MAEG4070 Engineering Optimization

Lecture 2 Linear Programming

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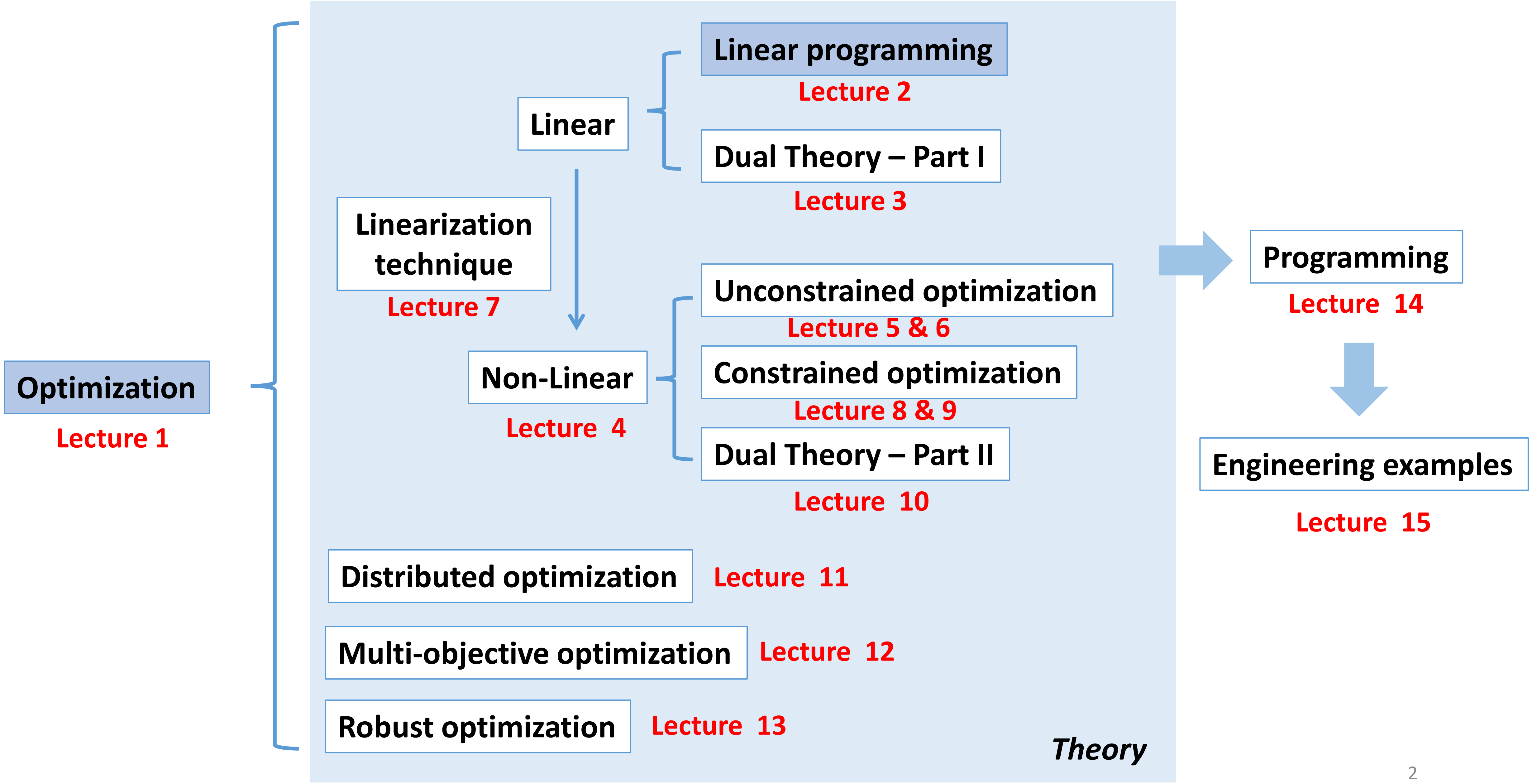
Content of this course (tentative)



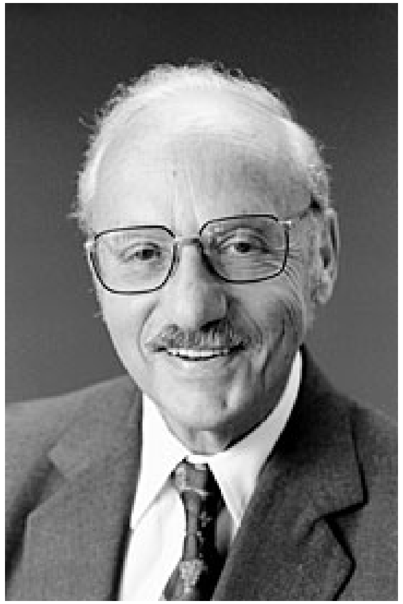
History of Linear Programming

Linear programming was developed as a discipline in the 1940's, to solve complex planning problems in wartime operations.

Founders of LP are generally regarded as **George B. Dantzig** (1914-2005)

Contributions:

- Developed the simplex algorithm
- Dantzig-Wolfe decomposition algorithm
- Stochastic programming



Publications:
Linear Programming and Extensions, 1963.

Honors:

- John von Neumann Theory Prize, 1974
- National Medal of Science, 1975

Pictures from Google image

History of Linear Programming

In 1947, John von Neumann established the theory of duality.

In 1975, The Nobel prize in economics to **Leonid Kantorovich** (1912-1986)

Contributions:

- Linear Programming
- Functional analysis
- Infinite-dimensional optimization problems

Publications:

- The Mathematical Method of Production Planning and Organization, 1939
- The Best Uses of Economic Resources, 1959

Honors:

- Stalin Prize, 1949; Nobel Prize, 1975

Pictures from Google image

LP examples

Recall the “production planning”, “transportation” examples in our previous lecture. One more example as follows:

A health center wants to provide a healthy breakfast for citizens.

Breakfast Food	Cal	Fat (g)	Cholesterol (mg)	Iron (mg)	Calcium (mg)	Protein (g)	Fiber (g)	Cost ($)
1. Bran cereal (cup)	90	0	0	6	20	3	5	0.18
2. Dry cereal (cup)	110	2	0	4	48	4	2	0.22
3. Oatmeal (cup)	100	2	0	2	12	5	3	0.10
4. Oat bran (cup)	90	2	0	3	8	6	4	0.12
5. Egg	75	5	270	1	30	7	0	0.10
6. Bacon (slice)	35	3	8	0	0	2	0	0.09
7. Orange	65	0	0	1	52	1	1	0.40
8. Milk-2% (cup)	100	4	12	0	250	9	0	0.16
9. Orange juice (cup)	120	0	0	0	3	1	0	0.50
10. Wheat toast (slice)	65	1	0	1	26	3	3	0.07

Breakfast should include at least 420 calories, 5mg iron, 400mg calcium, 20g protein, 12g fiber; and no more than 20g fat and 30mg cholesterol.

LP examples

Objective: minimizes the total cost

Variables? Constraints?

Breakfast Food	Cal	Fat (g)	Cholesterol (mg)	Iron (mg)	Calcium (mg)	Protein (g)	Fiber (g)	Cost ($)
1. Bran cereal (cup)	90	0	0	6	20	3	5	0.18
2. Dry cereal (cup)	110	2	0	4	48	4	2	0.22
3. Oatmeal (cup)	100	2	0	2	12	5	3	0.10
4. Oat bran (cup)	90	2	0	3	8	6	4	0.12
5. Egg	75	5	270	1	30	7	0	0.10
6. Bacon (slice)	35	3	8	0	0	2	0	0.09
7. Orange	65	0	0	1	52	1	1	0.40
8. Milk-2% (cup)	100	4	12	0	250	9	0	0.16
9. Orange juice (cup)	120	0	0	0	3	1	0	0.50
10. Wheat toast (slice)	65	1	0	1	26	3	3	0.07

$$\min \quad 0.18x_1 + 0.22x_2 + 0.10x_3 + 0.12x_4 + 0.10x_5 + 0.09x_6 + 0.40x_7 + 0.16x_8 + 0.50x_9 + 0.07x_{10}$$

s.t.

$$90x_1 + 110x_2 + 100x_3 + 90x_4 + 75x_5 + 35x_6 + 65x_7 + 100x_8 + 120x_9 + 65x_{10} \geq 420 \text{ calories}$$

$$2x_2 + 2x_3 + 2x_4 + 5x_5 + 3x_6 + 4x_8 + x_{10} \leq 20 \text{ g fat}$$

$$270x_5 + 8x_6 + 12x_8 \leq 30 \text{ mg cholesterol}$$

$$6x_1 + 4x_2 + 2x_3 + 3x_4 + x_5 + x_7 + x_{10} \geq 5 \text{ mg iron}$$

$$20x_1 + 48x_2 + 12x_3 + 8x_4 + 30x_5 + 52x_7 + 250x_8 + 3x_9 + 26x_{10} \geq 400 \text{ mg calcium}$$

$$3x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 + 2x_6 + x_7 + 9x_8 + x_9 + 3x_{10} \geq 20 \text{ g protein}$$

$$5x_1 + 2x_2 + 3x_3 + 4x_4 + x_7 + 3x_{10} \geq 12 \text{ g fiber}$$

$$x_i \geq 0, \text{ for all } i$$

Standard Form of Linear Programming

Element-wise form

$$\min_{x} \ z = \sum_{i=1}^{N} c_i x_i$$

$$\text{s.t.} \ \sum_{i=1}^{N} a_{ji} x_i = b_j, \forall j = 1, ..., M$$

$$x_i \geq 0, \forall i = 1, ..., N$$

Compact form

$$\min_{x} \ z = cx$$

$$\text{s.t.} \ Ax = b$$

$$x \geq 0$$

$\boldsymbol{c}_{\mathbf{1}\times \boldsymbol{N}}$ $\boldsymbol{x}_{\boldsymbol{N}\times\mathbf{1}}$

$\boldsymbol{A}_{\boldsymbol{M}\times\boldsymbol{N}}$ $\boldsymbol{b}_{\boldsymbol{M}\times\mathbf{1}}$

Features:

- Decision variables are continuous
- Objective function is linear
- Constraints are linear

$$f(a+b) = f(a) + f(b)$$

$$f(ka) = kf(a)$$

What if the form is not so standard?

Identification of LP

Are the following problems LPs?

(1)

$$\begin{aligned} \min_{x_1,x_2} \ & 4x_1 + 2x_2 \\ \text{s.t.} \ & 3x_1 + 4x_2 \geq 0 \\ & 6x_1 + 4x_2 \leq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(2)

$$\begin{aligned} \min_{x_1,x_2} \ & 4x_1 + 2x_2 \\ \text{s.t.} \ & 3x_1x_2 + 4x_2 \geq 0 \\ & 6x_1 + 4x_2^2 \leq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(3)

$$\begin{aligned} \max_{x_1,x_2} \ & 4x_1 + 2x_2 \\ \text{s.t.} \ & 6x_1 + 5x_2 = 0 \\ & 7x_1 + 11x_2 \leq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(4)

$$\begin{aligned} \max_{x_1,x_2} \ & 4|x_1| + 2x_2 \\ \text{s.t.} \ & 6x_1 + 5|x_2| = 0 \\ & 7x_1 + 11x_2 \leq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Identification of LP

Are the following problems LPs?

(1) ✓

$$\min_{x_1,x_2} \; 4x_1 + 2x_2$$
$$\text{s.t. } 3x_1 + 4x_2 \geq 0$$
$$6x_1 + 4x_2 \leq 0$$
$$x_1 \geq 0, x_2 \geq 0$$

(2) ✗

$$\min_{x_1,x_2} \; 4x_1 + 2x_2$$
$$\text{s.t. } 3x_1x_2 + 4x_2 \geq 0$$
$$6x_1 + 4x_2^2 \leq 0$$
$$x_1 \geq 0, x_2 \geq 0$$

(3) ✓

$$\max_{x_1,x_2} \; 4x_1 + 2x_2$$
$$\text{s.t. } 6x_1 + 5x_2 = 0$$
$$7x_1 + 11x_2 \leq 0$$
$$x_1 \geq 0, x_2 \geq 0$$

(4) ✗

$$\max_{x_1,x_2} \; 4|x_1| + 2x_2$$
$$\text{s.t. } 6x_1 + 5|x_2| = 0$$
$$7x_1 + 11x_2 \leq 0$$
$$x_1 \geq 0, x_2 \geq 0$$

$$f(a+b) = f(a) + f(b)$$
$$f(ka) = kf(a)$$

Identification of LP

Are the following problems LPs?

(1)
$$\begin{aligned} \max_{x_1,x_2} \quad & x_1x_2 + x_1 \\ \text{s.t.} \quad & \sin(x_1) + 3x_2 \leq 5 \\ & x_1 \geq 0 \end{aligned}$$

(2)
$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & x_1 + x_2x_3 \\ \text{s.t.} \quad & 7x_1 + 3x_2x_3 \geq 0 \\ & x_1 \geq 0 \end{aligned}$$

(3)
$$\begin{aligned} \max_{x_1,x_3} \quad & x_1 + 7x_3 \\ \text{s.t.} \quad & 8x_1 - 5x_3 \leq 0 \\ & 9x_1 + 3x_3 \geq 0 \\ & x_1 \geq 0, x_3 \geq 0 \end{aligned}$$

Identification of LP

Are the following problems LPs?

(1) ✗

$$\max_{x_1,x_2} \; x_1x_2 + x_1$$
$$\text{s.t.} \; \sin(x_1) + 3x_2 \le 5$$
$$x_1 \ge 0$$

(2) ✗

$$\min_{x_1,x_2,x_3} \; x_1 + x_2x_3$$
$$\text{s.t.} \; 7x_1 + 3x_2x_3 \ge 0$$
$$x_1 \ge 0$$

(3) ✓

$$\max_{x_1,x_3} \; x_1 + 7x_3$$
$$\text{s.t.} \; 8x_1 - 5x_3 \le 0$$
$$9x_1 + 3x_3 \ge 0$$
$$x_1 \ge 0, x_3 \ge 0$$

Can the above nonlinear problems be turned into a linear programming?

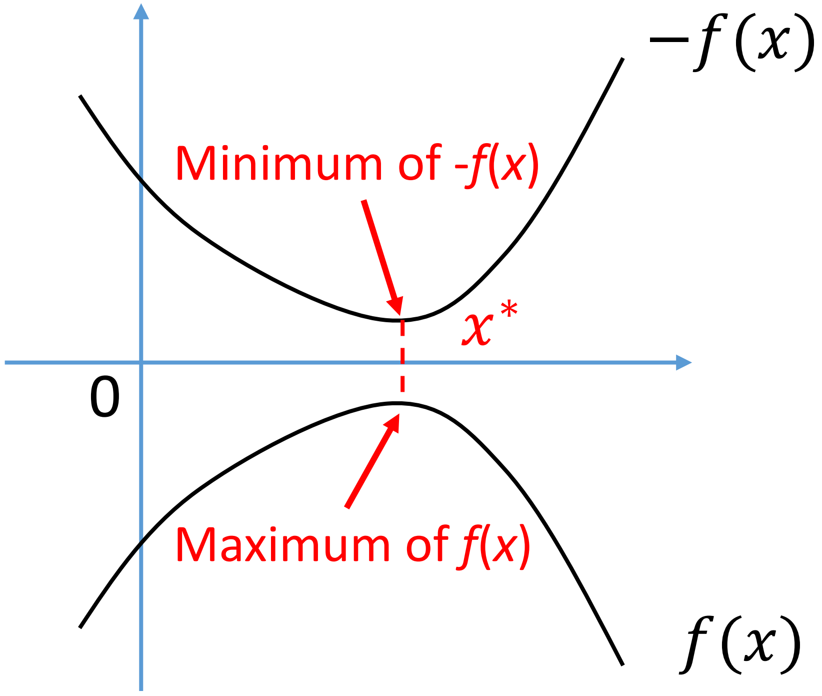
From non-standard form to standard form

1. From “maximization” problem to “minimization” problem

$$\max_{x_1,x_2} 4x_1 + 2x_2$$
$$\text{s.t. } 6x_1 + 5x_2 = 0$$
$$x_1 \geq 0, x_2 \geq 0$$

Let $z' = -z$, then

$$\min_{x_1,x_2} \; -4x_1 - 2x_2$$
$$\text{s.t. } 6x_1 + 5x_2 = 0$$
$$x_1 \geq 0, x_2 \geq 0$$



From non-standard form to standard form

2. Inequality constraints to equality constraints

For inequality ≤, introduce ***slack variable***

$$6x_1 + 5x_2 \leq 0$$

$$6x_1 + 5x_2 + y_1 = 0, \; y_1 \geq 0$$

For inequality ≥, introduce ***surplus variable***

$$7x_1 + 3x_2 \geq 0$$

$$7x_1 + 3x_2 - y_2 = 0, \; y_2 \geq 0$$

$$\begin{aligned} \min_{x_1, x_2} \quad & -4x_1 - 2x_2 \\ \text{s.t.} \quad & 6x_1 + 5x_2 \leq 0, \\ & 7x_1 + 3x_2 \geq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min_{x_1, x_2, y_1, y_2} \quad & -4x_1 - 2x_2 \\ \text{s.t.} \quad & 6x_1 + 5x_2 + y_1 = 0 \\ & 7x_1 + 3x_2 - y_2 = 0 \\ & x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0 \end{aligned}$$

From non-standard form to standard form

3. About the last constraint on x

- If there is no limit on x_1

$$\min_{x_1,x_2} \quad -4x_1 - 2x_2$$
$$\text{s.t. } 6x_1 + 5x_2 \leq 0,$$
$$7x_1 + 3x_2 \geq 0$$
$$x_2 \geq 0$$

Let $x_1 = y_1 - y_2$
$y_1 \geq 0, y_2 \geq 0$

$$\min_{x_1,x_2,y_1,y_2} \quad -4y_1 + 4y_2 - 2x_2$$
$$\text{s.t. } 6y_1 - 6y_2 + 5x_2 \leq 0,$$
$$7y_1 - 7y_2 + 3x_2 \geq 0$$
$$y_1 \geq 0, y_2 \geq 0, x_2 \geq 0$$

- If general range constraint on x

$$\min_{x_1,x_2} \quad -4x_1 - 2x_2$$
$$\text{s.t. } 6x_1 + 5x_2 \leq 0,$$
$$7x_1 + 3x_2 \geq 0$$
$$x_1 \geq 5, x_2 \geq 0$$

Let $y_1 = x_1 - 5$

$$\min_{x_1,x_2,y_1,y_2} \quad -4y_1 - 2x_2 - 20$$
$$\text{s.t. } 6y_1 + 5x_2 \leq -30,$$
$$7y_1 + 3x_2 \geq -35$$
$$y_1 \geq 0, x_2 \geq 0$$

From non-standard form to standard form

4. Problem with absolute value |x|

- Absolute values in constraints

$$|2x_1 + 3x_2| \leq 2$$

is equivalent to

$$-2 \leq 2x_1 + 3x_2 \leq 2$$

- Absolute values in the objective function

If it is "$\min\ |x_1| + \cdots$" or "$\max\ -|x_1| + \cdots$", we can

Let $u = |x_1|$ and add constraints $x_1 \leq u, -x_1 \leq u$

However, if "$\min\ -|x_1| + \cdots$" or "$\max\ |x_1| + \cdots$", it cannot be turned into an LP

Nonconvex problem (future lectures)

Proof:

If $x_1 > 0$, then $x_1 = u$ and $-x_1 \leq 0 \leq u$

If $x_1 = 0$, then $u \geq 0$, since it is minimize over u, we have $u = 0 = x_1$

If $x_1 < 0$, then $-x_1 = u, x_1 \leq 0 \leq u$

Example

Try to turn this example into a standard form and write its compact form

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 2x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & |3x_1 + 4x_2| + 2x_3 \le 600 \\ & 2x_1 + x_2 + 3x_3 \ge 400 \\ & x_1 \ge 0, x_2 \ge 3 \end{aligned}$$

1. From “maximization” problem to “minimization” problem
2. Inequality constraints to equality constraints
3. About the last constraint on x
4. Absolute value in constraints

Exercise

$$\max_{x_1,x_2,x_3} \quad 2x_1 + 4x_2 + 3x_3$$
$$\text{s.t.} \quad |3x_1 + 4x_2| + 2x_3 \leq 600$$
$$2x_1 + x_2 + 3x_3 \geq 400$$
$$x_1 \geq 0, x_2 \geq 3$$

$$\min_{x_1,x_2,x_3} \quad -2x_1 - 4x_2 - 3x_3$$
$$\text{s.t.} \quad |3x_1 + 4x_2| + 2x_3 \leq 600$$
$$2x_1 + x_2 + 3x_3 \geq 400$$
$$x_1 \geq 0, x_2 \geq 3$$

$$\min_{x_1,x_2,x_3} \quad -2x_1 - 4x_2 - 3x_3$$
$$\text{s.t.} \quad -600 + 2x_3 \leq 3x_1 + 4x_2$$
$$3x_1 + 4x_2 \leq 600 - 2x_3$$
$$2x_1 + x_2 + 3x_3 \geq 400$$
$$x_1 \geq 0, x_2 \geq 3$$

$$\min_{x_1,x_2,x_3,y_1,y_2,y_3} \quad -2x_1 - 4x_2 - 3x_3$$
$$\text{s.t.} \quad -3x_1 - 4x_2 + 2x_3 + y_1 = 600$$
$$3x_1 + 4x_2 + 2x_3 + y_2 = 600$$
$$2x_1 + x_2 + 3x_3 - y_3 = 400$$
$$x_1 \geq 0, x_2 \geq 3, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

Exercise

$$\min_{x_1,x_2,x_3,y_1,y_2,y_3} \quad -2x_1 - 4x_2 - 3x_3$$
$$\text{s.t.} \quad -3x_1 - 4x_2 + 2x_3 + y_1 = 600$$
$$3x_1 + 4x_2 + 2x_3 + y_2 = 600$$
$$2x_1 + x_2 + 3x_3 - y_3 = 400$$
$$x_1 \geq 0, x_2 \geq 3, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

$$\min_{x_1,u_1,x_3,y_1,y_2,y_3} \quad -2x_1 - 4u_1 - 3x_3 - 12$$
$$\text{s.t.} \quad -3x_1 - 4u_1 + 2x_3 + y_1 = 600 + 12$$
$$3x_1 + 4u_1 + 2x_3 + y_2 = 600 - 12$$
$$2x_1 + u_1 + 3x_3 - y_3 = 400 - 3$$
$$x_1 \geq 0, u_1 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

$$\min_{x_1,u_1,v_1,v_2,y_1,y_2,y_3} \quad -2x_1 - 4u_1 - 3v_1 + 3v_2 - 12$$
$$\text{s.t.} \quad -3x_1 - 4u_1 + 2v_1 - 2v_2 + y_1 = 612$$
$$3x_1 + 4u_1 + 2v_1 - 2v_2 + y_2 = 588$$
$$2x_1 + u_1 + 3v_1 - 3v_2 - y_3 = 397$$
$$x_1 \geq 0, u_1 \geq 0, v_1 \geq 0, v_2 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

$$x = [x_1, u_1, v_1, v_2, y_1, y_2, y_3]^T$$
$$c = [-2, -4, -3, 3, 0, 0, 0]$$
$$b = [612, 588, 397]^T$$
$$A = \begin{bmatrix} -3 & -4 & 2 & -2 & 1 & 0 & 0 \\ 3 & 4 & 2 & -2 & 0 & 1 & 0 \\ 2 & 1 & 3 & -3 & 0 & 0 & -1 \end{bmatrix}$$

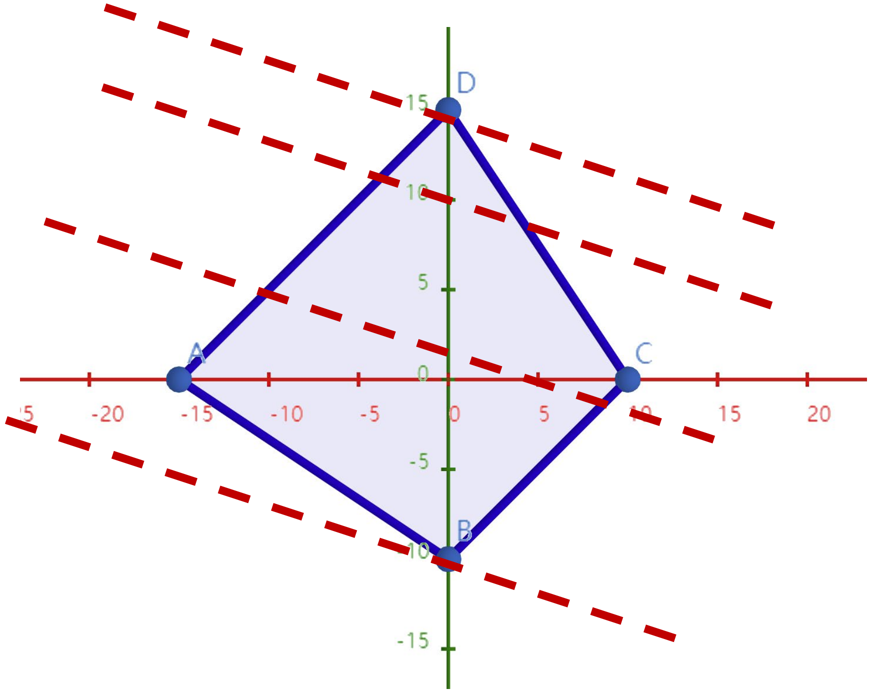
Graphical method for solving LP

Procedure:

Step 1: Draw the feasible region of the LP problem

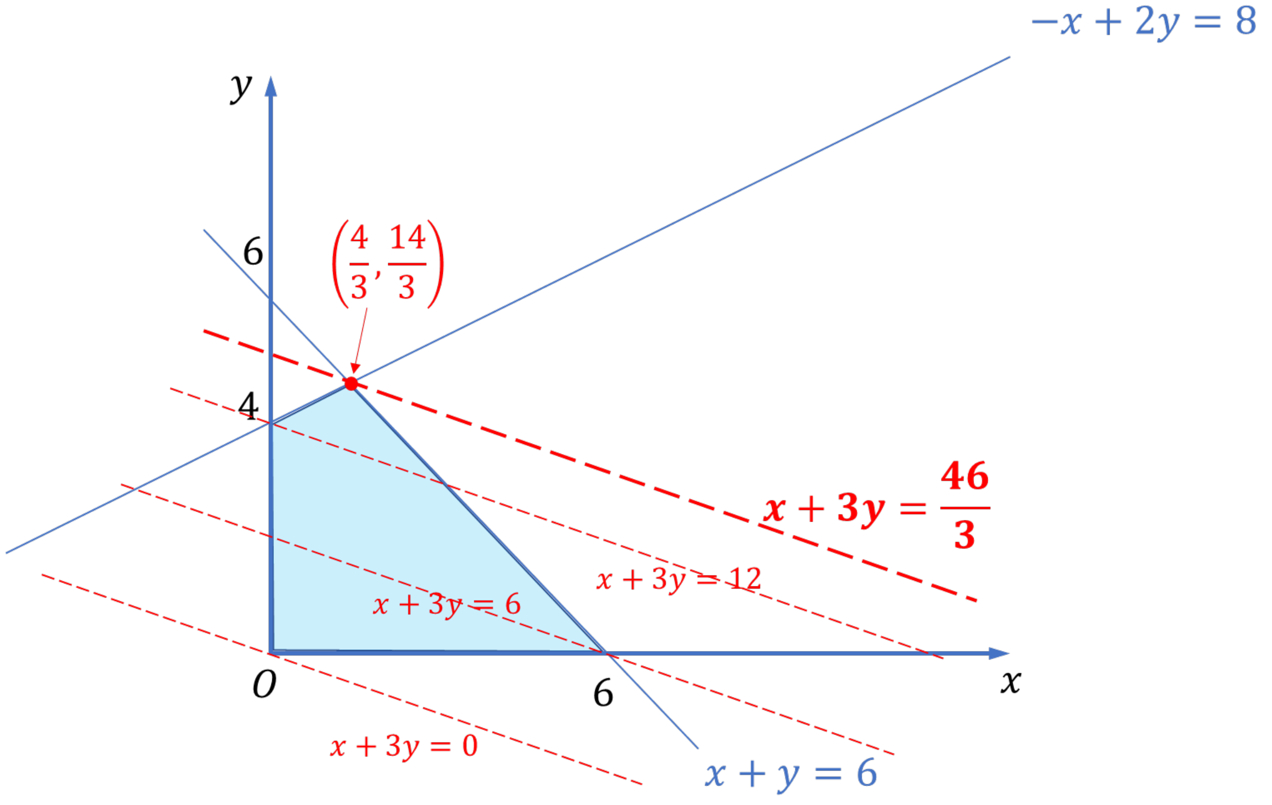
Step 2: Draw the contours of the objective function

Step 3: Move the contour until it reaches the optimal point



graphical method for solving LP - example 1

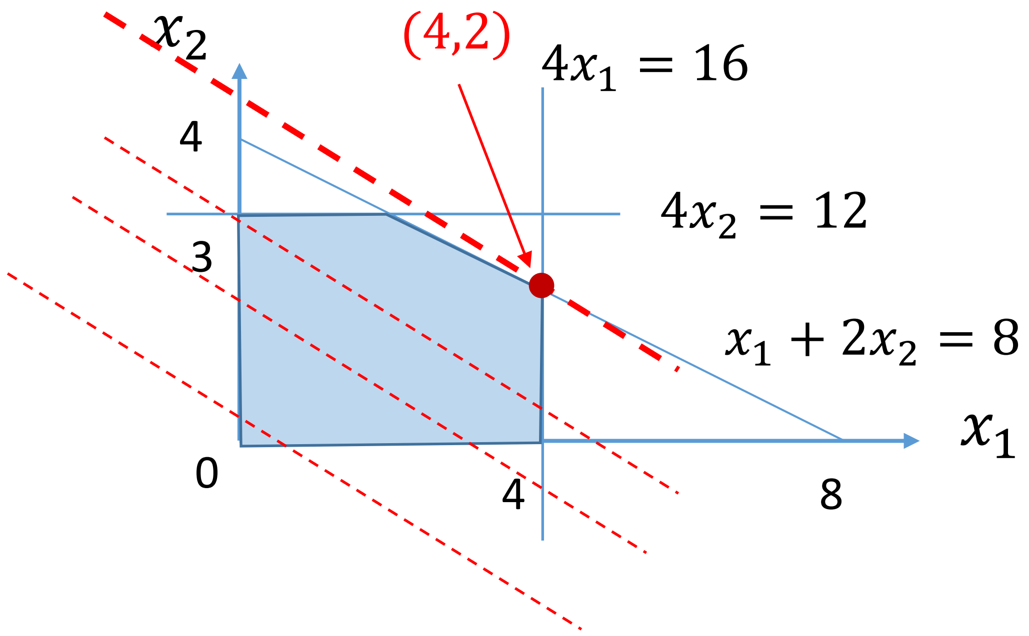
$$\max_{x,y} \; x + 3y$$
$$\text{s.t. } x + y \leq 6$$
$$-x + 2y \leq 8$$
$$x \geq 0, y \geq 0$$



Vertices are
$(0,0), (6,0), \left(\frac{4}{3}, \frac{14}{3}\right), (0,4)$
The contour is $x + 3y = const$
The optimal point is $x^* = \left(\frac{4}{3}, \frac{14}{3}\right)$

Graphical method for solving LP – example 2

$$\max_{x_1,x_2} \; 2x_1 + 3x_2$$
$$\text{s.t. } x_1 + 2x_2 \le 8$$
$$4x_1 \le 16$$
$$4x_2 \le 12$$
$$x_1 \ge 0, x_2 \ge 0$$

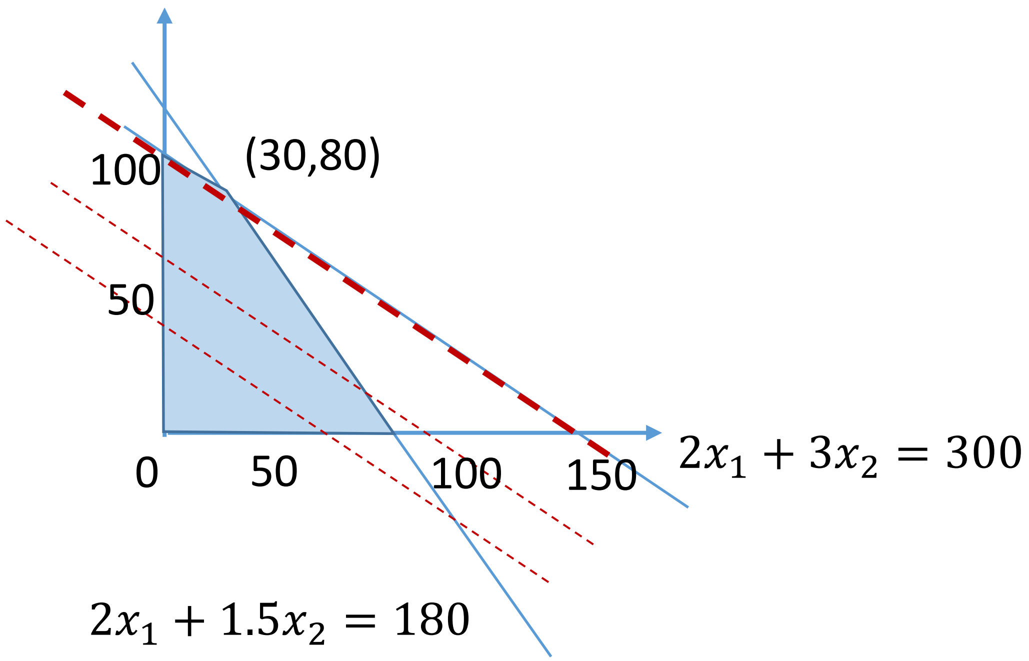


Vertices are
$(0,0), (4,0), (0,3), (4,2), (2,3)$
The contour is $2x_1 + 3x_2 = const$
The optimal point is $x^* = (4,2)$.

graphical method for solving LP - special case 1

The LP might have multiple solutions

$$\min_{x_1,x_2} \quad -10x_1 - 15x_2$$
$$\text{s.t. } 2x_1 + 3x_2 \leq 300$$
$$2x_1 + 1.5x_2 \leq 180$$
$$x_1 \geq 0, x_2 \geq 0$$



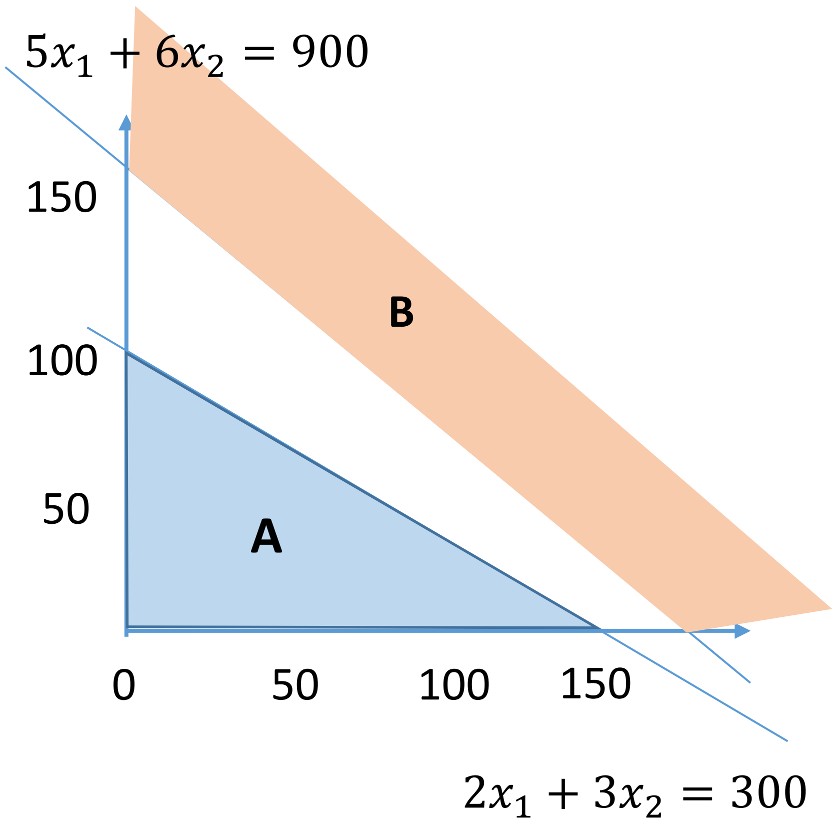
This happens when the contour is parallel to one of the boundary

graphical method for solving LP - special case 2

The LP might be infeasible

$$\min_{x_1,x_2} \quad -10x_1 - 12x_2$$
$$\text{s.t. } 5x_1 + 6x_2 \geq 900$$
$$2x_1 + 3x_2 \leq 300$$
$$x_1 \geq 0, x_2 \geq 0$$

This happens when the feasible set of the LP is empty.

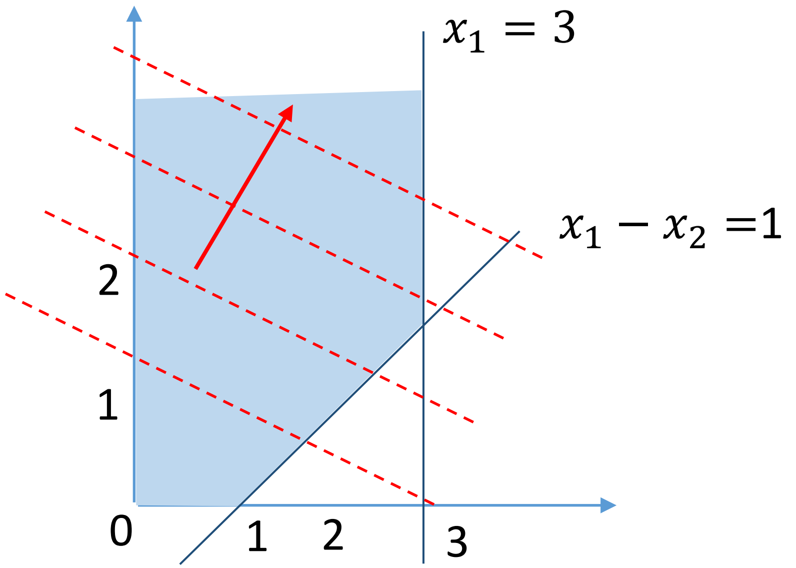


A∩B=∅

graphical method for solving LP - special case 3

The LP might have unbounded solution

$$\min_{x_1,x_2} \ -3x_1 - 4x_2$$
$$\text{s.t. } x_1 \leq 3$$
$$x_1 - x_2 \leq 1$$
$$x_1 \geq 0, x_2 \geq 0$$



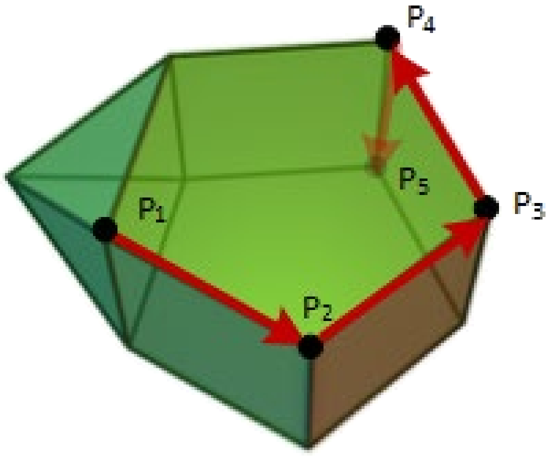
This happens when the feasible set of LP is unbounded.

Solutions for LP

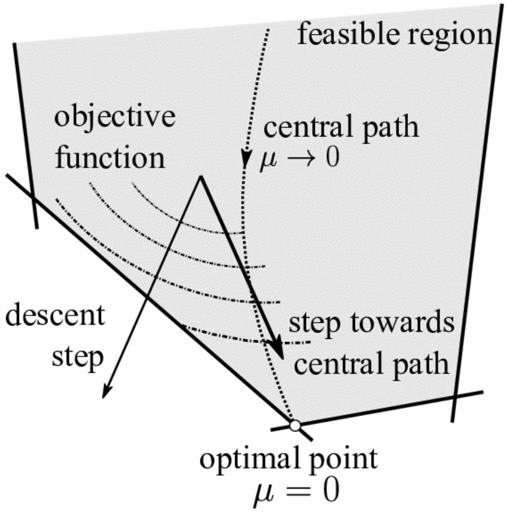
- LP problem
  - feasible
    - has optimal solution
      - unique
        Example 1 & 2
      - multiple
        Special case 1
    - no optimal solution (unbounded)
      Special case 3
  - infeasible
    Special case 2

Algorithms for solving LPs

Algorithm	Complexity
Simplex algorithm	Non-polynomial
Ellipsoid	Polynomial
Interior point algorithm	Polynomial



Simplex algorithm



Interior point algorithm

Pictures from Google image

Algorithms for solving LPs - Simplex

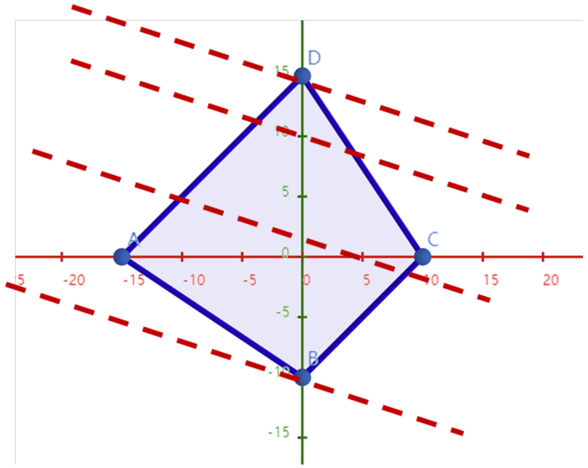
From the above examples, we notice that the optimal point is located at a vertex of the feasible region. Recall the standard form:

$$\min_{x} \quad z = cx \qquad \text{s.t.} \quad Ax = b \qquad x \geq 0$$

$$\Rightarrow$$

$$\min_{y,s} \quad c_y y + c_s s \qquad \text{s.t.} \quad A_y y + Is = b \qquad y, s \geq 0$$

- $Ax = b$ defines the feasible region (M equations in N unknowns, $N > M$).
- To find a vertex of the feasible region, we can set $N - M$ variables to zero and solve for the others.
- If all the obtained variables are non-negative, this point is in fact a vertex of the feasible region.



Algorithms for solving LPs - Simplex

We could generate all possible vertexes (basic feasible solutions), check the value of the cost function, and find the optimum by enumeration.

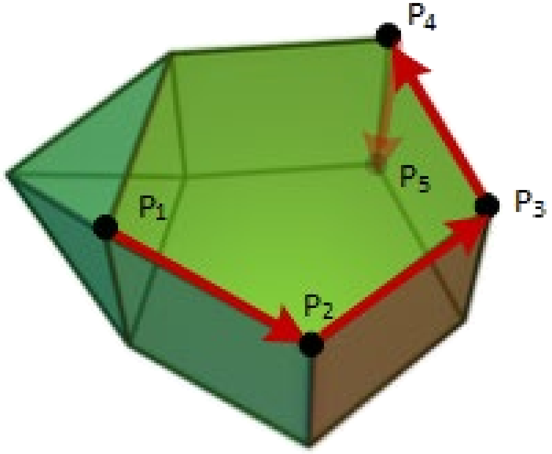
However…

- There are $\binom{N}{N-M} = \frac{N!}{M!(N-M)!}$ candidates
- Even for a small problem, we need to try many times. For example, $N = 10, M = 4$, we get 210 candidates.
- The typical size of realistic LPs is such that N, M are often in the range of several hundreds or even thousands.
- Computational inefficient

Algorithms for solving LPs - Simplex

Idea of the Simplex:

- Start at a vertex of the feasible region, e.g. P_1
- While there is an adjacent vertex that has a better objective value, move to that vertex, e.g. P_2
- Usually, the algorithm reaches an optimal solution in a finite number of steps



Key points:

- A vertex is identified by setting $N - M$ variables to zero and compute the others (basic variables) uniquely.
- Two vertices are adjacent if they share all their (non-) basic variables, except one.
- So: in order to find neighbors to a vertex, remove one of the (non-) basic variables and add another one.

Pictures from Google image

Algorithms for solving LPs - Interior point algorithm

Consider optimization problem:

$$\min_x \sum_{i=1}^{N} c_i x_i$$

$$\text{s.t. } \sum_{i=1}^{N} a_{ij} x_i \leq b_j, \forall j = 1, ..., M$$

Convert the above problem into an unconstrained minimization problem by adding barrier functions.

$$\min_x \sum_{i=1}^{N} c_i x_i + \sum_{j=1}^{M} \mathbb{I}(\sum_{i=1}^{N} a_{ij} x_i - b_j)$$

$$\mathbb{I}(u) = \begin{cases} 0, & \text{if } u < 0 \\ \infty, & \text{if } u \geq 0 \end{cases}$$

or

$$\mathbb{I}_t(u) = -\frac{1}{t} \log(-u)$$ convex (future lectures)

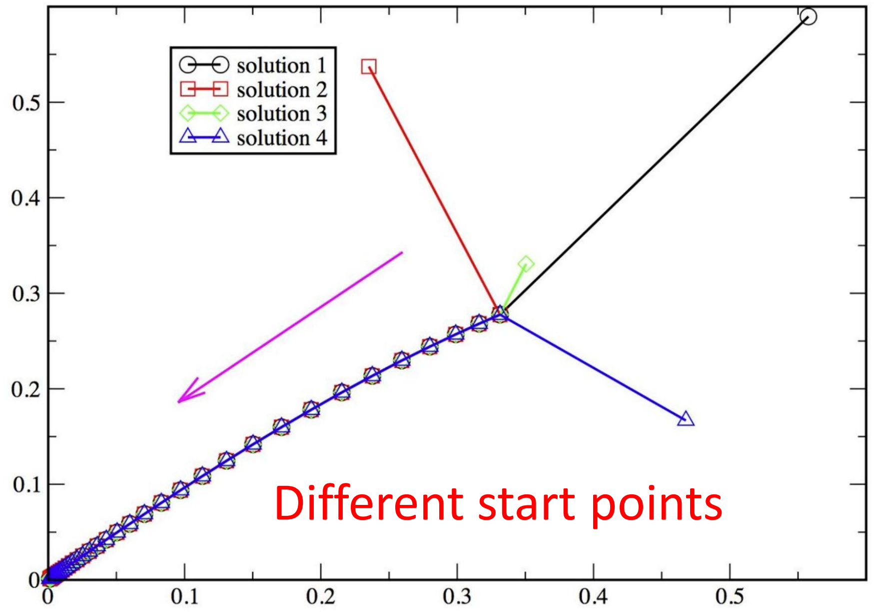
Increase *t* to get more accurate solution

Algorithms for solving LPs - Interior point algorithm

Interior point methods are widely applicable to convex optimization problems (future lectures), linear programs are just an example.

For example:

$$\min_{x,y} \; x + y$$
$$\text{s.t. } x + 2y \leq 1$$
$$2x + y \leq 1$$
$$x, y \geq 0$$



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Solve LPs using Solvers

Free LP Solvers:

- CDD, CLP, GLPK, LPSOLVE, QSOPT, SCIP
- LINPROG (not free but is embedded in MATLAB)

Commercial LP Solvers (free for academia):

- CPLEX, GUROBI
- MOSEK

How to choose a solver?
How to describe the model in coding?
…
Will discuss in future lectures.

Thanks！