MAEG4070 Engineering Optimization

Lecture 9 Convex Optimization

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Oct 26, 2022

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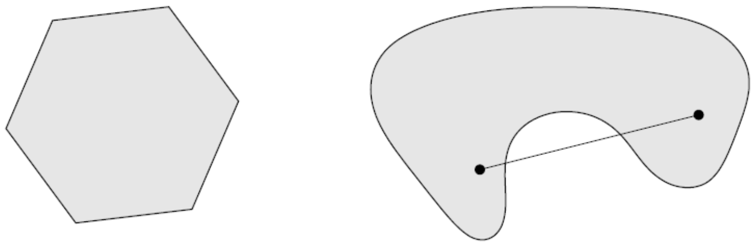
Convex Sets

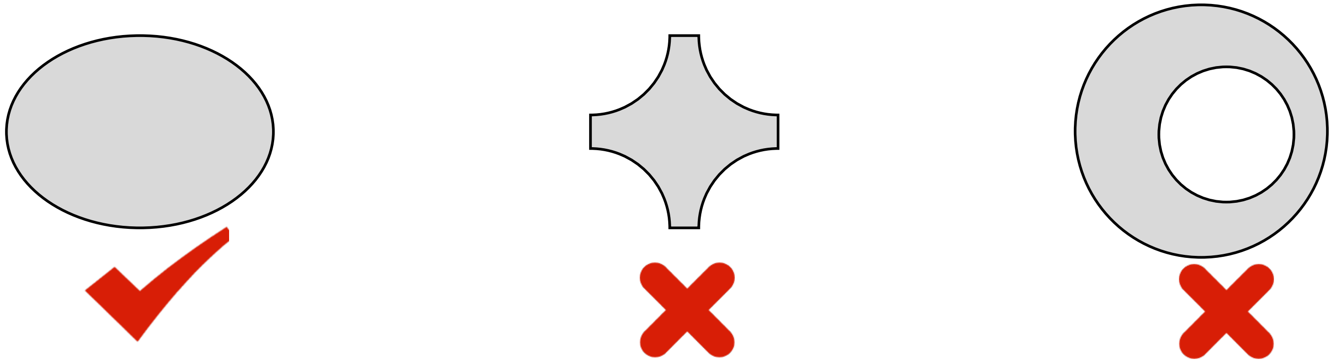
Convex set: the set that contains all line segment between any two distinct points in the set $\mathcal{C}$

$$\forall x_1, x_2 \in \mathcal{C}, \theta \in [0,1] \ \Rightarrow \ \theta x_1 + (1-\theta) x_2 \in \mathcal{C}$$



Intuitive explanation: in a convex set, you can see everywhere wherever you stand

Try it yourself: Are the following sets convex?



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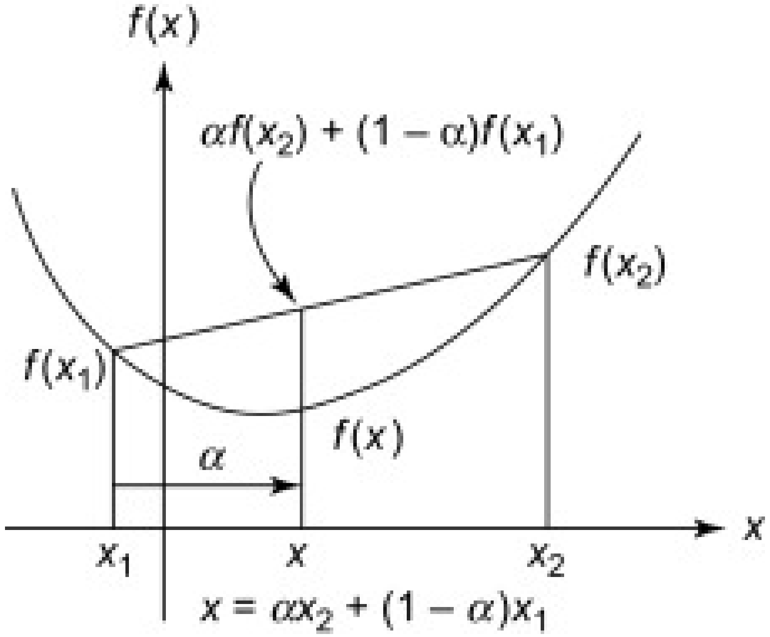
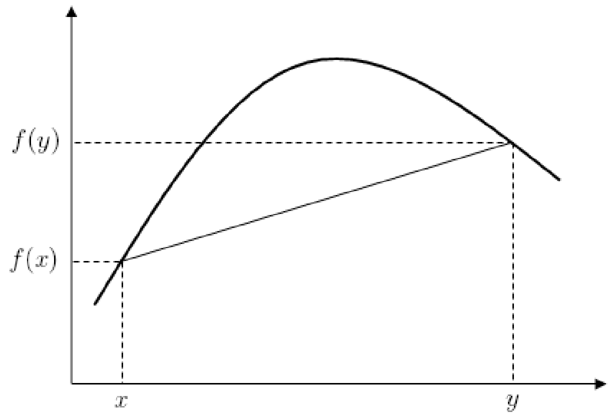
Convex function

Function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if dom(f) is a convex set, and the following inequality holds

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2), \forall \theta \in [0,1], \forall x_1, x_2 \in dom(f)$$



If we change ≤ into ≥, then it is **concave**



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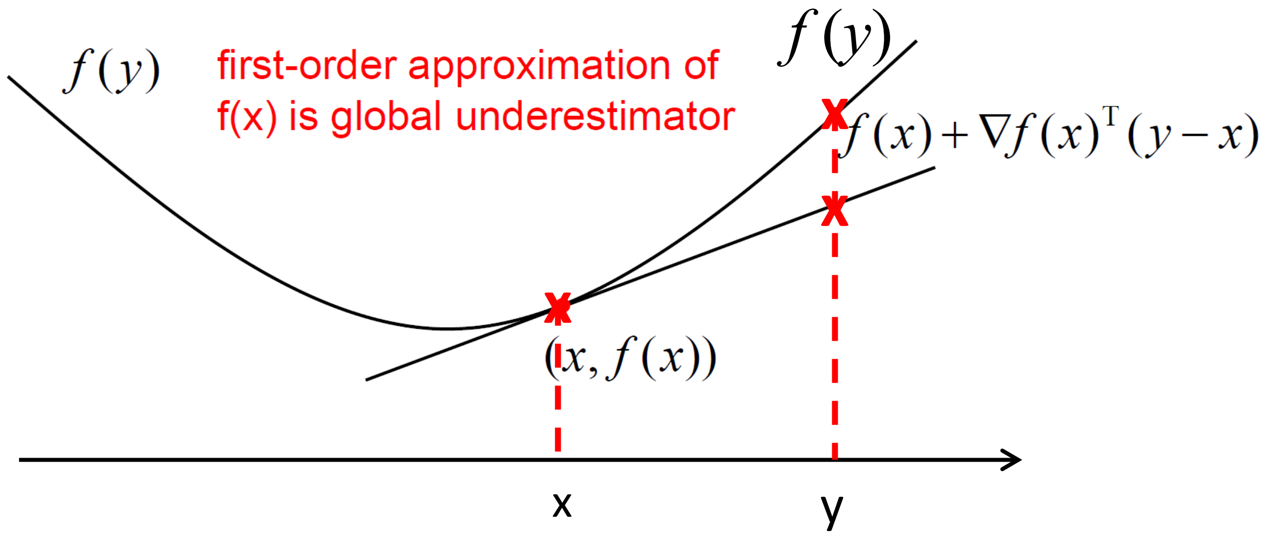
Convex function

Apart from proving the convexity by definition, in the following, we provide two conditions, i.e. first-order condition & second-order condition

Suppose f is differentiable and $\nabla f(x)$ exists at each $x \in dom(f)$

First-order condition f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y \in dom(y)$$



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Convex function

Suppose f is twice differentiable and the Hessian $H(x)$ exists at every $x \in dom(f)$.

Second-order condition function f with convex domain is

- convex iff

positive semidefinite

$$H(x) \succeq 0, \forall x \in dom(f)$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is positive semidefinite iff $a \geq 0$ and $ad - bc \geq 0$

- Strictly convex iff

positive definite

$$H(x) \succ 0, \forall x \in dom(f)$$

- Strongly convex iff

$$H(x) - \alpha I \succeq 0, \forall x \in dom(f)$$

Review – KKT point

$$\min_x \ f(x)$$
$$\text{s.t. } h(x) = 0, g(x) \le 0$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^r$ are continuously differentiable.

Lagrangian function

$$L(x, \lambda, \mu) = f(x^*) + \sum_{i=1}^{m} \lambda_i^* h_i(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*)$$

KKT point satisfies

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$
$$h_i(x^*) = 0, \forall i = 1, ..., m$$
$$0 \le -g_j(x^*) \perp \mu_j^* \ge 0, \forall j = 1, ..., r$$

Convex Optimization

Convex optimization

Case 1

$$\min_x \ f(x)$$
$$\text{s.t. } a_i^T x - b_i = 0, \forall i = 1, \ldots, m$$
$$g_j(x) \leq 0, j = 1, \ldots, r$$

is convex optimization if $f(x)$ and $g_j(x), \forall j$ are all convex functions.

Case 2

$$\max_x \ f(x)$$
$$\text{s.t. } a_i^T x - b_i = 0, \forall i = 1, \ldots, m$$
$$g_j(x) \geq 0, j = 1, \ldots, r$$

is convex optimization if $f(x)$ and $g_j(x), \forall j$ are all concave functions.

Convex Optimization

Necessary condition

Let x^* be a local minimum and a regular point.
Then there exist unique Lagrange multiplier vectors $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$, $\mu^* = (\mu_1^*, ..., \mu_r^*)$

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$
$$h_i(x^*) = 0, \forall i = 1, ..., m$$
$$0 \leq -g_j(x^*) \perp \mu_j^* \geq 0, \forall j = 1, ..., r$$

Sufficient condition

If the optimization is a convex optimization, and point x^* is a regular and KKT point, then x^* is a global optimum.

Unconstrained optimization is a special case

Constrained optimization	**Unconstrained optimization**
KKT point $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$ $h_i(x^*) = 0, \forall i = 1, \ldots, m$ $0 \leq -g_j(x^*) \perp \mu_j^* \geq 0, \forall j = 1, \ldots, r$	Stationary point $\nabla f(x^*) = 0$
Convex optimization & global optimum $\min_x \; f(x)$ ← **convex** $\text{s.t. } a_i^T x - b_i = 0, \forall i = 1, \ldots, m$ $g_j(x) \leq 0, j = 1, \ldots, r$ ↑ **convex**	Convex function & global optimum $\min_x f(x)$ **Hessian matrix positive semi-definite → $f(x)$ convex → global minimum**

Example

Determine the optimal solution of

$$\begin{aligned}\min_{x_1,x_2} \quad & -2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \geq 4 \\ & x_1 + 2x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

Solution:
Obviously, this optimization problem is a convex optimization.
First, let's turn it into a standard form

$$\begin{aligned}\min_{x_1,x_2} \quad & -2x_1 + x_2 \\ \text{s.t.} \quad & -x_1 - x_2 - x_3 \leq -4 \\ & x_1 + 2x_2 + 2x_3 \leq 6 \\ & -x_1 \leq 0, -x_2 \leq 0, -x_3 \leq 0\end{aligned}$$

Example

The Lagrangian function is

$$
\begin{aligned}
L(x,\mu) = \; & -2x_1 + x_2 + \mu_1(-x_1 - x_2 - x_3 + 4) + \mu_2(x_1 + 2x_2 + 3x_3 - 6) \\
& + \mu_3(-x_1) + \mu_4(-x_2) + \mu_5(-x_3)
\end{aligned}
$$

The KKT point satisfies

$$
\begin{aligned}
-2 - \mu_1 + \mu_2 - \mu_3 &= 0 \\
1 - \mu_1 + 2\mu_2 - \mu_4 &= 0 \\
-\mu_1 + 3\mu_2 - \mu_5 &= 0 \\
0 \leq (x_1 + x_2 + x_3 - 4) \perp \mu_1 &\geq 0 \\
0 \leq (-x_1 - 2x_2 - 2x_3 + 6) \perp \mu_2 &\geq 0 \\
0 \leq x_1 \perp \mu_3 &\geq 0 \\
0 \leq x_2 \perp \mu_4 &\geq 0 \\
0 \leq x_3 \perp \mu_5 &\geq 0
\end{aligned}
$$

The KKT point is $x^* = (6,0,0)^T$, which is a global optimum.

Example

Determine the optimal solution of

$$\min_{x_1,x_2} \; x_1 + x_2$$
$$\text{s.t. } x_1^2 + x_2^2 \leq 2$$

Solution:
First, we need to check if $g(x) = x_1^2 + x_2^2$ is a convex function.
The Hessian matrix is

$$H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \succ 0$$

Therefore, $g(x)$ is convex and the problem is a convex optimization.
The Lagrange function is

$$L(x, \mu) = x_1 + x_2 + \mu(x_1^2 + x_2^2 - 2)$$

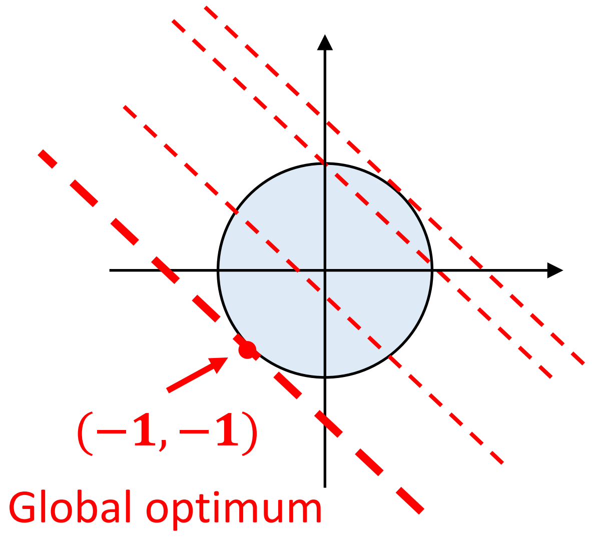
Example

The Lagrange function is

$$L(x,\mu) = x_1 + x_2 + \mu(x_1^2 + x_2^2 - 2)$$

The KKT point satisfies

$$1 + 2\mu x_1 = 0$$
$$1 + 2\mu x_2 = 0$$
$$0 \le (2 - x_1^2 - x_2^2) \perp \mu \ge 0$$



If $\mu = 0$, we have $1 = 0$, contradiction!
If $\mu \neq 0$, then $x_1^2 + x_2^2 = 2$, together with $x_1 = x_2 = -\frac{1}{2\mu}$.
We have $(x^*, \mu^*) = (-1, -1, \frac{1}{2})$, Which is a global optimum.

Example

Determine the optimal solution of

$$\min_{x_1, x_2} \; x_1 + x_2$$
$$\text{s.t. } 1 \le x_1^2 + x_2^2 \le 2$$

Solution:
First, we rewrite the problem in a standard form

$$\begin{aligned} \min_{x_1, x_2} \;& x_1 + x_2 \\ \text{s.t. } & x_1^2 + x_2^2 \le 2 \\ & -x_1^2 - x_2^2 \le -1 \end{aligned}$$

let $g_1(x) = x_1^2 + x_2^2$ and $g_2(x) = -x_1^2 - x_2^2$.
By checking their Hessian matrix, we know that $g_1(x)$ is convex and $g_2(x)$ is not.
Therefore, the above problem is not a convex optimization.

Example

The Lagrangian function is

$$L(x, \mu) = x_1 + x_2 + \mu_1(x_1^2 + x_2^2 - 2) + \mu_2(-x_1^2 - x_2^2 + 1)$$

The KKT point satisfies

$$\begin{aligned} 1 + 2\mu_1 x_1 - 2\mu_2 x_1 &= 0 \\ 1 + 2\mu_1 x_2 - 2\mu_2 x_2 &= 0 \\ 0 \leq (2 - x_1^2 - x_2^2) \perp \mu_1 &\geq 0 \\ 0 \leq (x_1^2 + x_2^2 - 1) \perp \mu_2 &\geq 0 \end{aligned}$$

Example

For μ_1 and μ_2, at least one of them equals to zero.
Case 1: $\mu_1 = \mu_2 = 0$, we have $1 = 0$, contradiction!
Case 2: $\mu_1 = 0, \mu_2 \neq 0$, we have

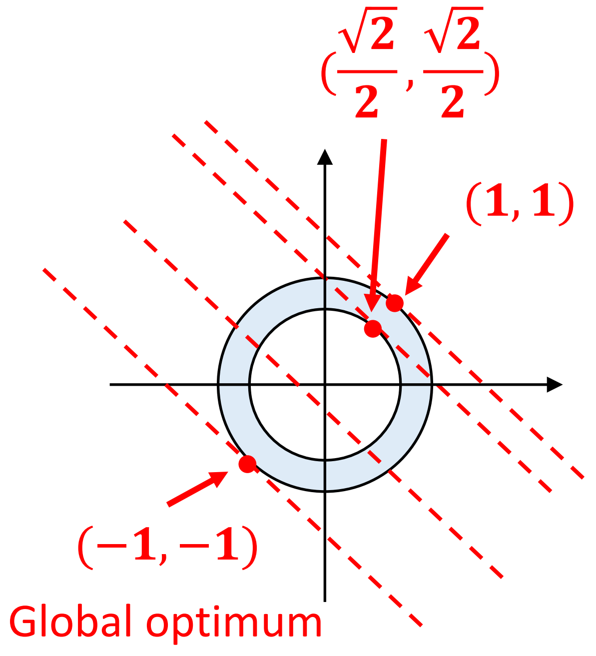
$$x_1 = x_2 = \frac{1}{2\mu_2}$$
$$x_1^2 + x_2^2 = 1$$

Therefore, $x^* = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\mu^* = (0, \frac{\sqrt{2}}{2})$.

Case 3: $\mu_1 \neq 0, \mu_2 = 0$, we have

$$x_1 = x_2 = -\frac{1}{2\mu_1}$$
$$x_1^2 + x_2^2 = 2$$

Therefore, $x^* = (-1, -1)$, $\mu^* = (\frac{1}{2}, 0)$.



KKT point is not global optimum

Example

Determine the optimal solution of

$$\min_{x_1, x_2} \; x_1 x_2$$
$$\text{s.t. } x_1^2 + x_2^2 \leq 2$$

Solution:
We already know $g(x) = x_1^2 + x_2^2 - 2$ is a convex function.
For $f(x) = x_1 x_2$, its Hessian matrix is

$$H(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is not semi-positive definite.}$$

Therefore, $f(x)$ is not convex and the problem is not a convex optimization.

Example

The Lagrangian function is

$$L(x,\mu) = x_1 x_2 + \mu(x_1^2 + x_2^2 - 2)$$

The KKT point satisfies

$$\begin{aligned} x_2 + 2x_1\mu &= 0 \\ x_1 + 2x_2\mu &= 0 \\ 0 \le (2 - x_1^2 - x_2^2) \perp \mu &\ge 0 \end{aligned}$$

Case 1: $\mu^* = 0$, then we have $x^* = (0,0)$.
Case 2: $\mu \neq 0$, then $(x^*, \mu^*) = (1, -1, \frac{1}{2})$ or $(x^*, \mu^*) = (-1, 1, \frac{1}{2})$.
Note that

$$(x_1 + x_2)^2 \ge 0 \iff x_1 x_2 \ge -\frac{1}{2}(x_1^2 + x_2^2) \ge -1$$

Therefore, points $(x^*, \mu^*) = (1, -1, \frac{1}{2})$ and $(x^*, \mu^*) = (-1, 1, \frac{1}{2})$ are global optimum.

KKT point happens to be global optimum

Thanks！