MAEG4070 Engineering Optimization

Lecture 13 Robust Optimization

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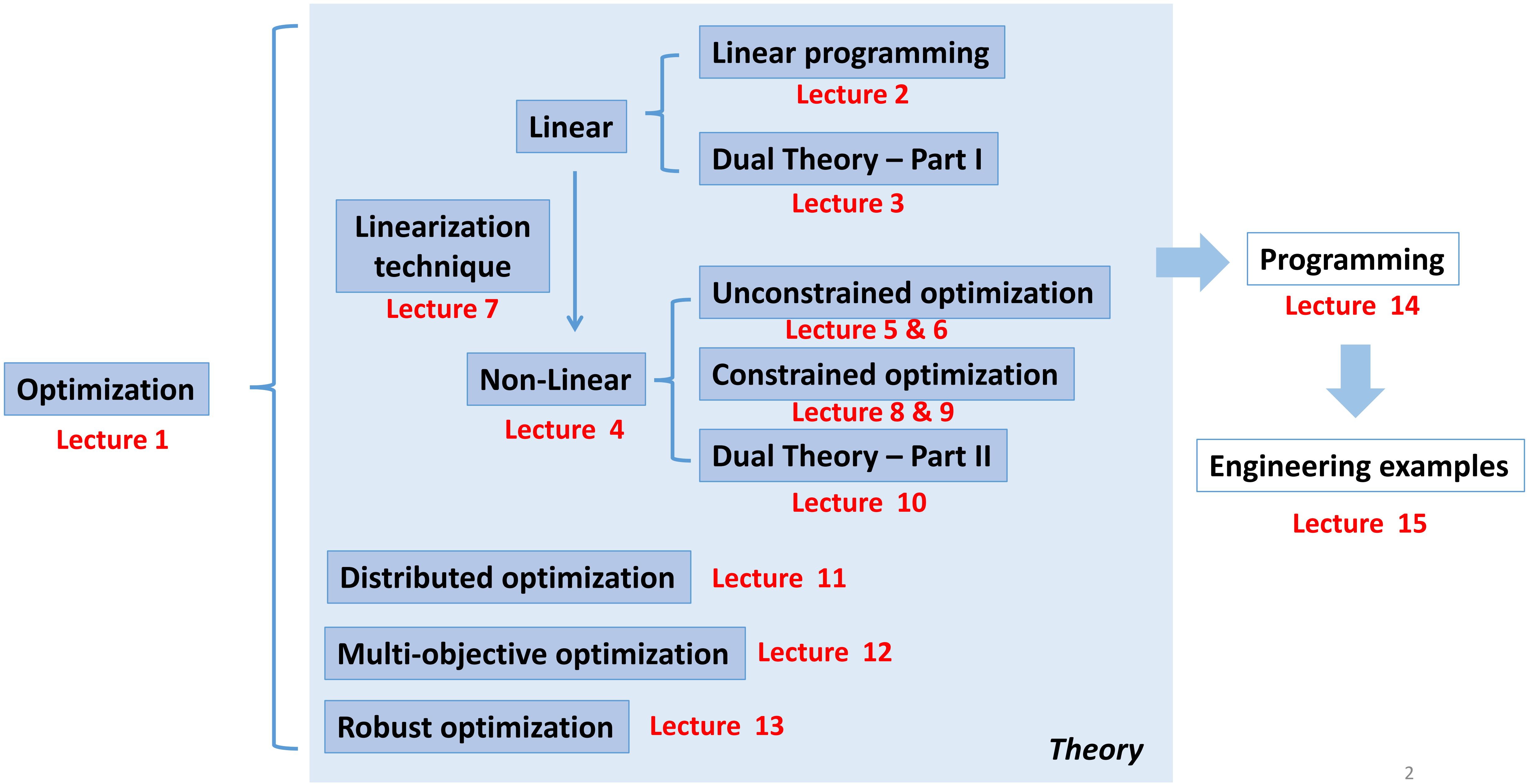
Content of this course (tentative)



Motivation— Why optimization under uncertainty?

Deterministic optimization models assume known parameters

Precise data is unavailable in most practical cases:

- Numerical error (data precision)
- Measurement error (cost coefficients)
- Forecast error (wind/solar power prediction)
- Changing environment (the attitudes and ability of the decision maker)

How much will the solution be affected by uncertainty

- In 13 of 90 benchmark LPs, 0.01% perturbations of uncertain data result in severe constraint violations
- 0.1% coefficient perturbation makes 50% optimal solutions infeasible

Reference: Arkadi N. Lectures on Robust Convex Optimization. Available at: https://www.ims.nus.edu.sg/oldwww/Programs/012opti/files/arkadi1p.pdf

Approaches to handle uncertainty

Q: How to handle uncertainty in an optimal way?

- Stochastic optimization
- Robust optimization

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Methods

- Generate scenarios to represent possible realizations of uncertain data
- Assign a probability to each scenario

Advantages

- Various models(LP, MILP, SDP...)
- Less conservative in the sense of statistics

Challenges

- Scenario generation and reduction (proper scenarios and their probabilities)
- Computationally challenging

Reference: Conejo A J, Carrión M, Morales J M. Decision making under uncertainty in electricity markets[M]. New York: Springer, 2010.

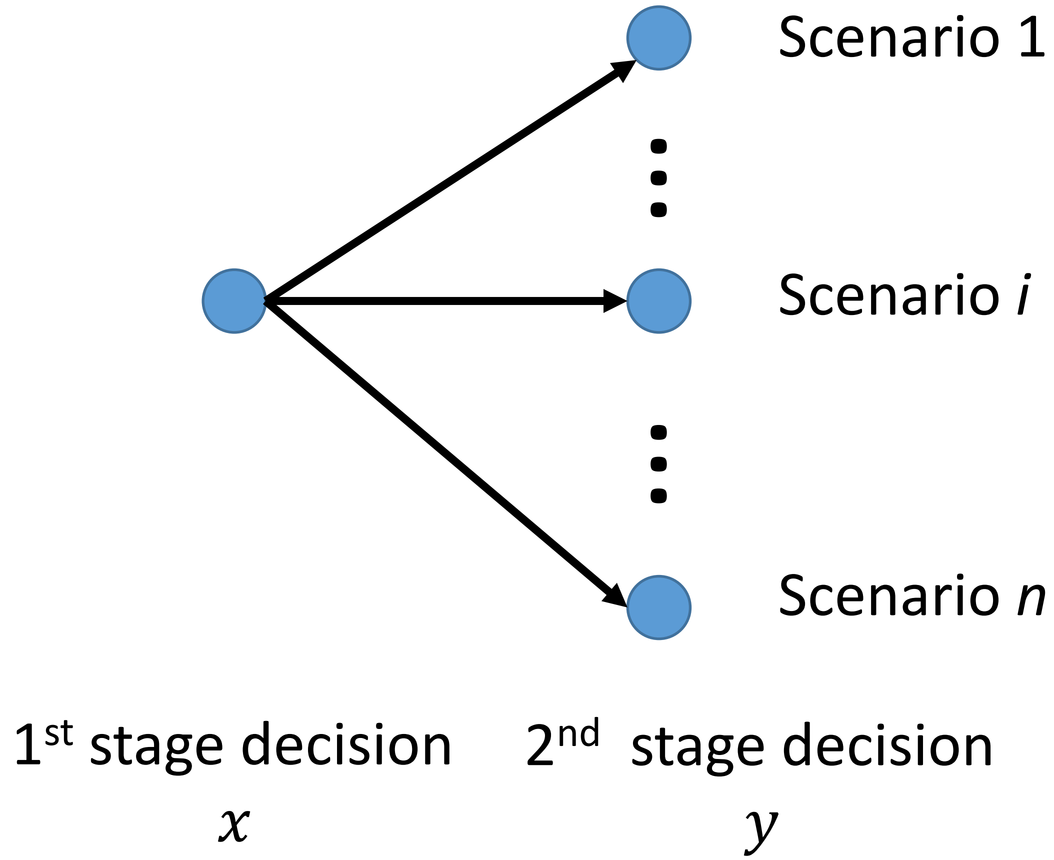
Stochastic optimization

Expectation

$$\min_{x,y} \; \mathbb{E}_w f(x,y,w)$$
$$\text{s.t. } h(x,y,w) = 0$$
$$g(x,y,w) \leq 0$$

x - Here-and-now variables
w – stochastic vector
y – Wait-and-see variables (optional)

Decision tree



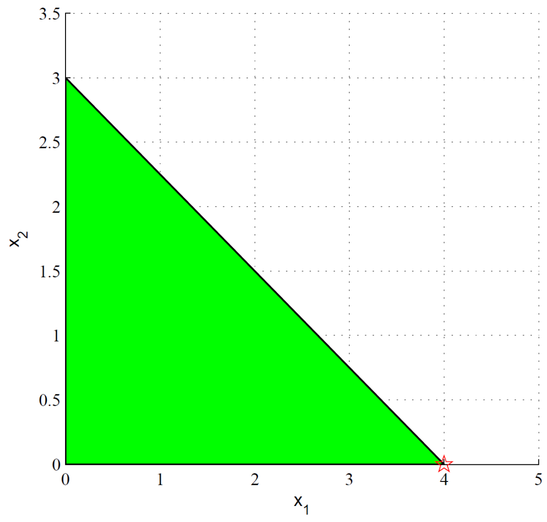
An example on uncertain LP

Consider this optimization problem:

$$\max_{x_1,x_2} \; z = x_1 + x_2$$

$$\text{s.t. } ax_1 + bx_2 \le 12$$

$$x_1 \ge 0, x_2 \ge 0$$



If $a = 3, b = 4$ are deterministic, then

$$x^* = (4,0), z^* = 4$$

If parameters a, b are uncertain may take the values of

$$(a, b) = (2,6), (3,5), (5,3), (6,2)$$

Each scenario has a probability of 0.25.

Try to formulate a stochastic optimization for this problem?

Approaches to handle uncertainty

Q: How to handle uncertainty in an optimal way?

- Stochastic optimization
- Robust optimization

Methods

- Use a pre-specified set to model uncertain data
- Optimize the outcome in the worst case

Advantages

- guarantee in the absence of exact input data, provable performance
- More tractable than SO

Challenges

- Conservativeness
- restrict recourse policy

Robust optimization

$$
\begin{aligned}
\min_{x} \; \max_{w} \; \min_{y} \quad & f(x,y,w) \\
\text{s.t.} \quad & h^2(x,w,y) = 0 \\
& g^2(x,w,y) \leq 0 \\
& y \in \mathcal{Y} \\
\text{s.t.} \quad & w \in \mathcal{W} \\
\text{s.t.} \quad & h^1(x) = 0 \\
& g^1(x) \leq 0 \\
& x \in \mathcal{X}
\end{aligned}
$$

Uncertainty set
(contains all scenarios)

x - Here-and-now variables
w – stochastic vector
y – Wait-and-see variables (optional)

When we do NOT have “wait-and-see variables”, it is called “**static** robust optimization”

When we have “wait-and-see variables”, it is called “**adjustable** robust optimization”

Main contributors

Aharon Ben-Tal (1946-)

- Convex optimization
- Robust optimization
- INFORMS Khachiyan Prize, 2016
- INFORMS Fellow, SIAM Fellow
- EURO Gold Medal, 2007

Dimitris Bertsimas (1962-)

- Optimization, Applied Probability
- Transportation, Finance
- Farkas Prize
- INFORMS Fellow
- Member of National Academy of Engineering

An example on uncertain LP

Consider this optimization problem:

$$\max_{x_1, x_2} \; z = x_1 + x_2$$

$$\text{s.t. } ax_1 + bx_2 \le 12$$

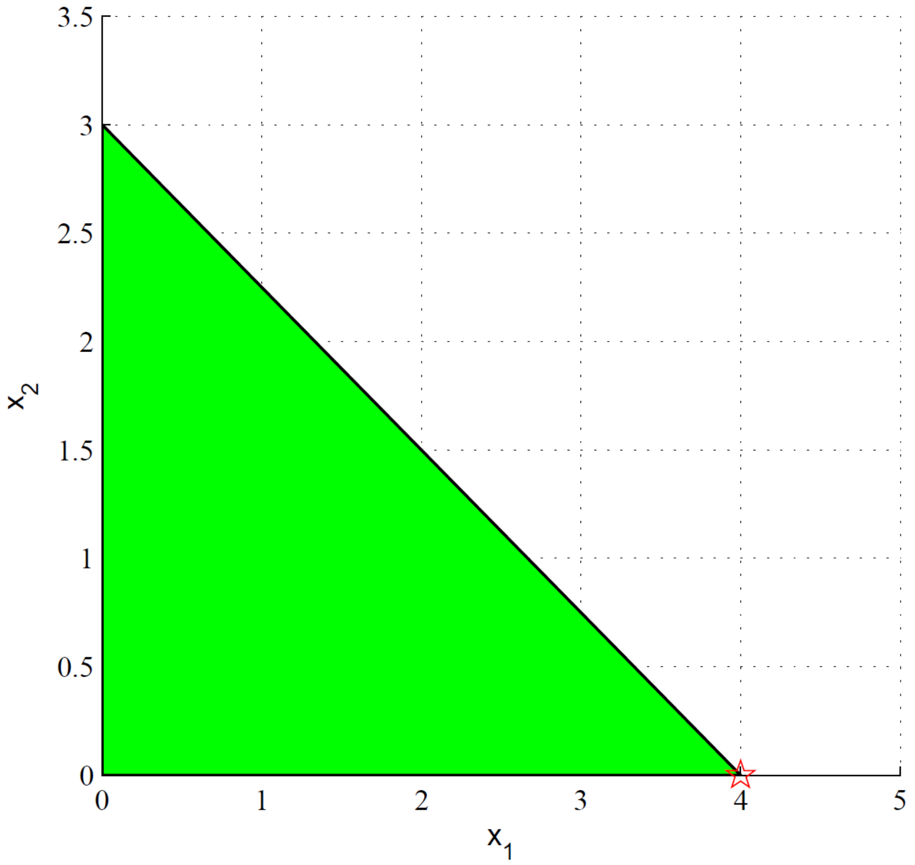
$$x_1 \ge 0, x_2 \ge 0$$

If $a = 3, b = 4$ are deterministic, then

$$x^* = (4{,}0), z^* = 4$$

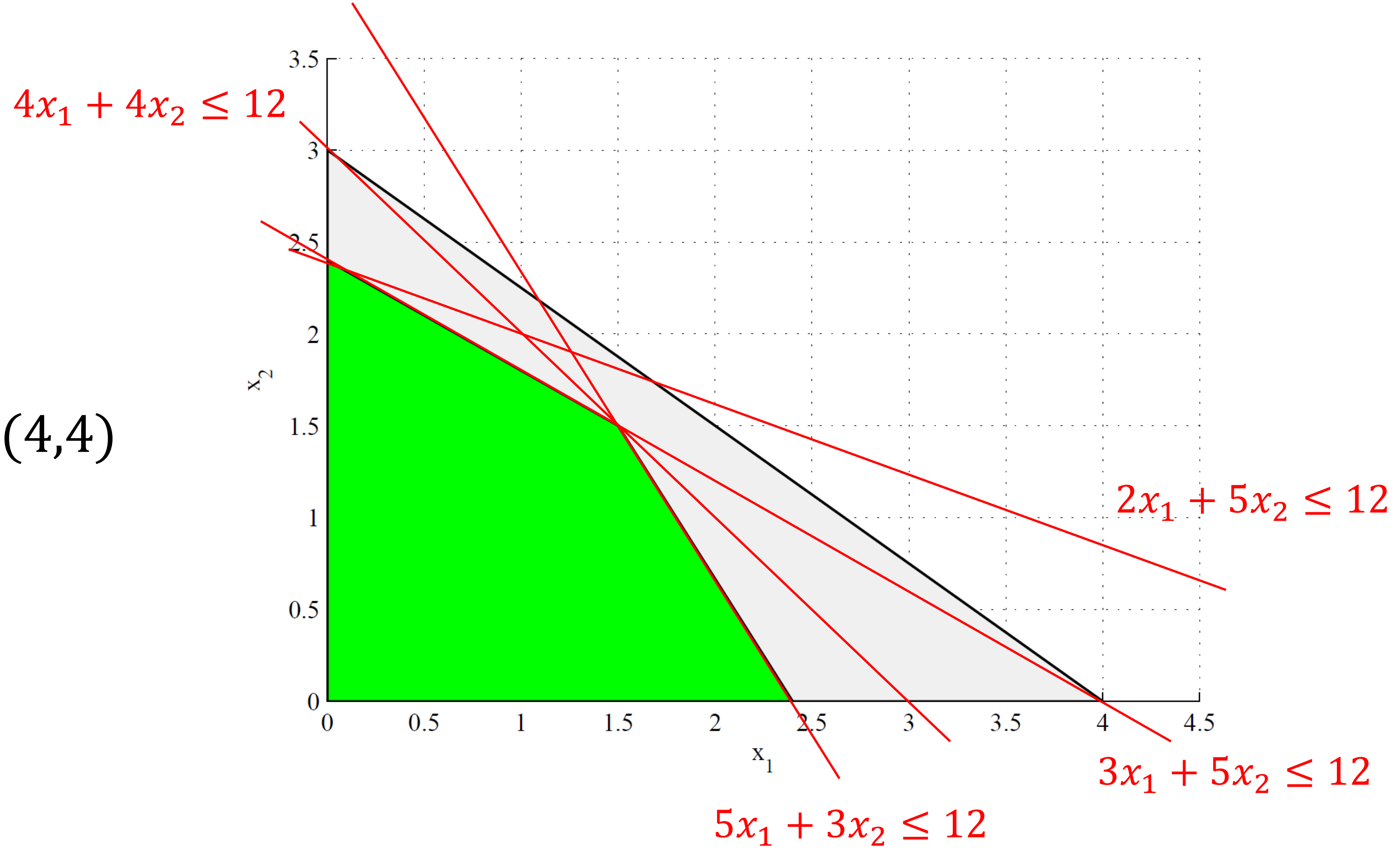
If parameters a, b are uncertain, and

$$a \ge 2, b \ge 2, 6 \le a + b \le 8$$



An example on uncertain LP

If we let
$(a, b) = (2,5), (5,3), (3,5), (4,4)$



An example on uncertain LP

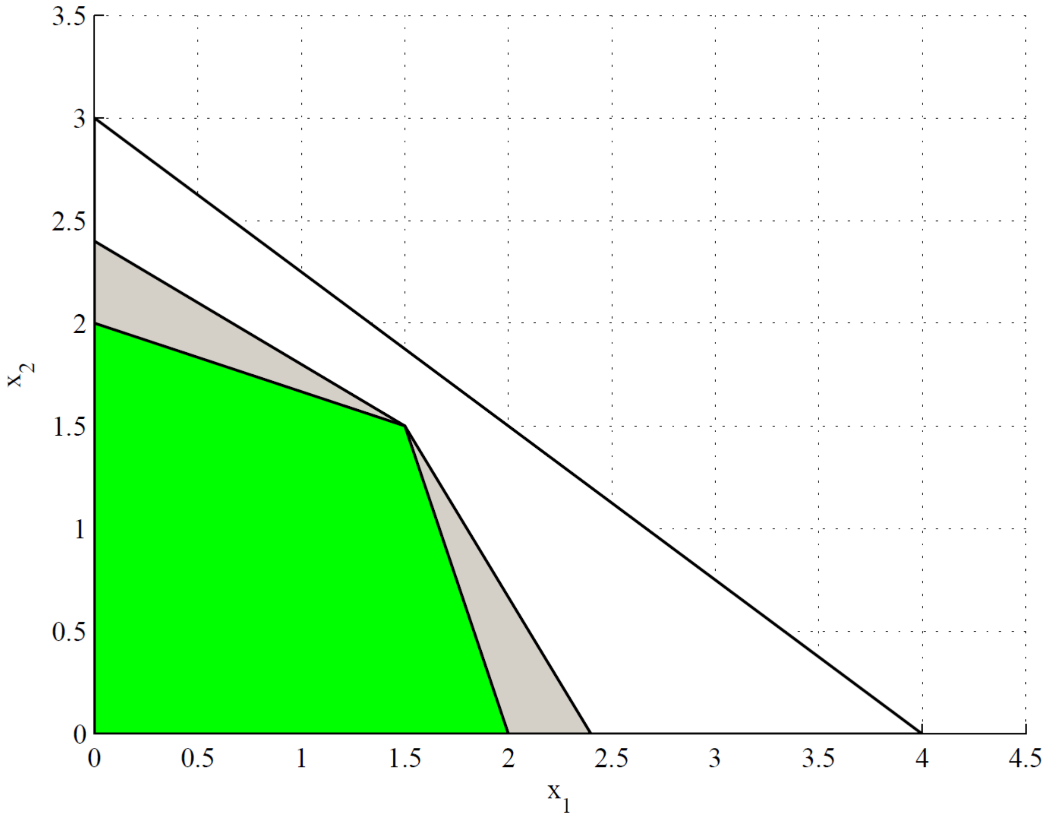
If we let

$$(a, b) = (2,6), (6,2)$$

The feasible region further ↓

In fact, the "robust" optimal solution is

$$x^* = (1.5, 1.5), z^* = 3$$



An example on uncertain LP

Mathematical Analysis:

- The worst data must satisfy $a + b = 8$. Why?

- The constraints yields a cluster of lines

$$ax_1 + (8 - a)x_2 \leq 12$$

All the lines in the cluster pass through point $(1.5, 1.5)$

- The extreme scenarios are $(a, b) = (2,6); (6,2)$

- The robust feasible set is given by (regardless of the objective function)

$$x_1 \geq 0, 6x_1 + 2x_2 \leq 12$$
$$x_2 \geq 0, 2x_1 + 6x_2 \leq 12$$

Static robust optimization

$$\min_{x} \; c^\top x$$
$$\text{s.t. } Ax \leq b, \forall A \in W$$

where x represent "here-and-now" decisions: they should be specified before the actual data is known.

Constraints are mandatory, i.e., constraint violation is not tolerable when $A \in W$.

Without loss of generality, we assume

1. the objective is deterministic
2. the right-hand side of each constraint is deterministic
3. W is compact and convex
4. the uncertainty is constraint-wise

Static robust optimization

$$\min_x \; c^\top x$$
$$\text{s.t. } Ax \le b, \forall A \in W$$

Suppose c is uncertain and $c \in \mathcal{C}$, we have the equivalent form:

$$\begin{aligned} \min_{x,\sigma} \;& \sigma \\ \text{s.t. } & c^\top x \le \sigma : \forall c \in \mathcal{C} \\ & Ax \le b : \forall A \in W \end{aligned}$$

Suppose b is uncertain, then we can introduce an additional variable $y = -1$ and get

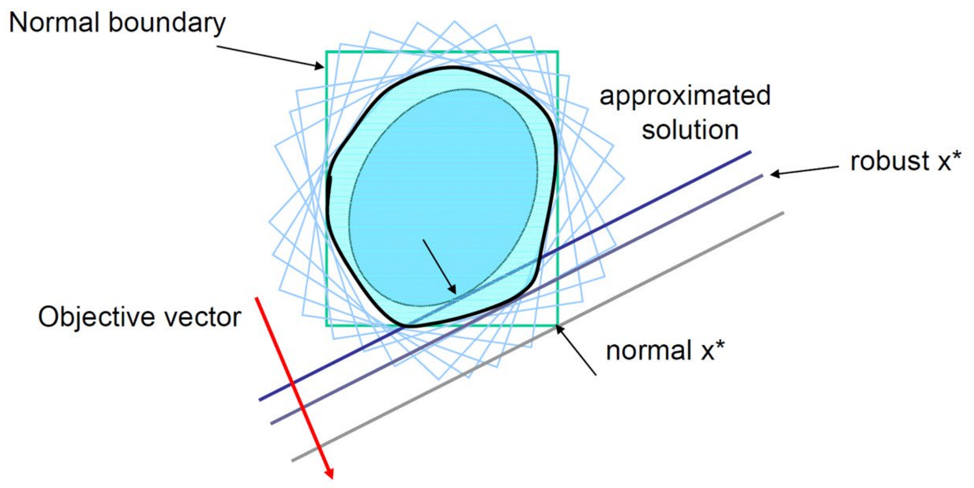
$$\begin{aligned} \min_x \;& c^\top x \\ \text{s.t. } & [A, b] \begin{bmatrix} x \\ -1 \end{bmatrix} \le 0 : \forall (A, b) \in W \end{aligned}$$

Static robust optimization

Definition (**Feasibility**): A solution is robust feasible if it stays feasible for all possible realizations of the uncertain data.

Definition (**Optimality**): A robust feasible solution is optimal if it minimizes the objective function for all possible realizations of the uncertain data.

$$\min_{x} \; c^\top x$$
$$\text{s.t. } Ax \leq b, \forall A \in W$$



Static robust optimization

W is called an "uncertainty set".
Different types of uncertainty set:
1. Discrete uncertainty set

How to deal with discrete uncertainty set?

$$a \in [\bar{a}_1, \bar{a}_2, ..., \bar{a}_N]$$

2. Box uncertainty set

$$a \in \{a \mid ||a_i|| \leq \tau, \forall i = 1, ..., N\}$$

3. Ellipsoidal uncertainty set

$$a \in \{a \mid (a - \bar{u})^\top R^{-1}(a - \bar{u}) \leq \Omega^2\}$$

4. Polyhedral uncertainty set

$$a \in \{a \mid a = \bar{a} + Pu, Du + q \geq 0\}$$

*Adjustable robust optimization**

Feasible set of static robust optimization:

$$X_S := \{x \mid \forall A \in W: Ax \leq b\}$$

All decisions must be made before A is known exactly.

Feasible set of adjustable robust optimization:

$$X_A \coloneqq \{x \mid \forall A \in W, \exists y: Ax + By \leq b\}$$

The choice of y is a function (feedback, recourse) of A.

x: here-and-now variable

y: wait-and-see variable

X_A is usually larger than X_N

*Static v.s. Adjustable robust optimization**

$$\begin{aligned}\min_{x_1,x_2} \quad & x_1 \\ \text{s.t.} \quad & x_2 \geq 0.5\xi x_1 + 1, \forall \xi \in \Xi \\ & x_1 \geq (2-\xi)x_2, \forall \xi \in \Xi \\ & x_1 \geq 0, x_2 \geq 0\end{aligned}$$

where $\Xi = \{\xi \mid 0 \leq \xi \leq \rho, \rho \in (0,1)\}$

Static case: both x_1 and x_2 are determined prior to knowing the exact ξ

- When $\xi = \rho$, $x_2 \geq 0.5\rho x_1 + 1$
- When $\xi = 0$, $x_1 \geq 2x_2$

Hence $x_1 \geq \rho x_1 + 2$ or $x_1 \geq 2/(1-\rho)$
When $\rho \rightarrow 1, x_1 \rightarrow \infty$

*Static v.s. Adjustable robust optimization**

$$\min_{x_1,x_2} \ x_1$$
$$\text{s.t. } x_2 \geq 0.5\xi x_1 + 1, \forall \xi \in \Xi$$
$$x_1 \geq (2-\xi)x_2, \forall \xi \in \Xi$$
$$x_1 \geq 0, x_2 \geq 0$$

$$\text{where } \Xi = \{\xi \mid 0 \leq \xi \leq \rho, \rho \in (0,1)\}$$

Adaptive case: x_2 is the "wait-and-see" variable

We can let $x_2 = 0.5\xi x_1 + 1$, then $x_1 \geq (2-\xi)(0.5\xi x_1 + 1), \forall \xi \in [0,\rho]$

Since $4 \geq (2-\xi)(2\xi+1) = -2\xi^2 + 3\xi + 2, \forall \xi \in [0,\rho]$
$x_1 = 4$ is a robust feasible solution
We know that the optimal value of the robust optimization is ≤ 4.

*Example – Robust energy and reserve dispatch**

Traditional Economics Dispatch:

$$\min F = \sum_{g=1}^{N_G} \left(a_g p_g^2 + b_g p_g\right)$$

$$\text{s.t.} \quad P_g^l \le p_g \le P_g^u \quad \forall g$$

$$\sum_{g=1}^{N_G} p_g = \sum_{q=1}^{N_Q} p_q$$

$$-F_l \le \sum_{g=1}^{N_G} \pi_{gl} p_g - \sum_{q=1}^{N_Q} \pi_{ql} p_q \le F_l, \quad \forall l$$

With renewable energy…

$$\min F = \sum_{g=1}^{N_G} \left(a_g p_g^2 + b_g p_g\right)$$

reserve

$$\text{s.t.} \quad P_g^l + r_g \le p_g \le P_g^u - r_g \quad \forall g$$

$$\sum_{g=1}^{N_G} p_g + \sum_{m=1}^{N_W} p_m^{we} = \sum_{q=1}^{N_Q} p_q$$

$$-F_l \le \sum_{g=1}^{N_G} \pi_{gl} p_g + \sum_{m=1}^{N_W} \pi_{ml} p_m^{we} - \sum_{q=1}^{N_Q} \pi_{ql} p_q \le F_l,$$

$$W^D = \left\{ \begin{array}{l} p_m^w = p_m^{we} + z_m^+ p_m^{wh} \\ \quad -z_m^- p_m^{wh}, \quad \forall m \end{array} \middle| \begin{array}{l} z_m^+, z_m^- \in \{0,1\}, \quad \forall m \\ z_m^+ + z_m^- \le 1, \quad \forall m \\ \sum_{m=1}^{N_W} z_m^+ + z_m^- \le \Gamma^S \end{array} \right\}$$

Uncertainty budget

Reference: Wei W, Liu F, Mei S, et al. Robust energy and reserve dispatch under variable renewable generation[J]. IEEE Transactions on Smart Grid, 2014, 6(1): 369-380.

*Example – Robust energy and reserve dispatch**

Set-point stage

$$\min_{p_g^f,\, r_g} \quad F = \sum_{g=1}^{N_G} \left(a_g \left(p_g^f\right)^2 + b_g p_g^f + c_g r_g \right) \tag{6a}$$

$$\text{s.t.} \quad p_g^f + r_g \leq P_g^u \quad \forall g \tag{6b}$$

$$P_g^l \leq p_g^f - r_g \quad \forall g \tag{6c}$$

$$\sum_{g=1}^{N_G} p_g^f + \sum_{m=1}^{N_W} p_m^{we} = \sum_{q=1}^{N_Q} p_q \tag{6d}$$

$$-F_l \leq \sum_{g=1}^{N_G} \pi_{gl} p_g^f + \sum_{m=1}^{N_W} \pi_{ml} p_m^{we} - \sum_{q=1}^{N_Q} \pi_{ql} p_q \leq F_l, \quad \forall l \tag{6e}$$

$$0 \leq r_g \leq \min\left\{ R_g^- \Delta t,\ R_g^+ \Delta t \right\} \quad \forall g \tag{6f}$$

Feasibility check

$$\forall \{p_m^w\} \in W^D, \exists \{\Delta p_g^+, \Delta p_g^-\} \text{ such that}$$

$$0 \leq \Delta p_g^+ \leq r_g,\ 0 \leq \Delta p_g^- \leq r_g \quad \forall g \tag{6g}$$

$$p_g^c = p_g^f + \Delta p_g^+ - \Delta p_g^- \quad \forall g \tag{6h}$$

$$\sum_{g=1}^{N_G} p_g^c + \sum_{m=1}^{N_W} p_m^w = \sum_{q=1}^{N_Q} p_q \tag{6i}$$

$$-F_l \leq \sum_{g=1}^{N_G} \pi_{gl} p_g^c + \sum_{m=1}^{N_W} \pi_{ml} p_m^w - \sum_{q=1}^{N_Q} \pi_{ql} p_q \leq F_l, \quad \forall l \tag{6j}$$

Reference: Wei W, Liu F, Mei S, et al. Robust energy and reserve dispatch under variable renewable generation[J]. IEEE Transactions on Smart Grid, 2014, 6(1): 369-380.

*Example – Robust energy and reserve dispatch**

Algorithms for solving robust optimization

- Benders decomposition, D Bertsimas, E Litvinov, etc. 2012
- Column & Constraint Generation, B. Zeng, L. Zhao, 2013

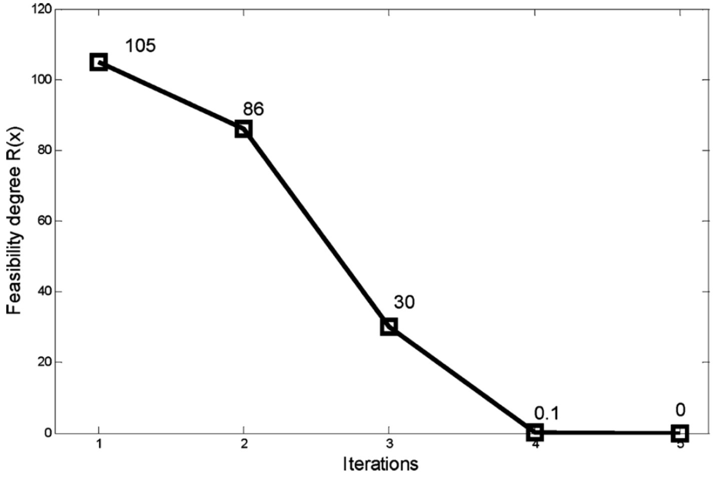


Fig. 5. $R(\boldsymbol{x})$ as a feasibility degree in each iteration.

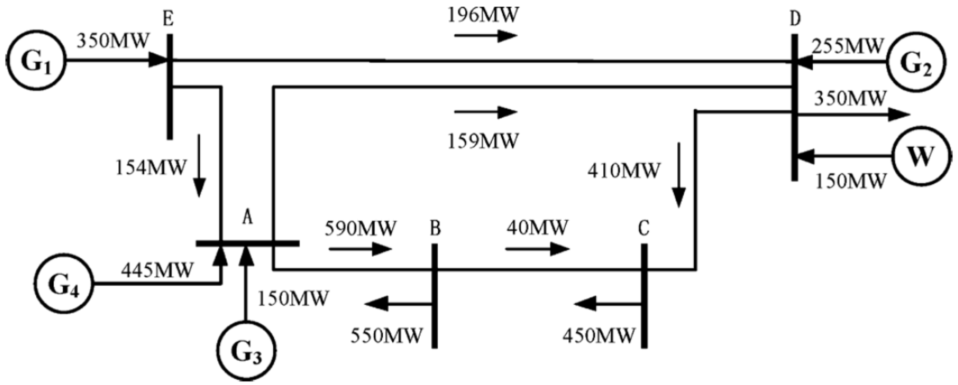
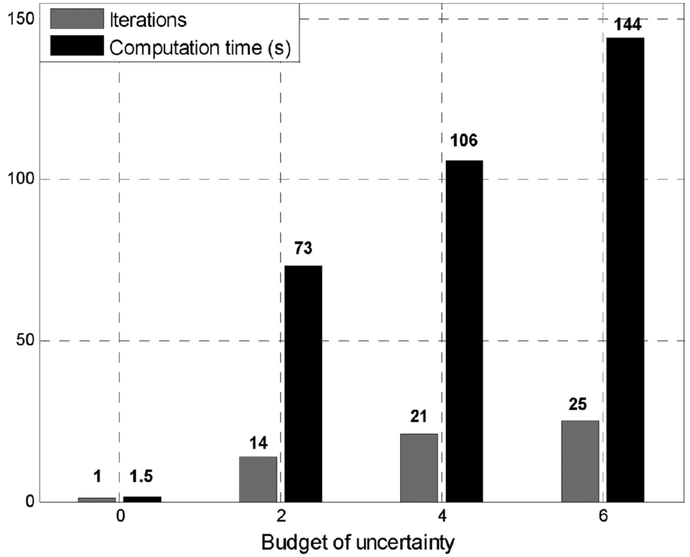


Fig. 4. Power flow for RERD under ±105 MW uncertainty.



Reference: Wei W, Liu F, Mei S, et al. Robust energy and reserve dispatch under variable renewable generation[J]. IEEE Transactions on Smart Grid, 2014, 6(1): 369-380.

Thanks！