MAEG4070 Engineering Optimization

Solution 1

1. For the original problem, first, we turn the "max" into "min"

$$\begin{aligned}\min_{x_1,x_2} \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 + |x_2 - 3| \leq 10 \\ & x_1 + 5x_2 \geq 3 \\ & x_1 \geq 1\end{aligned}$$

Then we eliminate the absolute value in constraints

$$\begin{aligned}\min_{x_1,x_2} \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 13 \\ & -2x_1 + x_2 \geq -7 \\ & x_1 + 5x_2 \geq 3 \\ & x_1 \geq 1\end{aligned}$$

Introduce slack or surplus variables to turn the inequalities into equalities

$$\begin{aligned}\min_{x_1,x_2,y_1,y_2,y_3} \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 + y_1 = 13 \\ & -2x_1 + x_2 - y_2 = -7 \\ & x_1 + 5x_2 - y_3 = 3 \\ & x_1 \geq 1, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\end{aligned}$$

Let $v_1 = x_1 - 1, v_1 \geq 0$, we have

$$\begin{aligned}\min_{v_1,x_2,y_1,y_2,y_3} \quad & -3v_1 - x_2 - 3 \\ \text{s.t.} \quad & 2v_1 + x_2 + y_1 = 11 \\ & -2v_1 + x_2 - y_2 = -5 \\ & v_1 + 5x_2 - y_3 = 2 \\ & v_1 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\end{aligned}$$

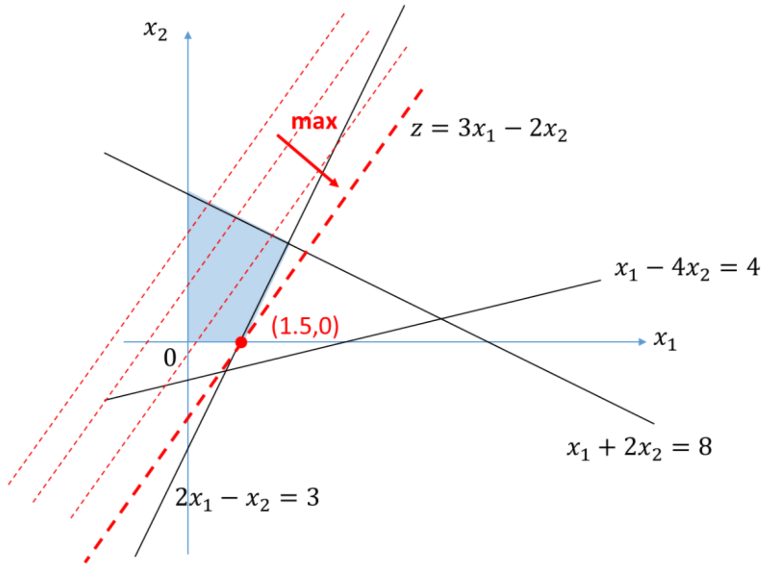
Finally, let $x_2 = u_1 - u_2, u_1 \geq 0, u_2 \geq 0$, we have

$$\begin{aligned}\min_{v_1,u_1,u_2,y_1,y_2,y_3} \quad & -3v_1 - u_1 + u_2 - 3 \\ \text{s.t.} \quad & 2v_1 + u_1 - u_2 + y_1 = 11 \\ & -2v_1 + u_1 - u_2 - y_2 = -5 \\ & v_1 + 5u_1 - 5u_2 - y_3 = 2 \\ & v_1 \geq 0, u_1 \geq 0, u_2 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\end{aligned}$$

Therefore, the compact form is

$$
\begin{aligned}
x = & \ [v_1, u_1, u_2, y_1, y_2, y_3]^\top \\
c = & \ [-3, -1, 1, 0, 0, 0] \\
A = & \begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & -1 & 0 \\ 1 & 5 & -5 & 0 & 0 & -1 \end{bmatrix} \\
b = & \ [11, -5, 2]^\top
\end{aligned}
$$

2. The feasible region of this LP is



The optimal solution is $x^* = (1.5,0)$, the optimal value is $f^* = 3 \times 1.5 - 2 \times 0 = 4.5$

3. The dual problem is

$$
\begin{aligned}
\max_{\lambda_1,\lambda_2,\lambda_3} \quad & 2\lambda_1 + 10\lambda_2 + 5\lambda_3 \\
\text{s.t.} \quad & \lambda_1 + \lambda_2 \le 2 \\
& \lambda_1 + \lambda_3 = 0 \\
& \lambda_1 - 2\lambda_2 + 2\lambda_3 \ge 3 \\
& \lambda_1 \ge 0, \lambda_2 \le 0
\end{aligned}
$$

4. Given any two point $x = [x_1, x_2] \in S, y = [y_1, y_2] \in S$. For any $0 \le \lambda \le 1$, we have

$$\lambda x + (1-\lambda)y = [\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2]$$

Then we check whether $\lambda x + (1-\lambda)y$ is in S.

$$
\begin{aligned}
& 3[\lambda x_1 + (1-\lambda)y_1] - [\lambda x_2 + (1-\lambda)y_2] \\
\ge & \ \lambda(3x_1 - x_2) + (1-\lambda)(3y_1 - y_2) \\
\ge & \ \lambda + 1 - \lambda = 1
\end{aligned}
$$

$$\begin{aligned} & [\lambda x_1 + (1-\lambda)y_1] + [\lambda x_2 + (1-\lambda)y_2] \\ \leq\ & \lambda(x_1 + x_2) + (1-\lambda)(y_1 + y_2) \\ \leq\ & \lambda + 1 - \lambda = 1 \end{aligned}$$

Therefore, $\lambda x + (1-\lambda)y \in S$, so S is a convex set.

5. The function is convex.

   First, obviously, the domain of $f(x_1, x_2)$ is a convex set.

   Second, the gradient of $f(x_1, x_2) = x_1^2 - 4x_1x_2 + 4x_2^2 + 3x_1 + x_2$ is

$$\frac{\partial f}{\partial x_1} = 2x_1 - 4x_2 + 3, \frac{\partial f}{\partial x_2} = -4x_1 + 8x_2 + 1$$

   The Hessian is

$$H(x) = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$$

$H(x)$ is positive semi-definite. Therefore, $f(x_1, x_2)$ is a convex function.