ or $\mathcal{S}^+(\Sigma)$ spans the entire state space $\mathcal{X}$ of the given system, which means the almost disturbance decoupling problem is solvable for any disturbance entering the system.

Such a system is **good** for disturbance rejection under state feedback.

We conclude this part on linear systems theory by noting that the topics covered in this course are pretty elementary, but sufficient for students to understand basic linear system theory and to grasp basic ideas and solutions to many linear control problems.

Some advanced topics such as the geometric subspaces of linear systems, which are instrumental in developing many control theories (including some nonlinear control theories), are left out as there is too much mathematics involved.

Interested readers can find more detailed information in the text by Chen, Lin and Shamash (2004). One can also utilize a Linear Systems Toolkit developed by Lin, Chen and Liu, available for free by request, for computing all the structural decompositions and geometric subspaces of general linear systems.



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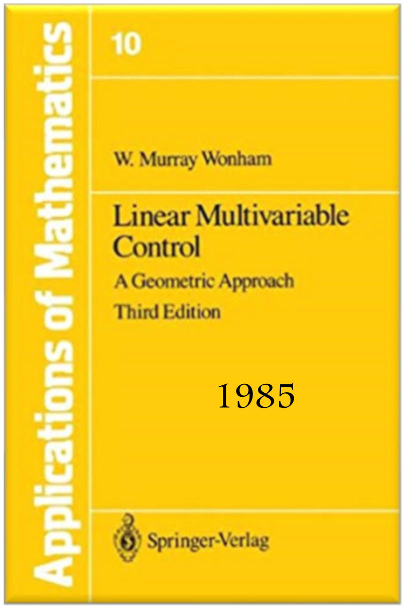
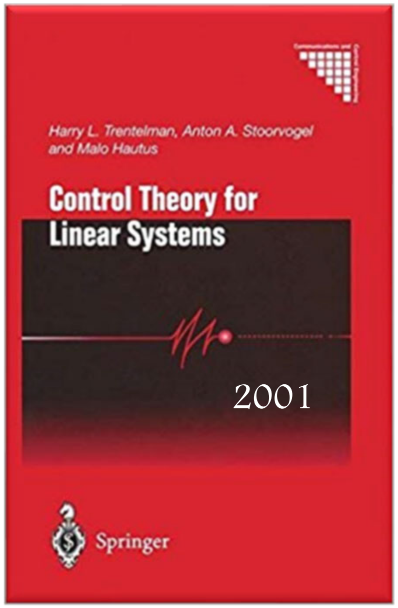
Other advanced linear systems theory for control using a geometric approach can also be found in the literature, e.g., the texts listed on the right.

Finally, we note that the control performance of a system depends more on its system structural properties rather than control methodologies used.

Don't expect to have a good performance if the system to be controlled is bad!





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www.mae.cuhk.edu.hk/~bmchen/linsyskit/index.html

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Linear Systems Toolkit

Zongli Lin, Ben M. Chen, Xinmin Liu

The Linear Systems Toolkit contains 66 m-functions that realize all the structural decompositions of linear systems, and their properties (such as finite and infinite zero structures, invertibility structure and geometric subspaces) as well as applications (such as system factorizations and sensor selection), documented in the monograph, ***Linear Systems Theory: A Structural Decomposition Approach***, authored by B. M. Chen, Z. Lin and Y. Shamash (Birkhauser, Boston, 2004).

The beta version of this toolkit is currently available for free. Interested readers might wish to register below. A zipped file that contains all m-functions of the toolkit will then be sent to the registered email addresses. Registered users will also automatically receive any advanced version of the toolkit through email. Nonetheless, the owners of the toolkit reserve all the rights. Users should bear in mind that the toolkit downloaded from the web site or received through email is free for use in research and academic work only. Uses for other purposes, such as commercialization, commercial development and redistribution without permission from the owners, are strictly prohibited.

The contents of the toolkit can be viewed by clicking this link. Some of these m-functions are interactive, which require users to enter desired parameters when executed. Some are implemented in a way that can return results either in a symbolic or numerical form. Detailed descriptions of the toolkit and the user manual can be found in Chapter 12 of the monograph.

Interested readers please **send us an email** with (1) your name; (2) email address; (3) institution; and (4) country. A zipped file, **linsyskit.zip**, containing all the m-functions of the toolkit will be sent to your email address. Please note that we might verify your information first before sending out the package to you. Once again, note that your information will be added to our database for distribution of future versions.

> > ... **This link leads to the list of errata for the monograph mentioned above ...**

> > ... **This link leads to bmchen.net (= www.mae.cuhk.edu.hk/~bmchen) for other toolkits ...**

Linear Systems & Control Toolkit m-functions

For control system design in Part 2…

Tzwo.m
Tzws.m
addps.m
atea.m
bdcsd.m
bdosd.m
blkz.m

csd.m
ctridx.m
daddps.m
dare.m
datea.m
ddpcm.m
dgm2star.m
dgm8star.m
dh2state.m

dh8state.m
diofact.m
dmpfact.m
dscb.m
dssd.m
ea_ds.m

gcfact.m
gm2sos.m
gm2star.m
gm8sos.m
gm8star.m
h2care.m
h2dare.m
h2out.m
h2state.m
h8care.m
h8dare.m
h8out.m

h8state.m
infz.m
infz_ds.m
invz.m
invz_ds.m
iofact.m
jcf.m
kcf.m
l_invt.m
l_invt_ds.m
ltrloops.m
morseidx.m
mpfact.m
n_star.m
normrank.m
obvidx.m
osd.m
r_invt.m
r_invt_ds.m
v_plus.m

Geometric subspaces…

r_star.m
rjd.m
rosys4ddp.m
s_lambda.m
s_minus.m
s_plus.m
s_star.m
sa_act.m
sa_sen.m
scb.m
scbraw.m
sd_ds.m
ss2tf_ds.m
ssadd.m
ssd.m
ssintsec.m
ssorder.m
v_lambda.m
v_minus.m
v_star.m

What is a system?

A system is a set of integrated chains of things.

In this course,

A system is a set of chains of integrators...

End of Part One…