MAEG4070 Engineering Optimization

Solution 2

1. Decision variables: let K, L denote the amount of capital and labor the firm invests.
   Constraints: The production should be at least 10. Both K, L are non-negative
   Objective: minimize the cost $30K + 30L$

   The optimization problem is

$$\min_{K,L} \; 30K + 30L$$
$$\text{s.t. } K^{\frac{1}{4}} L^{\frac{1}{4}} \geq 10$$
$$K \geq 0, L \geq 0$$

2. The gradient of $f(x)$ is

$$\frac{\partial f}{\partial x_1} = x_1^2 - 1$$
$$\frac{\partial f}{\partial x_2} = x_2^2 - 2x_2$$

Let $\nabla f = \mathbf{0}$, we have

$$x_1^2 - 1 = 0$$
$$x_2^2 - 2x_2 = 0$$

Thus, we have four candidate optimal solutions $x^{(1)} = (1,0)^T$, $x^{(2)} = (1,2)^T, x^{(3)} = (-1,0)^T, x^{(4)} = (-1,2)^T$.

The Hessian matrix is

$$H(x) = \begin{pmatrix} 2x_1 & 0 \\ 0 & 2x_2 - 2 \end{pmatrix}$$

Since the function $f(x)$ is not a convex function, what we get will be "local" instead of "global" optimum. The value of Hessian matrix at different candidate points are

Candidate point	$H(x)$	Nature of H
$x^{(1)} = (1,0)^T$	$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$	Indefinite
$x^{(2)} = (1,2)^T$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	Positive definite
$x^{(3)} = (-1,0)^T$	$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$	Negative definite
$x^{(4)} = (-1,2)^T$	$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$	Indefinite

Therefore, $x^{(1)}$ and $x^{(4)}$ are neither maximum nor minimum; $x^{(2)}$ is strict local minimum; $x^{(3)}$ is strict local maximum.

3. Denote the objective function as $f(x) = (x_1 - 1)^3 + 2x_2^2$.

The gradient and Hessian matrix of $f(x)$ are

$$\nabla f(x) = \begin{bmatrix} 3(x_1-1)^2 \\ 4x_2 \end{bmatrix},\ H(x) = \begin{bmatrix} 6(x_1-1) & 0 \\ 0 & 4 \end{bmatrix}$$

Iteration 1:

$$\nabla f(x^{(0)}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix},\ H(x^{(0)}) = \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{aligned} x^{(1)} &= x^{(0)} - \left[H(x^{(0)})\right]^{-1} \nabla f(x^{(0)}) \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \end{aligned}$$

Iteration 2:

$$\nabla f(x^{(1)}) = \begin{bmatrix} 0.75 \\ 0 \end{bmatrix},\ H(x^{(1)}) = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{aligned} x^{(2)} &= x^{(1)} - \left[H(x^{(1)})\right]^{-1} \nabla f(x^{(1)}) \\ &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0.75 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0 \end{bmatrix} \end{aligned}$$

4. (1) Method 1: Let $t = 5x \in [15,50]$, then $z = ty$ can be linearized as

$$\begin{aligned} &15y \le z \le 50y \\ &15(1-y) \le t - z = 5x - z \le 50(1-y) \end{aligned}$$

Method 2: Let $z' = xy$ then we have

$$3y \le z' \le 10y$$
$$3(1-y) \le x - z' \le 10(1-y)$$

Moreover, as $z' = z/5$, then substitute this into the above functions

$$3y \le z/5 \le 10y$$
$$3(1-y) \le x - z/5 \le 10(1-y)$$

(2) Let $L = \max\{10,8,12\} = 12$, then

$$\begin{aligned} &1 \le x_1 \le 10, 2 \le x_2 \le 8, 3 \le x_3 \le 12 \\ &y \ge x_1, y \ge x_2, y \ge x_3 \\ &x_1 + (12-1)(1-z_1) \ge y \\ &x_2 + (12-2)(1-z_2) \ge y \\ &x_3 + (12-3)(1-z_3) \ge y \\ &z_1, z_2, z_3 \in \{0,1\}, z_1 + z_2 + z_3 = 1 \end{aligned}$$