MAEG4070 Engineering Optimization

Final Review

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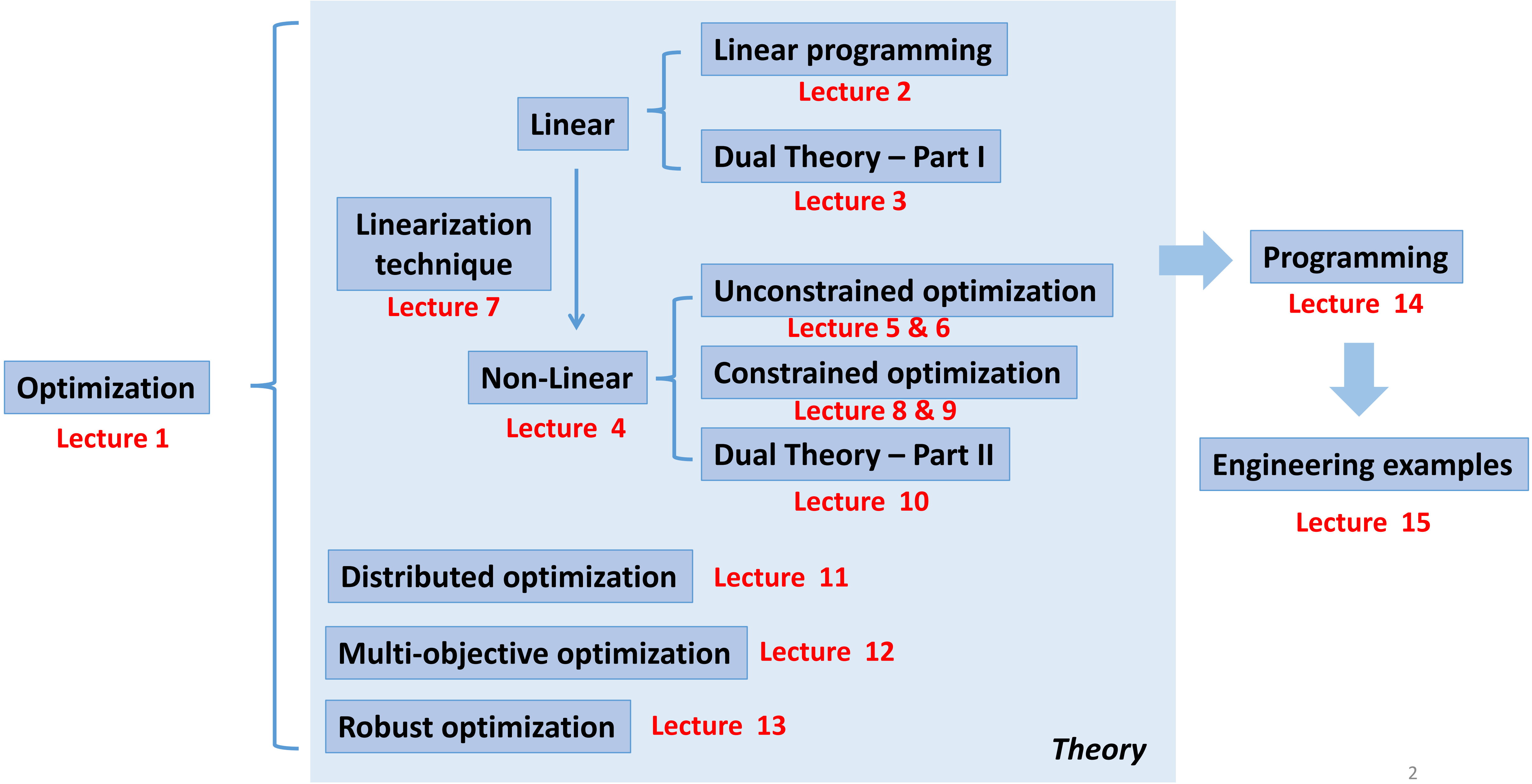
Content of this course



Online Office Hours Before Final Exam

Time: **Dec 12, 2022 (Monday) 13:00-14:00**

https://cuhk.zoom.us/j/95450969213?pwd=ZlZkWC9wMjlhOG5Da1RkMS82d0V6QT09

Meeting ID： 954 5096 9213
Passcode： 021836

Key points

The Final exam will focus on the lectures after mid-term exam (Lecture 8-13), but some key points learned before mid-term might also be tested.

Lecture 1 - Introduction

- **How to model an engineering problem into an optimization**
  - ✓ Lecture 1, pp. 22-27
  - ✓ Lecture 2, pp. 5-6
  - ✓ Lecture 3, pp. 3-5
  - ✓ Summary of Lecture 1-4, pp. 3-4
  - ✓ Homework-2, Q1
  - ✓ Lecture 8, pp. 3

Key points

Lecture 2 – Linear Programming

- **Check whether an optimization is a linear program or not**
  - ✓ Lecture 2, pp. 8-11
  - ✓ Mid-term, Q1

- **How to turn an optimization into a standard form**
  - ✓ Lecture 2, pp. 12-18
  - ✓ Summary of Lecture 1-4, pp. 5
  - ✓ Homework-1, Q1

- **Solving an LP using graphical method**
  - ✓ Lecture 2, pp. 20-24
  - ✓ Summary of Lecture 1-4, pp. 7-8
  - ✓ Homework-1, Q2
  - ✓ Mid-term, Q2

Key points

Lecture 3 – Dual Theory Part I

- **Write down the dual problem of a linear program**
  - ✓ Lecture 3, pp. 13-20
  - ✓ Summary of Lecture 1-4, pp. 11-12
  - ✓ Homework-1, Q3
  - ✓ Mid-term, Q5

- **Weak & Strong duality**
  - ✓ Lecture 3, pp. 23-28

- **Apply the complementary and slackness condition**
  - ✓ Lecture 3, pp. 29-31
  - ✓ Summary of Lecture 1-4, pp. 14-15
  - ✓ Mid-term, Q5

Key points

Lecture 4 – Convex sets and Convex functions

- **How to prove a set is a convex set**
  - ✓ Lecture 4, pp. 12-13, 16-20
  - ✓ Summary of Lecture 1-4, pp. 19-21
  - ✓ Homework-1, Q4
  - ✓ Homework-3, Q3
  - ✓ Mid-term, Q3

- **How to prove a function is a convex function**
  - ✓ Lecture 4, pp. 32-34
  - ✓ Summary of Lecture 1-4, pp. 26-27
  - ✓ Homework-1, Q5
  - ✓ Mid-term, Q1, Q4

Key points

Lecture 5 – Unconstrained Optimization (Basics)

- **How to determine the candidate optimal points; local optimum or global optimum? relative optimum or strict optimum?**
  - ✓ Lecture 5, pp. 8-9, 22-23, 26-29
  - ✓ Summary of Lecture 5-7, pp. 5, 9-15
  - ✓ Homework-2, Q2
  - ✓ Mid-term, Q4

Lecture 6 – Unconstrained Optimization (Gradient Based Methods)

- **How to apply the gradient descent method**
  - ✓ Lecture 6, pp. 17-18
  - ✓ Summary of Lecture 5-7, pp. 20
  - ✓ Mid-term, Q4

Key points

Lecture 7 – Linearization Techniques

- **Linearization technique for "minimizing a convex objective function"**
  - ✓ Lecture 7, pp. 12
  - ✓ Summary of Lecture 5-7, pp. 27

- **Big-M method**
  - ✓ Lecture 7, pp. 20-21
  - ✓ Mid-term, Q2

Lecture 8 - Constrained Optimization (Lagrange Multiplier)

- **Graphical methods to solve nonlinear optimization**
  - ✓ Lecture 8, pp. 4-5
  - ✓ Homework-3, Q2, Q3

Key points

Lecture 8 - Constrained Optimization (Lagrange Multiplier)

- **How to write the KKT condition of an optimization; determine whether a point is a KKT point or not.**
  - ✓ Lecture 8, pp. 11-16, 18-19, 28-29
  - ✓ Summary of Lecture 8-10, pp. 8-12
  - ✓ Homework-3, Q1

Lecture 9 – Convex Optimization

- **Determine whether an optimization is a convex optimization and solve it**
  - ✓ Lecture 9, pp. 11-19
  - ✓ Summary of Lecture 8-10, pp. 15-18
  - ✓ Homework-3, Q2, Q3

Key points

Lecture 10 – Dual Theory Part II

- **Write down the dual problem of a nonlinear optimization**
  - ✓ Lecture 10, pp. 7, 12-14, 16-17
  - ✓ Summary of Lecture 8-10, pp. 20-21
  - ✓ Homework-3, Q2

Lecture 11 – Distributed Optimization

- **How to apply the dual decomposition method**
  - ✓ Lecture 11, pp. 10-13
  - ✓ Homework-4, Q1

Key points

Lecture 12 – Multi-objective Optimization

- **Determine the dominance relationship between two points**
  - ✓ Lecture 12, pp. 10
  - ✓ Homework-4, Q2

- **Pros and Cons of different multi-objective optimization methods**
  - ✓ Lecture 12, pp. 19, 21

Lecture 13 – Robust Optimization

- **How to solve a static robust optimization with discrete uncertainty set**
  - ✓ The "cake" example last week
  - ✓ Homework-4, Q3
- **Understand the concept of "robust feasible" and "robust optimal"**
  - ✓ Lecture 13, pp. 17

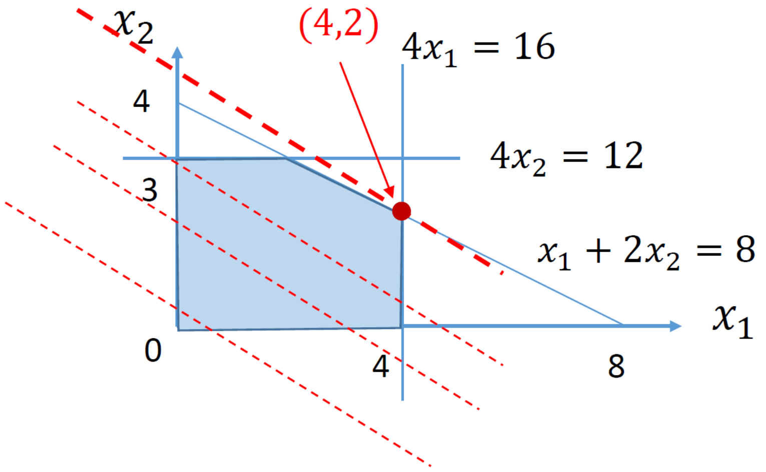
Comparison 1 – Unconstrained v.s. Constrained

Unconstrained optimization	Constrained optimization
Stationary point $\nabla f(x^*) = 0$	KKT point **Regular point** $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$ $h_i(x^*) = 0, \forall i = 1, ..., m$ $0 \leq -g_j(x^*) \perp \mu_j^* \geq 0, \forall j = 1, ..., r$
Convex function & global optimum $\min_x f(x)$ **Hessian matrix positive semi-definite** **→ $f(x)$ convex → global minimum**	Convex optimization & global optimum $\min_x \; f(x)$ ← **convex** s.t. $a_i^T x - b_i = 0, \forall i = 1, ..., m$ ← **linear** $g_j(x) \leq 0, j = 1, ..., r$ ← **convex**

Comparison 2 – Linear Programming v.s. Nonlinear Optimization

Linear Programming

Graphical method



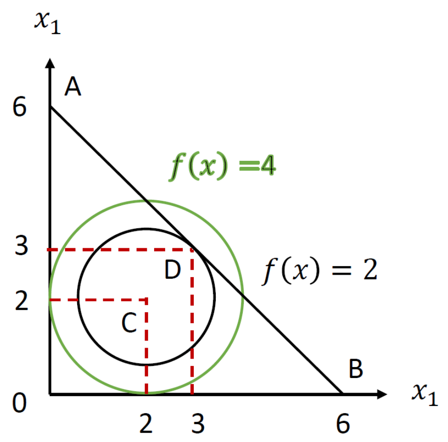
Strong duality

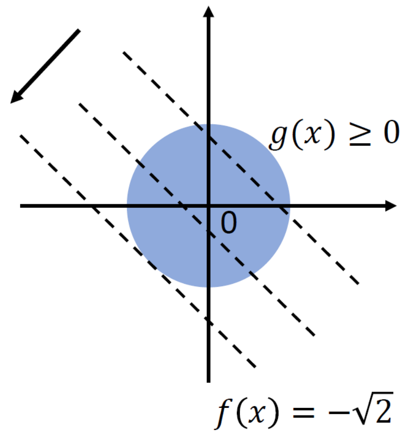
Suppose x^* and λ^* are the optimal solutions of the primal and dual problems, respectively, then

$$c^T x^* = d^T \lambda^*$$

Nonlinear Optimization

Graphical method





Strong duality

Suppose f^* and Q^* are the primal and dual optimal objective values, however,

f^* may not equal to Q^*

Comparison 2 – Linear Programming v.s. Nonlinear Optimization

Construct the dual optimization of this LP:

Primal

$$\begin{aligned} \min_{x} \;& c^T x \\ \text{s.t. } & A_1 x \geq b_1 \text{ or } -A_1 x \leq -b_1 \\ & A_2 x = b_2 \text{ or } -A_2 x = -b_2 \end{aligned}$$

Dual

$$\begin{aligned} \max_{\mu,\lambda} \;& \mu^T b_1 + \lambda^T b_2 \\ \text{s.t. } & A_1^T \mu + A_2^T \lambda = c \\ & \mu \geq 0 \end{aligned}$$

The Lagrangian is

$$\begin{aligned} L(x,\mu,\lambda) =& \; c^T x - \mu^T (A_1 x - b_1) - \lambda^T (A_2 x - b_2) \\ =& \; (c^T - \mu^T A_1 - \lambda^T A_2) x + \mu^T b_1 + \lambda^T b_2 \end{aligned}$$

If any element i of $c^T - \mu^T A_1 - \lambda^T A_2 > 0$, let $x_i \to -\infty$, $x_{j\neq i} = 0$, then $L = -\infty$.
If any element i of $c^T - \mu^T A_1 - \lambda^T A_2 < 0$, let $x_i \to \infty$, $x_{j\neq i} = 0$, then $L = -\infty$.
Therefore, $c^T - \mu^T A_1 - \lambda^T A_2 = 0$ and $\min_x L = \mu^T b_1 + \lambda^T b_2$.
The dual problem is

$$\begin{aligned} \max_{\mu,\lambda} \;& \mu^T b_1 + \lambda^T b_2 \\ \text{s.t. } & A_1^T \mu + A_2^T \lambda = c \\ & \mu \geq 0 \end{aligned}$$

Comparison 2 – Linear Programming v.s. Nonlinear Optimization

Linear Programming

Complementary and Slackness

Primal problem

$$\min_x \; c^T x$$
$$\text{s.t.} \; Ax \geq b$$
$$x \geq 0$$

Dual problem

$$\max_\lambda \; b^T \lambda$$
$$A^T \lambda \leq c$$
$$\lambda \geq 0$$

$$0 \leq \lambda \perp (Ax - b) \geq 0$$

$$0 \leq \lambda \leq Mz$$
$$0 \leq (Ax - b) \leq M(1 - z)$$
$$z \in \{0,1\}$$

Nonlinear Optimization

Complementary and Slackness

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$
$$h_i(x^*) = 0, \forall i = 1, \dots, m$$
$$0 \leq -g_j(x^*) \perp \mu_j^* \geq 0, \forall j = 1, \dots, r$$

If $g_j(x)$ is linear, then it can be linearized:

$$0 \leq \mu_j^* \leq Mz$$
$$0 \leq -g_j(x^*) \leq M(1 - z)$$
$$z \in \{0,1\}$$

Comparison 3 – Gradient descent v.s. Dual Decomposition

Gradient descent

$$x_k = x_{k-1} - \alpha \nabla f(x_{k-1})$$

Dual ascent

$$\begin{aligned} x^{k+1} &= \text{argmin}_x L(x, \lambda^k) \\ \lambda^{k+1} &= \lambda^k + \alpha^k (A x^{k+1} - b) \end{aligned}$$

Dual decomposition

$$\begin{aligned} x_n^{k+1} &= \text{argmin}_{x_n} L_n(x_n, \lambda^k), \forall n = 1, ..., N \\ \lambda^{k+1} &= \lambda^k + \alpha^k (\sum_{n=1}^{N} A_n x_n^{k+1} - b) \end{aligned}$$

Comparison 4 – Single Objective & Multi-objective

$$\begin{aligned} &\min_x \ f(x) \\ &\text{s.t. } g_i(x) \leq 0, \forall i = 1, ..., I \\ &\quad\ h_j(x) = 0, \forall j = 1, ..., J \end{aligned}$$

$$\begin{aligned} &\min_x \ f_1(x) \\ &\max_x \ \vdots \\ &\min_x \ f_n(x) \\ &\text{s.t. } g_i(x) \leq 0, \forall i = 1, ..., I \\ &\quad\ h_j(x) = 0, \forall j = 1, ..., J \end{aligned}$$

Better – dominant
Best solution – Pareto optimal sets
Optimal value – Pareto optimal front

Some Tips

1. Check clearly whether it is a “min” or “max” problem
2. Remember to turn it into a standard form, e.g., $g(x) \leq 0$
3. To solve a “$\max_x f(x)$” problem, you can solve “$\min_x -f(x)$” instead
4. To prove a matrix A is negative definite, you can try to prove that $-A$ is positive definite instead
5. …

Final exam

Time: **Dec 13, 2022 12:30-14:30**

Location: **Multi-purpose Hall, Pommerenke Student Center**

- **Closed-book, no calculator**.
- Formal answer books and scratch papers will be provided.
- **No discussion or communication** with others during the exam.
- Coverage: **Lecture 1-13**

Good luck!