MAEG4070 Engineering Optimization

Solution 3

1. The Lagrangian function is

$$L(x,\mu) = (x_1-4)^2 + (x_2-4)^2 + \mu_1(x_1+x_2-4) + \mu_2(x_1+3x_2-9)$$

Then the KKT condition is

$$\begin{aligned} 2(x_1-4)+\mu_1+\mu_2 &= 0 \\ 2(x_2-4)+\mu_1+3\mu_2 &= 0 \\ 0 \le (-x_1-x_2+4) \perp \mu_1 &\ge 0 \\ 0 \le (-x_1-3x_2+9) \perp \mu_2 &\ge 0 \end{aligned}$$

For the point $x^* = \left(\frac{3}{2},\frac{5}{2}\right)^T$, we can get that $\mu_1 = 6, \mu_2 = -1 < 0$.

Therefore, it is not a KKT point.

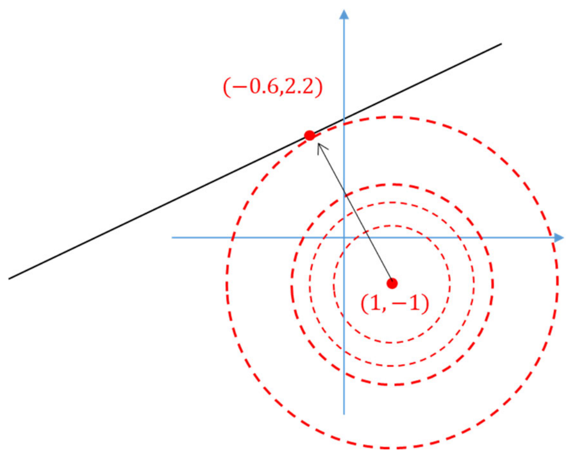
2. (1) Denote the objective function as $f(x) = (x_1-1)^2 + (x_2+1)^2$, then gradient is

$$\nabla f(x) = \begin{bmatrix} 2(x_1-1) \\ 2(x_2+2) \end{bmatrix}$$

And the Hessian matrix is

$$H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \succ 0$$

So $f(x)$ is a convex function. Denote $g(x) = x_1 - 2x_2 + 5$, then the constraint can be represented as $g(x) \le 0$.
Since both $f(x)$ and $g(x)$ are convex functions, the problem is a convex optimization.
(2) The feasible region and the contours of the objective function are



Therefore, $x^* = (-0.6, 2.2), f^* = 12.8$.

(3) The Lagrangian function is

$$L(x,\mu) = (x_1-1)^2 + (x_2+1)^2 + \mu(x_1 - 2x_2 + 5)$$

The KKT condition is

$$\begin{aligned} 2(x_1-1)+\mu &= 0 \\ 2(x_2+1)-2\mu &= 0 \\ 0 \le \mu \perp (-x_1+2x_2-5) &\ge 0 \end{aligned}$$

If $\mu^* = 0$, then $x_1^* = 1, x_2^* = -1$, and $-x_1^* + 2x_2^* - 5 = -1 - 2 - 5 = -8 < 0$,the constraint is not satisfied.

If $\mu^* \neq 0$, then we have $-x_1^* + 2x_2^* - 5 = 0$. Together with the first two equations, we can obtain $x^* = (-0.6, 2.2), \mu^* = 3.2, f^* = 12.8$.

(4) The Lagrangian function is

$$\begin{aligned} L(x,\mu) = &\ (x_1-1)^2 + (x_2+1)^2 + \mu(x_1-2x_2+5) \\ = &\ x_1^2 - 2x_1 + 1 + x_2^2 + 2x_2 + 1 + \mu x_1 - 2\mu x_2 + 5\mu \\ = &\ x_1^2 + (\mu-2)x_1 + \frac{(\mu-2)^2}{4} + x_2^2 + (2-2\mu)x_2 + (1-\mu)^2 + 5\mu - \frac{(\mu-2)^2}{4} - (1-\mu)^2 \\ = &\ \left(x_1 - \frac{\mu-2}{2}\right)^2 + (x_2-(1-\mu))^2 + 5\mu - \frac{(\mu-2)^2}{4} - (1-\mu)^2 \end{aligned}$$

Therefore,

$$\min_x L(x,\mu) = 5\mu - \frac{(\mu-2)^2}{4} - (1-\mu)^2 = -\frac{5}{4}\mu^2 + 8\mu - 2$$

The dual problem is

$$\begin{aligned} \max_\mu &\quad -\frac{5}{4}\mu^2 + 8\mu \\ \text{s.t.} &\ \ \mu \ge 0 \end{aligned}$$

3. (1) Given any two points $x = [x_1, x_2] \in S, y = [y_1, y_2] \in S$. For any $0 \le \lambda \le 1$, we have

$$\lambda x + (1-\lambda)y = [\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2]$$

Then we check whether $\lambda x + (1-\lambda)y$ is in S.

$$\begin{aligned} 2[\lambda x_1 + (1-\lambda)y_1] + [\lambda x_2 + (1-\lambda)y_2] &= \lambda(2x_1 + x_2) + (1-\lambda)(2y_1 + y_2) \\ &\le 4\lambda + 4(1-\lambda) = 4 \end{aligned}$$

Therefore, $\lambda x + (1-\lambda)y \in S$, so S is a convex set.

(2) We can minimize the square distance instead, and the optimization problem is

$$\begin{aligned}\min_{x_1,x_2} \ & (x_1-4)^2+(x_2-4)^2 \\ \text{s.t. } & 2x_1+x_2 \le 4\end{aligned}$$

The Lagrangian function is

$$L(x,\mu) = (x_1-4)^2+(x_2-4)^2+\mu(2x_1+x_2-4)$$

The KKT condition is

$$\begin{aligned}2(x_1-4)+2\mu &= 0 \\ 2(x_2-4)+\mu &= 0 \\ 0 \le \mu \perp (4-2x_1-x_2) &\ge 0\end{aligned}$$

If $\mu = 0$, we have $x_1 = x_2 = 4$, the constraint is not met.
If $\mu > 0$, we have

$$\begin{aligned}2x_1-8+2\mu &= 0 \\ 2x_2-8+\mu &= 0 \\ 2x_1+x_2-4 &= 0\end{aligned}$$

Therefore, $x^* = (\frac{4}{5}, \frac{12}{5})$.
Since the optimization is a convex optimization, x^* is global optimal.