MAEG4070 Engineering Optimization

Lecture 3 Dual Theory - Part I

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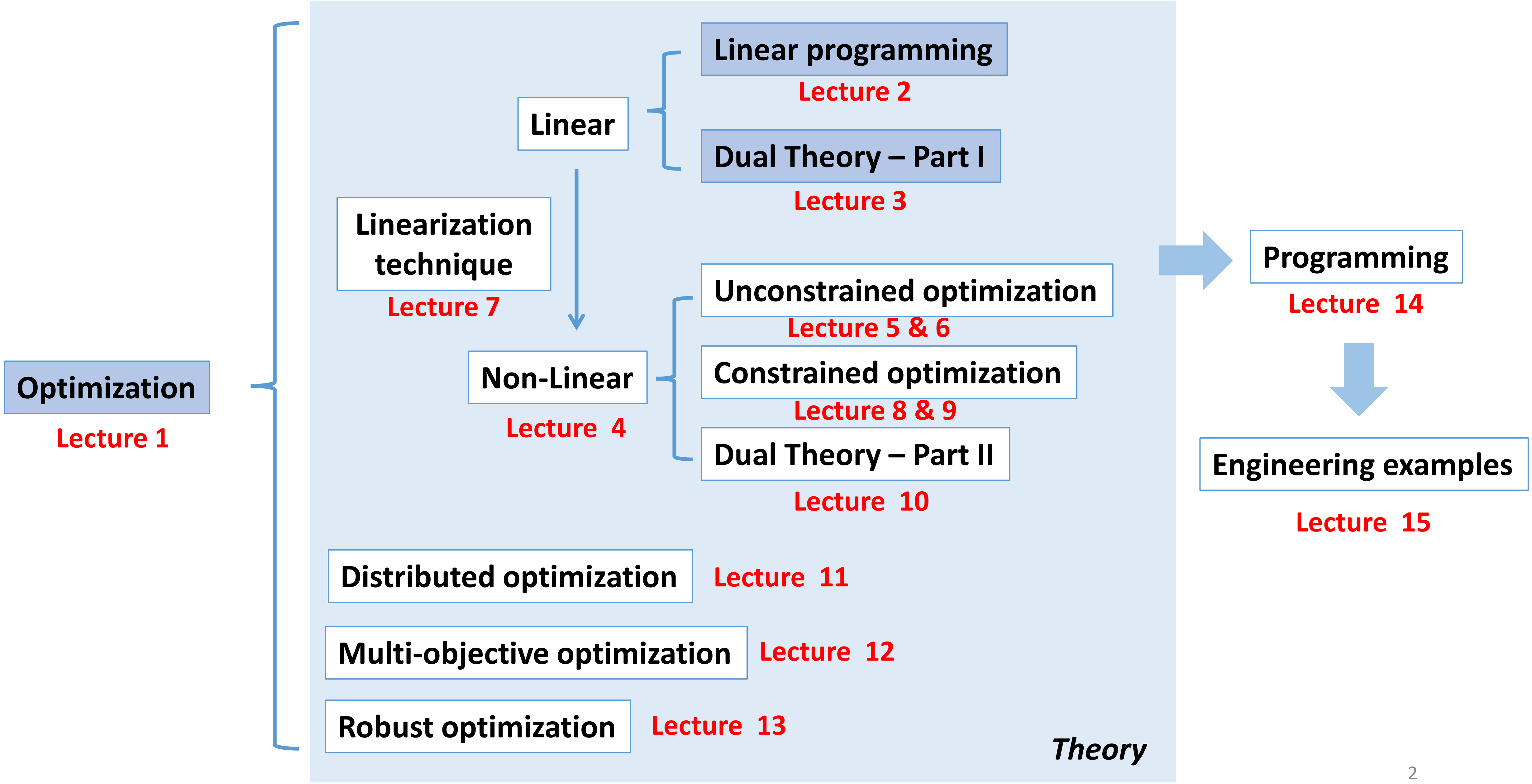
Content of this course (tentative)



Introduction – Simple example

A company has some resources to produce three products (denoted as A, B, C). Each product consumes a different mix of resources, and there will be a profit from selling the product. The endowment of resources and its relationship with products are:

Production: $y_1 \geq 0$ $y_2 \geq 0$ $y_3 \geq 0$

	A	B	C		Endowment
Steel	3	4	2	$\leq$	600
Wood	2	1	2	$\leq$	400
Label	1	3	3	$\leq$	300
Machine	1	4	4	$\leq$	200
Profit	2	4	3		

Maximize $2y_1 + 4y_2 + 3y_3$

Introduction - General case

A company has R kind of resources to produce N items. Item $n \in \{1, \dots, N\}$ needs a_{nr} unit of resource $r \in \{1, \dots, R\}$. The endowment of resource $r \in \{1, \dots, R\}$ is c_r. The price of item $n \in \{1, \dots, N\}$ is b_n. How to maximize the total profit?

Variable: y_n denotes the number of item n produced
Objective: total profit $\sum_{n=1}^{N} b_n y_n$
Constraints: 1) do not exceed the resource endowment $\sum_{n=1}^{N} a_{nr} y_n \le c_r, \forall r = 1, \dots, R$
2) $y_n \ge 0, \forall n = 1, \dots, N$

$$\max_{y_n, \forall n} \sum_{n=1}^{N} b_n y_n$$

$$\text{s.t.} \sum_{n=1}^{N} a_{nr} y_n \le c_r, \forall r = 1, ..., R$$

$$y_n \ge 0, \forall n = 1, ..., N$$

Introduction – General case

A company can also choose to sell all of its resource endowments $c_r, \forall r = 1, \dots, R$ to let another company to produce. Suppose the price for resource $r \in \{1, \dots, R\}$ is x_r. The company wants to ensure that the profit $\sum_{r=1}^{R} a_{nr} x_r$ of selling $a_{n1}, \dots, a_{nR}$ unit of resource $1, \dots, R$ respectively, which are just adequate to produce one unit of product n, is no less than the profit b_n of producing and selling this unit of product n by itself. Meanwhile, it tries to minimize the total payment $\sum_{r=1}^{R} c_r x_r$ by the other company for purchasing resources, so that the other company is willing to buy resources and produce.

$$
\begin{aligned}
\min_{x_r, \forall r} \quad & \sum_{r=1}^{R} c_r x_r \\
\text{s.t.} \quad & \sum_{r=1}^{R} a_{nr} x_r \geq b_n, \forall n = 1, \dots, N \\
& x_r \geq 0, \forall r = 1, \dots, R
\end{aligned}
$$

Basic models

Primal problem

$$\max_{x_n,\forall n} \sum_{n=1}^{N} b_n y_n \qquad \boldsymbol{b} = [\boldsymbol{b_1}, \dots, \boldsymbol{b_N}]^{\boldsymbol{T}} \quad \boldsymbol{y} = [\boldsymbol{y_1}, \dots, \boldsymbol{y_N}]^{\boldsymbol{T}}$$

$$\text{s.t. } \sum_{n=1}^{N} a_{nr} y_n \leq c_r, \forall r = 1, \dots, R$$

$$\boldsymbol{A}\text{: } \mathbb{R}^{\boldsymbol{N \times R}}, \boldsymbol{c}\text{: } \mathbb{R}^{\boldsymbol{r \times 1}}$$

$$y_n \geq 0, \forall n = 1, \dots, N$$

Dual problem

$$\min_{x_r,\forall r} \sum_{r=1}^{R} c_r x_r \qquad \boldsymbol{x} = [\boldsymbol{x_1}, \dots, \boldsymbol{x_R}]^{\boldsymbol{T}}$$

$$\text{s.t. } \sum_{r=1}^{R} a_{nr} x_r \geq b_n, \forall n = 1, \dots, N$$

$$x_r \geq 0, \forall r = 1, \dots, R$$

Compact form

$$\max_{y} \; b^T y \quad \text{s.t. } A^T y \leq c, \quad y \geq 0$$

$$\min_{x} \; c^T x \quad \text{s.t. } Ax \geq b, \quad x \geq 0$$

Another Example

Can you tell me the optimal value of the following LP in 10 seconds?

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ \text{s.t.} \quad & 2x_1 + x_2 + 3x_3 + 8x_4 + 5x_5 + 3x_6 \geq 5 \\ & 6x_1 + 2x_2 + 6x_3 + x_4 + x_5 + 4x_6 \geq 2 \\ & 2x_1 + 7x_2 + x_3 + x_4 + 4x_5 + 3x_6 \geq 1 \end{aligned}$$

Adding up all constraints, we have

$$10(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) \geq 8$$

We can easily find such a solution, so the optimal value is 0.8.

LP duality is a straightforward generalization of this simple trick!

Another Example

How to find a systematic way to estimate a lower bound of an LP's optimal value?

$$
\begin{aligned}
\min_{x}\quad & c_1x_1 + c_2x_2 + \dots + c_Rx_R \\
\text{s.t.}\quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1R}x_R \geq b_1 \qquad \times\ \boldsymbol{\lambda_1 \geq 0} \\
& a_{21}x_1 + a_{22}x_2 + \dots + a_{2R}x_R \leq b_2 \qquad \times\ \boldsymbol{\lambda_2 \leq 0} \\
& \vdots \\
& a_{N1}x_1 + a_{N2}x_2 + \dots + a_{NR}x_R = b_N \qquad \times\ \boldsymbol{\lambda_N} \\
& d_1x_1 + d_2x_2 + \dots + d_Rx_R \geq \sum_{n=1}^{N} b_n\lambda_n
\end{aligned}
$$

$$d_r = \sum_{n=1}^{N} a_{nr}\lambda_n$$

We want to determine the $\lambda_n, \forall n$, so that $c = d$ and $\sum_{n=1}^{N} b_n\lambda_n$ gives a lower bound

Another Example

To get the best (highest/tightest) lower bound, we maximize $\sum_{n=1}^{N} b_n\lambda_n$

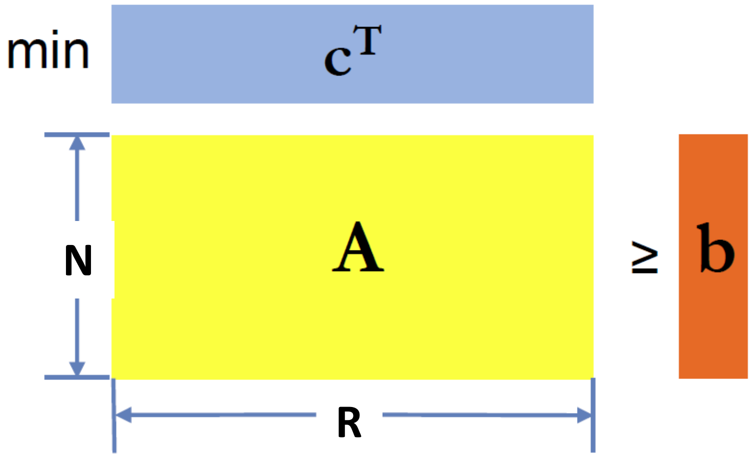
$$\max_{\lambda_n,\forall n} \; b_1\lambda_1 + b_2\lambda_2 + ... + b_N\lambda_N$$

$$\begin{aligned} \text{s.t.}\; & a_{11}\lambda_1 + a_{21}\lambda_2 + ... + a_{N1}\lambda_N = c_1 \\ & a_{12}\lambda_2 + a_{22}\lambda_2 + ... + a_{N2}\lambda_N = c_2 \\ & \\ & a_{1R}\lambda_N + a_{2R}\lambda_2 + ... + a_{NR}\lambda_N = c_R \\ & \lambda_1 \geq 0, \lambda_2 \leq 0, ..., \lambda_n \in \mathbb{R} \end{aligned}$$

Another Example

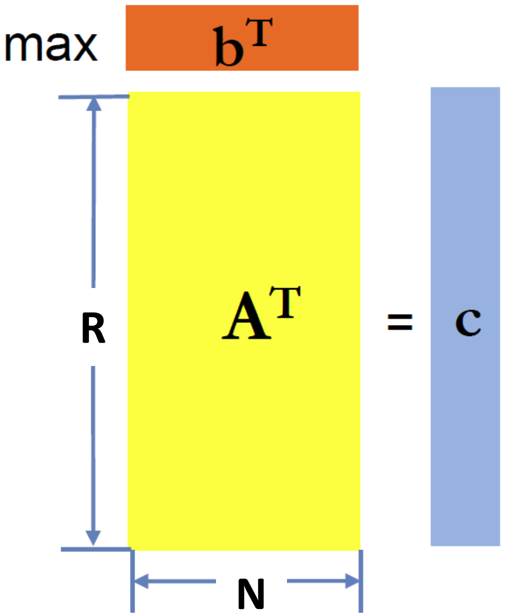
Primal problem

$$\min_{x} \; c^T x$$
$$\text{s.t. } Ax \geq b$$



Dual problem

$$\max_{\lambda} \; b^T \lambda$$
$$A^T \lambda = c$$
$$\lambda \geq 0$$



General Form

Primal problem

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & A_1 x \geq b_1 \ :\ \lambda_1 \\ & A_2 x \leq b_2 \ :\ \lambda_2 \\ & A_3 x = b_3 \ :\ \lambda_3 \\ & x \geq 0 \ :\ \lambda_4 \end{aligned}$$

Dual problem

$$\begin{aligned} \max\ & b_1^T \lambda_1 + b_2^T \lambda_2 + b_3^T \lambda_3 \\ \text{s.t. } & A_1^T \lambda_1 + A_2^T \lambda_2 + A_3^T \lambda_3 \leq c \\ & \lambda_1 \geq 0, \lambda_2 \leq 0, \lambda_3 \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & A_1 x \geq b_1 \ :\ \lambda_1 \\ & -A_2 x \geq -b_2 \ :\ \lambda_2 \\ & A_3 x \geq b_3 \ :\ \lambda_3 \\ & -A_3 x \geq -b_3 \ :\ \lambda_3' \\ & x \geq 0 \ :\ \lambda_4 \end{aligned}$$

$$\begin{aligned} \max\ & b_1^T \lambda_1 - b_2^T \lambda_2 + b_3^T (\lambda_3 - \lambda_3') + 0^T \lambda_4 \\ \text{s.t. } & A_1^T \lambda_1 - A_2^T \lambda_2 + A_3^T (\lambda_3 - \lambda_3') + I \lambda_4 = c \\ & \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_3' \geq 0, \lambda_4 \geq 0 \end{aligned}$$

Principles for LP duality

$$\min_x \; c^T x \quad \text{s.t. } Ax \geq b, \quad x \geq 0$$

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

$$\max_y \; b^T y \quad \text{s.t. } A^T y \leq c, \quad y \geq 0$$

Example - 1

Primal problem	**Dual problem**
min $5x_1 + 4x_2 + 3x_3$	max $4\lambda_1 + 5\lambda_2$
s.t. $x_1 + x_2 + x_3 = 4$	s.t. $\lambda_1 + 3\lambda_2 \leq 5$
$3x_1 + 2x_2 + x_3 = 5$	$\lambda_1 + 2\lambda_2 \leq 4$
$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$	$\lambda_1 + \lambda_2 \leq 3$

Example - 2

Primal problem

$$\min \quad x_1 + 2x_2 + x_3$$

$$\text{s.t.} \quad x_1 + x_2 - x_3 \geq 4$$

$$x_1 - x_2 + x_3 = 1$$

$$2x_1 + x_2 + x_3 \leq 1$$

$$x_1 \geq 0,\ x_2 \leq 0$$

Dual problem

$$\max \quad 4\lambda_1 + \lambda_2 + \lambda_3$$

$$\text{s.t.} \quad \lambda_1 + \lambda_2 + 2\lambda_3 \leq 1$$

$$\lambda_1 - \lambda_2 + \lambda_3 \geq 2$$

$$-\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_1 \geq 0,\ \lambda_3 \leq 0$$

Example - 3

Primal problem

$$\min \quad 2x_1 + 5x_2 + 3x_3$$

$$\text{s.t.} \quad x_1 + 3x_2 - x_3 \geq 4$$

$$x_1 - x_2 + 2x_3 = 1$$

$$2x_1 + 4x_2 + x_3 \leq 1$$

$$x_1 \leq 0,\ x_3 \geq 0$$

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

Example - 3

Primal problem	**Dual problem**
min $2x_1 + 5x_2 + 3x_3$	max $4\lambda_1 + \lambda_2 + \lambda_3$
s.t. $x_1 + 3x_2 - x_3 \geq 4$	s.t. $\lambda_1 + \lambda_2 + 2\lambda_3 \geq 2$
$x_1 - x_2 + 2x_3 = 1$	$3\lambda_1 - \lambda_2 + 4\lambda_3 = 5$
$2x_1 + 4x_2 + x_3 \leq 1$	$-\lambda_1 + 2\lambda_2 + \lambda_3 \leq 3$
$x_1 \leq 0,\ x_3 \geq 0$	$\lambda_1 \geq 0,\ \lambda_3 \leq 0$

Example - 4

Primal problem

$$\min \quad 5x_1 + x_2 + 9x_3$$

$$\text{s.t.} \quad 3x_1 + x_2 + 2x_3 = 27$$

$$2x_1 + 2x_2 + x_3 \geq 15$$

$$x_1 + 5x_2 + 7x_3 \leq 42$$

$$x_1 \leq 0, x_3 \geq 0$$

Primal LP		**Dual LP**	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

Example - 4

Primal problem

$$\begin{aligned} \min \quad & 5x_1 + x_2 + 9x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + 2x_3 = 27 \\ & 2x_1 + 2x_2 + x_3 \geq 15 \\ & x_1 + 5x_2 + 7x_3 \leq 42 \\ & x_1 \leq 0,\ x_3 \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \max \quad & 27\lambda_1 + 15\lambda_2 + 42\lambda_3 \\ \text{s.t.} \quad & 3\lambda_1 + 2\lambda_2 + \lambda_3 \geq 5 \\ & \lambda_1 + 2\lambda_2 + 5\lambda_3 = 1 \\ & 2\lambda_1 + \lambda_2 + 7\lambda_3 \leq 9 \\ & \lambda_2 \geq 0,\ \lambda_3 \leq 0 \end{aligned}$$

Example - 5

Primal problem

$$\text{max} \quad 2x_1 + 3x_2 + 7x_3$$

$$\text{s.t.} \quad x_1 + 2x_2 + x_3 \le 60$$

$$2x_1 + 3x_2 + x_3 \ge 15$$

$$5x_1 + x_2 + 2x_3 = 20$$

$$x_1 \ge 0,\ x_2 \le 0$$

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

Example - 5

Primal problem	**Dual problem**
max $2x_1 + 3x_2 + 7x_3$	min $60\lambda_1 + 15\lambda_2 + 20\lambda_3$
s.t. $x_1 + 2x_2 + x_3 \leq 60$	s.t. $\lambda_1 + 2\lambda_2 + 5\lambda_3 \geq 2$
$2x_1 + 3x_2 + x_3 \geq 15$	$2\lambda_1 + 3\lambda_2 + \lambda_3 \leq 3$
$5x_1 + x_2 + 2x_3 = 20$	$\lambda_1 + \lambda_2 + 2\lambda_3 = 7$
$x_1 \geq 0,\ x_2 \leq 0$	$\lambda_1 \geq 0,\ \lambda_2 \leq 0$

Why do we need dual problem?

1. When the **primal problem** has a lot of constraints and few variables, solving the dual problem can reduce the computation time
   (Because of the procedure of some well-known algorithms, e.g. Simplex)

2. Help to prove the primal problem is infeasible
   (When the dual problem is unbounded, the primal problem is infeasible)

3. Sensitivity analysis
   (Shadow price, the impact of constraint coefficients on the optimal objective value)

Duality Theorems

Symmetric: The dual of dual problem is the primal problem.

$$\min_{x} \ c^T x \quad \text{s.t. } Ax \geq b, \quad x \geq 0$$

primal → dual

dual ← primal

$$\max_{\lambda} \ b^T \lambda \quad A^T \lambda \leq c, \quad \lambda \geq 0$$

Proof:

$$\max_{\lambda} \ b^T \lambda \quad A^T \lambda \leq c, \quad \lambda \geq 0$$

$$\Rightarrow \quad \min_{\lambda} \ (-b^T)\lambda \quad \text{s.t. } (-A^T)\lambda \geq (-c), \quad \lambda \geq 0$$

$$\Rightarrow \quad \max_{x} \ (-c^T)x \quad \text{s.t. } (-A)x \leq (-b), \quad x \geq 0$$

$$\Rightarrow \quad \min_{x} \ c^T x \quad \text{s.t. } Ax \geq b, \quad x \geq 0$$

Duality Theorems

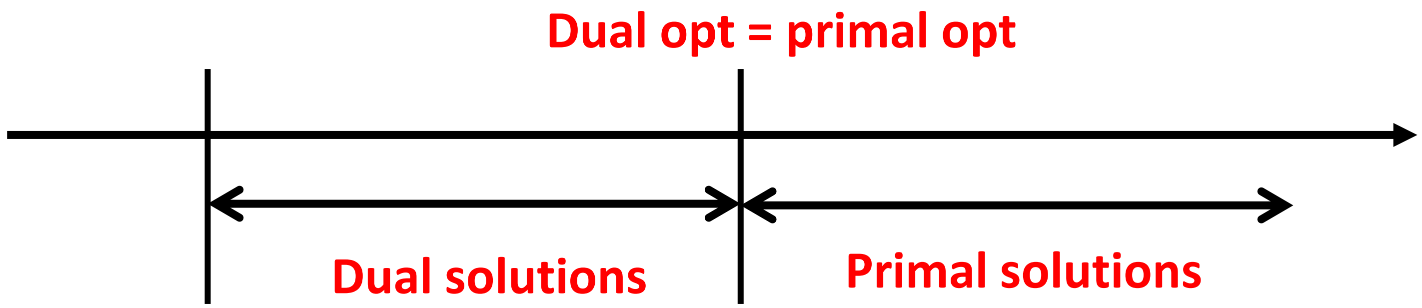
Weak duality: Let x_0, λ_0 be a feasible solution of the primal problem and the dual problem, respectively. We have $c^T x_0 \geq b^T \lambda_0$.

Proof: since x_0 is a feasible solution of the primal problem, we have

$$A x_0 \geq b, x_0 \geq 0$$

We also have $\lambda_0 \geq 0$, therefore, $\lambda_0^T A x_0 \geq \lambda_0^T b$. So

$$b^T \lambda_0 \leq x_0^T A^T \lambda_0 \leq x_0^T c = c^T x_0$$



An example

Primal problem

$$\min_{x_1,x_2} \ 20x_1 + 20x_2$$
$$\text{s.t. } x_1 + 2x_2 \geq 1$$
$$2x_1 + x_2 \geq 2$$
$$2x_1 + 3x_2 \geq 3$$
$$3x_1 + 2x_4 \geq 4$$
$$x_1, x_2 \geq 0$$

Dual problem

$$\max_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \ \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4$$
$$\text{s.t. } \lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 \leq 20$$
$$2\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 \leq 20$$
$$\lambda_j \geq 0, j = 1,2,3,4$$

Equal?

For the primal problem, $x_0 = (1,1)^T$ is a feasible solution
$c^T x_0 = 40$ is an upper bound of the objective value

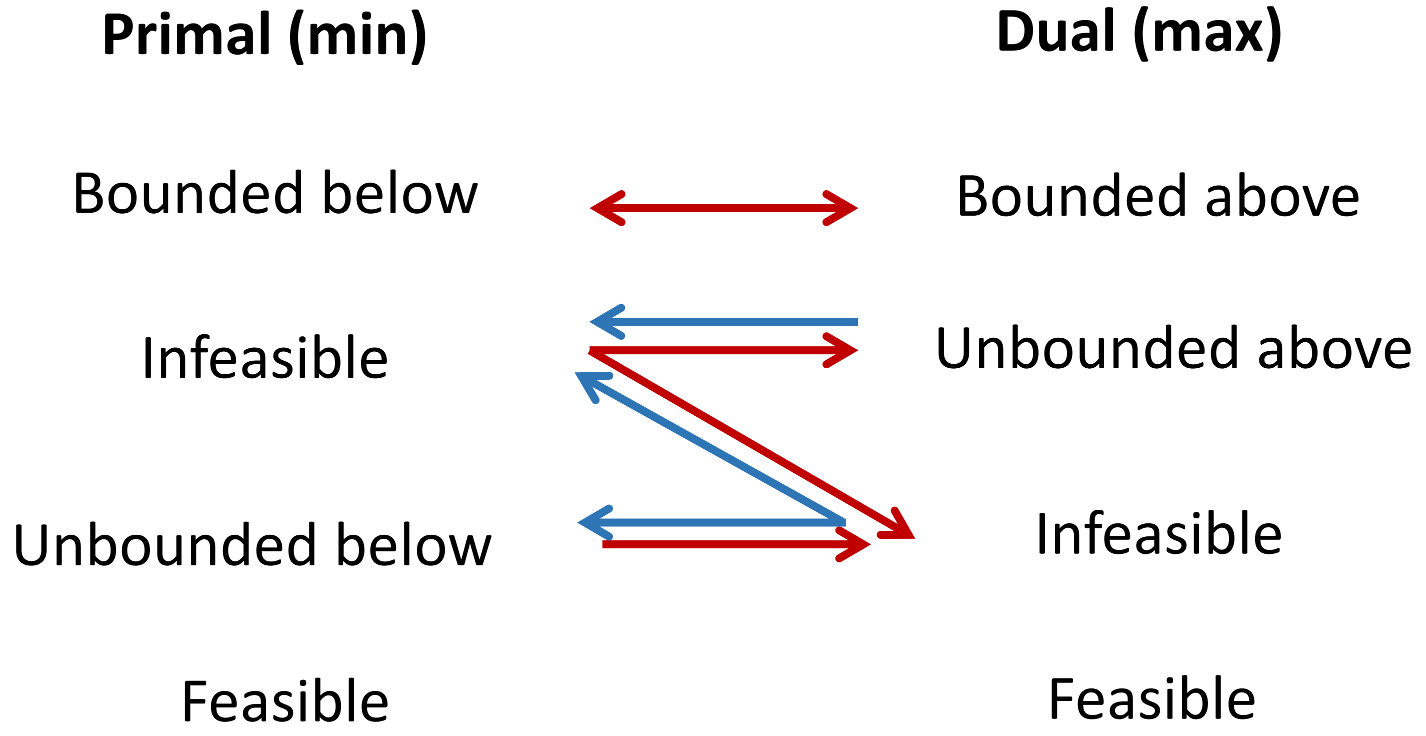
$x^* = \left(\frac{6}{5}, \frac{1}{5}\right), c^T x^* = 28$

$\lambda^* = (0,0,4,4), d^T \lambda^* = 28$

For the dual problem, $\lambda_0 = (1,1,1,1)^T$ is a feasible solution
$d^T \lambda_0 = 10$ is an lower bound of the objective value

Duality Theorems

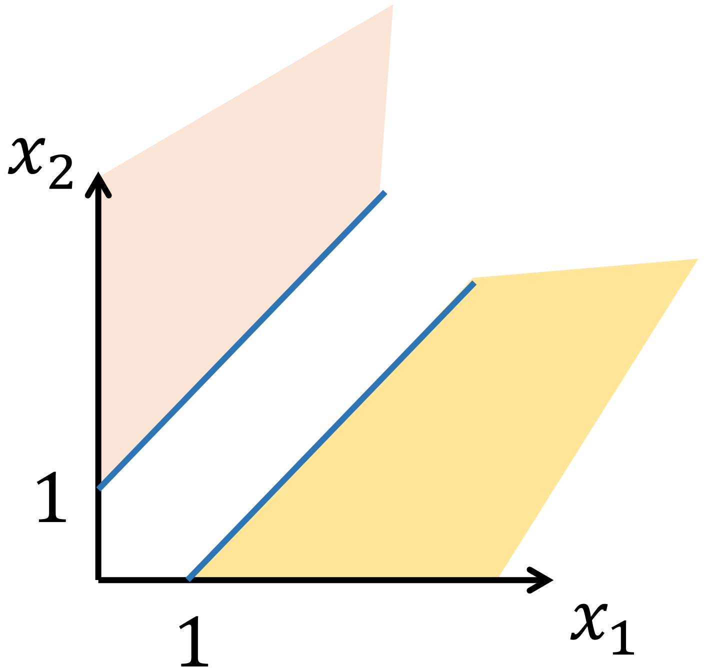
Weak duality: Let x_0, λ_0 be a feasible solution of the primal problem and the dual problem, respectively. We have $c^T x_0 \geq b^T \lambda_0$.



Duality Theorems

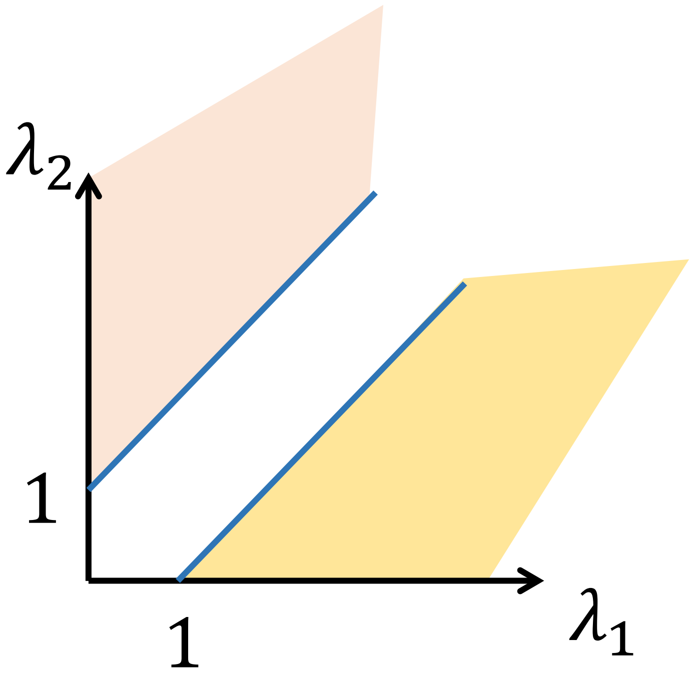
$$\min_{x_1,x_2} \; -x_1 - x_2$$
$$\text{s.t. } x_1 - x_2 \geq 1$$
$$-x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Infeasible



$$\max_{\lambda_1,\lambda_2} \; \lambda_1 + \lambda_2$$
$$\text{s.t. } \lambda_1 - \lambda_2 \leq -1$$
$$-\lambda_1 + \lambda_2 \leq -1$$
$$\lambda_1, \lambda_2 \geq 0$$

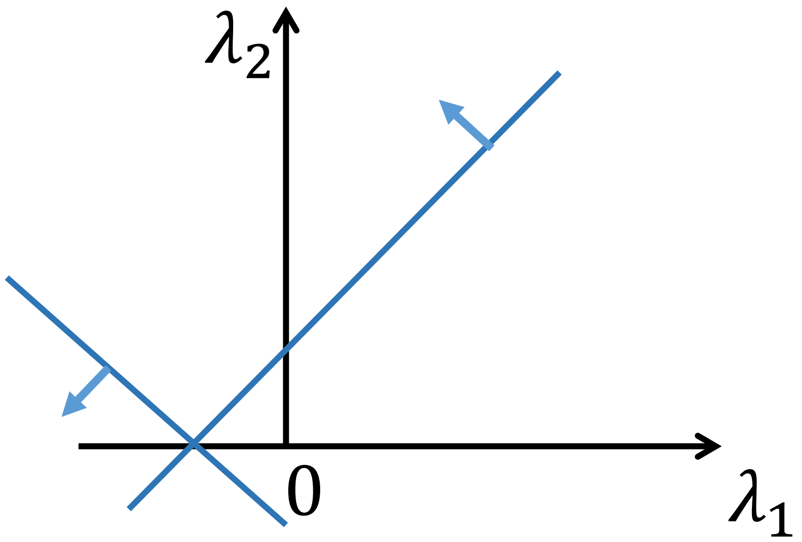
Infeasible



Duality Theorems

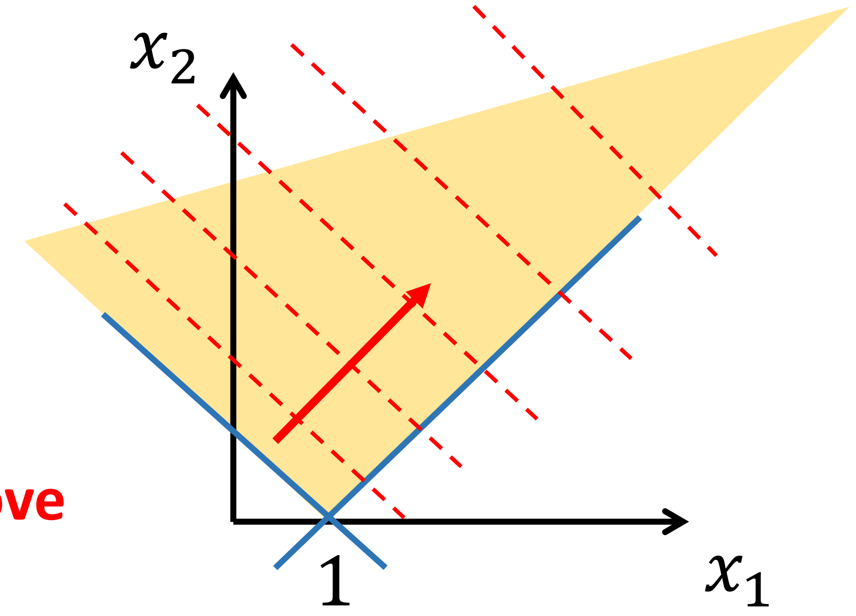
$$\min_{\lambda_1,\lambda_2} \; -\lambda_1+\lambda_2$$
$$\text{s.t.} \; -\lambda_1+\lambda_2 \geq 1$$
$$-\lambda_1-\lambda_2 \geq 1$$
$$\lambda_1,\lambda_2 \geq 0$$

Infeasible



$$\max_{x_1,x_2} \; x_1+x_2$$
$$\text{s.t.} \; -x_1-x_2 \leq -1$$
$$x_1-x_2 \leq 1$$
$$x_1,x_2 \geq 0$$

Unbounded above



Duality Theorems

- This lecture: focus on linear programs.
- For nonlinear programs, strong duality needs more conditions than feasibility (future lectures)

Optimality criterion: Let x_0, λ_0 be a feasible solution of the primal problem and the dual problem, respectively. If $c^T x_0 = b^T \lambda_0$, then x_0, λ_0 are the optimal solution of the primal and dual problems, respectively.

Proof:
According to weak duality, we have $c^T x \geq b^T \lambda_0, \forall x \in X$
Since $c^T x_0 = b^T \lambda_0$, we can know x_0 is the optimal solution.

Strong duality: If both the primal and dual problems are feasible, then both of them have an optimal solution, i.e. x^*, λ^*, and the optimal values are equal, i.e. $c^T x^* = d^T \lambda^*$.

Duality Theorems

Primal problem

$$\min_x \ c^T x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0$$

Dual problem

$$\max_\lambda \ b^T \lambda$$
$$A^T \lambda \leq c$$
$$\lambda \geq 0$$

Complementarity and slackness: Suppose x^*, λ^* are the primal and dual optimal solutions, respectively. Then, we have

$$a_n^T x^* > b \quad \Rightarrow \quad \lambda_n^* = 0$$
$$\lambda_n^* > 0 \quad \Rightarrow \quad a_n^T x^* = b$$

An Example

Primal problem

$$\min_{x_1,x_2} \quad 20x_1 + 20x_2$$
$$\text{s.t.} \quad x_1 + 2x_2 \geq 1$$
$$2x_1 + x_2 \geq 2$$
$$2x_1 + 3x_2 \geq 3$$
$$3x_1 + 2x_4 \geq 4$$
$$x_1, x_2 \geq 0$$

Dual problem

$$\max_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \quad \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4$$
$$\text{s.t.} \quad \lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 \leq 20$$
$$2\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 \leq 20$$
$$\lambda_j \geq 0, j = 1, 2, 3, 4$$

If we know the optimal solution of the primal problem is $x^* = \left(\frac{6}{5}, \frac{1}{5}\right)$,
Try to determine the optimal solution of dual problem based on Complementarity and slackness.

An Example

Since $x_1^* > 0, x_2^* > 0$, we have

$$\lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 = 20$$
$$2\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 = 20$$

Moreover, as $x_1^* + 2x_2^* = 1.6 > 1$ and $2x_1^* + x_2^* = 2.6 > 2$ we have $\lambda_1^* = 0, \lambda_2^* = 0$.
Therefore,

$$2\lambda_3 + 3\lambda_4 = 20$$
$$3\lambda_3 + 2\lambda_4 = 20$$

So $\lambda_3^* = 4, \lambda_4^* = 4$, dual optimal is $\lambda^* = (0, 0, 4, 4)^T$.

Primal problem

$$\min_{x_1,x_2} \; 20x_1 + 20x_2$$
$$\text{s.t. } x_1 + 2x_2 \geq 1 \quad \lambda_1$$
$$2x_1 + x_2 \geq 2 \quad \lambda_2$$
$$2x_1 + 3x_2 \geq 3 \quad \lambda_3$$
$$3x_1 + 2x_4 \geq 4 \quad \lambda_4$$
$$x_1, x_2 \geq 0$$

Dual problem

$$\max_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \; \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4$$
$$\text{s.t. } \lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 \leq 20 \quad x_1$$
$$2\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 \leq 20 \quad x_2$$
$$\lambda_j \geq 0, j = 1, 2, 3, 4$$

Economic Interpretation

According to Optimality criterion, we have

$$f^* = c^T x^* = b^T \lambda^* = \lambda_1^* b_1 + \cdots + \lambda_N^* b_N$$

If parameter b_n changes, what is the impact on the optimal value f^*?

$$\frac{\partial f^*}{\partial b_1} = \lambda_1^*, \ldots, \frac{\partial f^*}{\partial b_N} = \lambda_N^*$$

Therefore, λ_n^* can be interpreted as the change of f^* should there be 1 unit change of b_n. We call it "*shadow price*" in economics.

The scarcer the resource, the greater the impact of its changes on the objective function (cost), and therefore the higher the shadow price.

Thanks！