MAEG4070 Engineering Optimization

Mid-term Review

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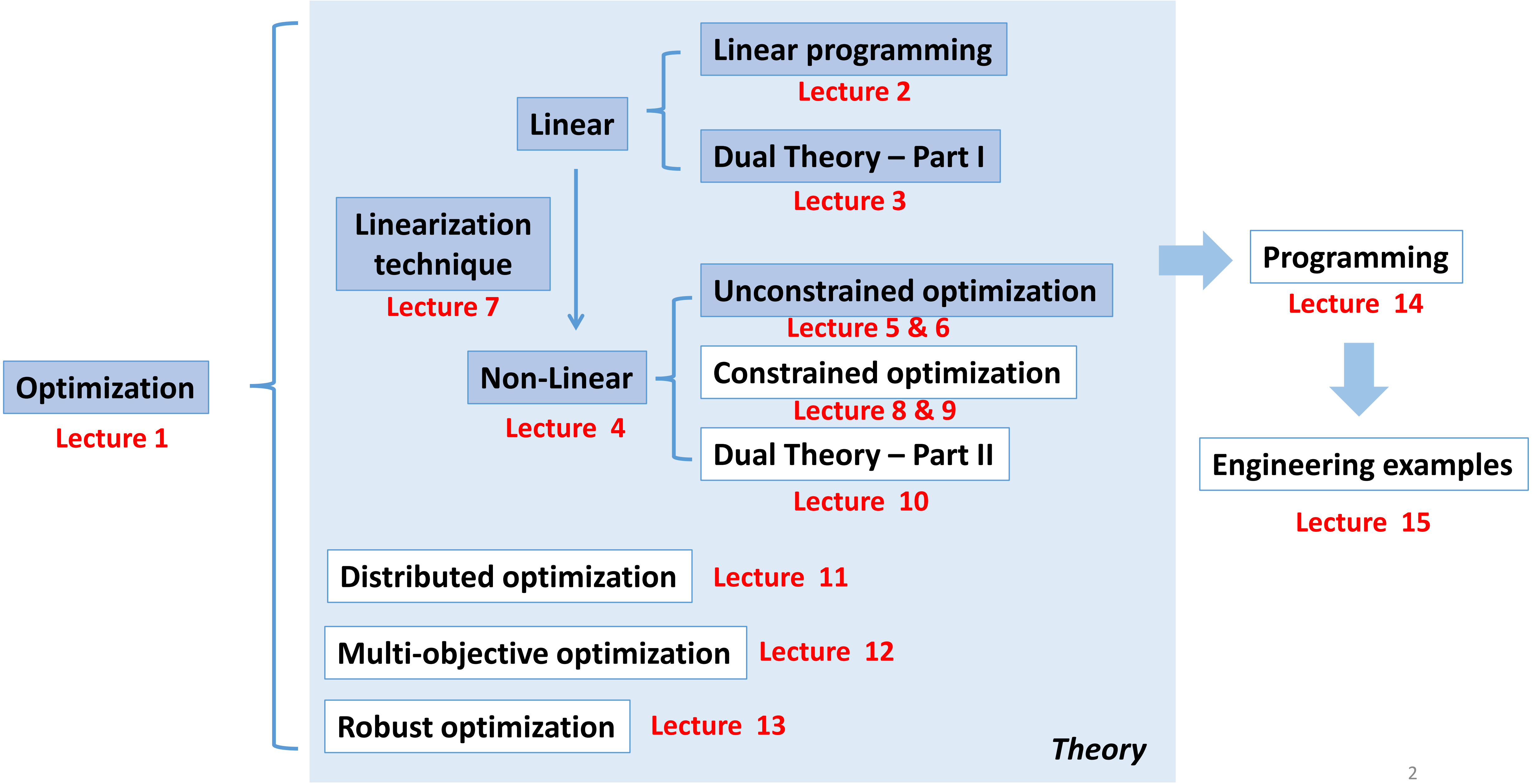
Content of this course (tentative)



Problem description

A company has some resources to produce three products (denoted as A, B, C). Each product consumes a different mix of resources, and there will be a profit from selling the product. The endowment of resources and its relationship with products are:

	A	B	C	Endowment
Steel	3	4	2	600
Wood	2	1	2	400
Label	1	3	3	300
Machine	1	4	4	200
Profit	2	4	3	

The company aims to maximize its profit. Please answer the following questions.

Questions:

1. Try to translate the above statement into an optimization problem. Is it a linear programming?
2. solve it using graphical method.
3. Write down the dual problem of the optimization.
4. Write down the complementary and slackness conditions. Linearize it?
5. Suppose we know the optimal solution of the primal problem, how about the optimal solution of the dual problem?
6. What if the profit per unit depends on the quantity? How to model the optimization?
7. Prove the feasible region is a convex set.
8. If we temporarily ignore the constraints, prove the new objective function is a convex function.
9. If we temporarily ignore the constraints, try to solve the problem using different methods.

Questions:

1. Try to translate the above statement into an optimization problem. Is it a linear programming?

Key points:

Lecture-1: model an engineering problem using optimization

- What are the decision variables?
- What are the constraints?
- What is the objective function?

Lecture-2 how to identify if a problem is an LP?

- Does it satisfy

$$f(a+b) = f(a) + f(b)$$

$$f(ka) = kf(a)$$

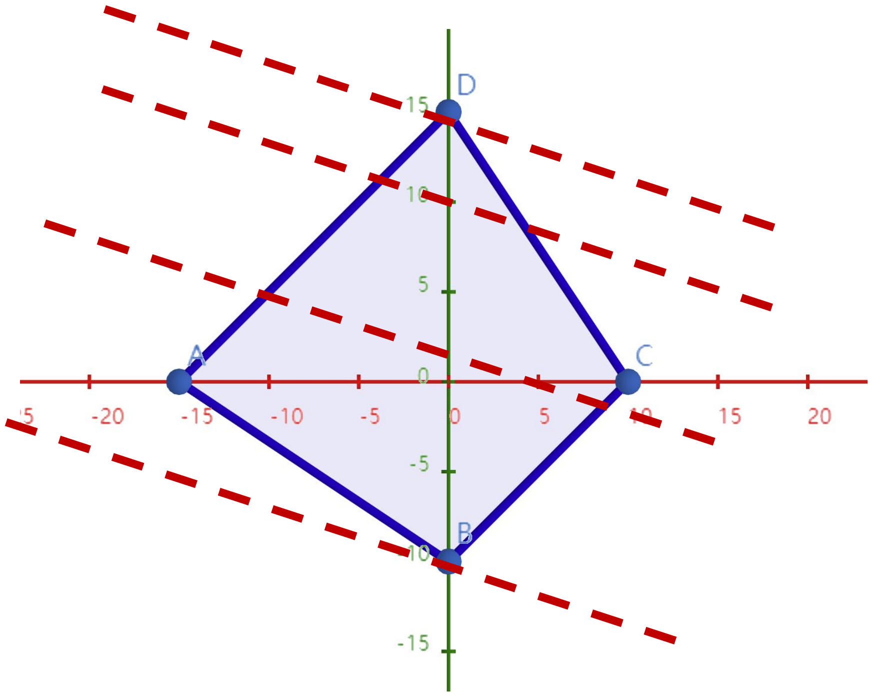
Questions:

2. Solve it using graphical method.

Key points:

Lecture-2: solve an optimization using graphical method

- How to draw the feasible region?
- How to draw the contours of objective?
- How to find the optimal point?



Questions:

3. Write down the dual problem of the optimization.

Key points:

Lecture-3: construct the dual problem of an LP.

- The objective - min or max?
- The constraints - $\geq$ or $\leq$ or $=$?
- The variables - $\geq$ or $\leq$ or $=$?

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

$$\min_x \; c^T x \quad \text{s.t. } Ax \geq b, \; x \geq 0$$

$$\max_y \; b^T y \quad \text{s.t. } A^T y \leq c, \; y \geq 0$$

Questions:

4. Write down the complementary and slackness conditions. Linearize it?

Key points:

Lecture-3: complementary slackness condition.

Complementarity slackness: Suppose x^*, λ^* are the primal and dual optimal solutions, respectively. Then, we have

$$a_n^T x^* > b \quad \Rightarrow \quad \lambda_n^* = 0$$
$$\lambda_n^* > 0 \quad \Rightarrow \quad a_n^T x^* = b$$

Lecture-7: Linearization of $0 \leq x \perp y \geq 0$

- Big-M method

Questions:

5. Suppose we know the optimal solution of the primal problem, how about the optimal solution of the dual problem?

Key points:

Lecture-3: complementary and slackness condition.

- Which of the x are >0 → which of the constraints in the dual problem is binding
- Which of the constraints in the primal problem is not binding → which of the λ equal to 0
- Solve for the λ

Questions:

6. What if the profit per unit depends on the quantity? How to model the optimization?

Key points:

Lecture-1: model an engineering problem using optimization.

	A	B	C	Endowment
Steel	3	4	2	600
Wood	2	1	2	400
Label	1	3	3	300
Machine	1	4	4	200
Profit	$2x_1$	$4x_2$	$3x_3$	

The objective will become $\max 2x_1^2 + 4x_2^2 + 3x_3^2$ Nonlinear optimization

Questions:

7. Prove the feasible region is a convex set.

Key points:

Lecture-4: how to prove a set is convex.

- For any two distinct points $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$
- For any $\theta \in [0,1]$
- Try to prove $\theta x^1 + (1-\theta)x^2$ is inside the set given that x^1, x^2 is inside the set.

$$3x_1 + 4x_2 + 2x_3 \leq 600$$
$$2x_1 + x_2 + 3x_3 \leq 400$$
$$x_1 + 3x_2 + 3x_3 \leq 300$$
$$x_1 + 4x_2 + 4x_3 \leq 200$$
$$x_1, x_2, x_3 \geq 0$$

Questions:

8. If we temporarily ignore the constraints, prove the new objective function is a convex function.

Key points:

Lecture-4: how to prove a function is convex.

- Apply definition
- First-order condition

  Suppose f is differentiable and $\nabla f(x)$ exists at each $x \in dom(f)$

  First-order condition f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y \in dom(y)$$

- Second-order condition

$$H(x) \succeq 0, \forall x \in dom(f)$$

Questions:

9. If we temporarily ignore the constraints, try to solve the problem using different methods.

Key points:

Method-1: Lecture-5 necessary and sufficient conditions

- Let the gradient equal to 0 → candidate point x^*
- Calculate the Hessian matrix → Is the function a convex function?
  → global optimum or local optimum?
- Substitute x^* into the Hessian matrix → Is it positive (semi-)definite?
  → relative minimum or strict minimum?

(attention: first turn the maximization problem into a minimization)

Questions:

9. If we temporarily ignore the constraints, try to solve the problem using different methods.

Key points:

Method-2: Lecture-6 gradient-based methods

- Calculate the gradient
- Calculate the Hessian, the inverted Hessian
- Apply gradient descent method/Newton method

Algorithm: Choose initial point $x_0 \in \mathbb{R}^n$, repeat:

Gradient Descent: $x_k = x_{k-1} - \alpha \nabla f(x_{k-1})$

Or Newton: $x_k = x_{k-1} - \left[\nabla^2 f(x_{k-1})\right]^{-1} \nabla f(x_{k-1})$

Stop until convergence, e.g. $\|x_k - x_{k-1}\| \leq \varepsilon$

Questions:

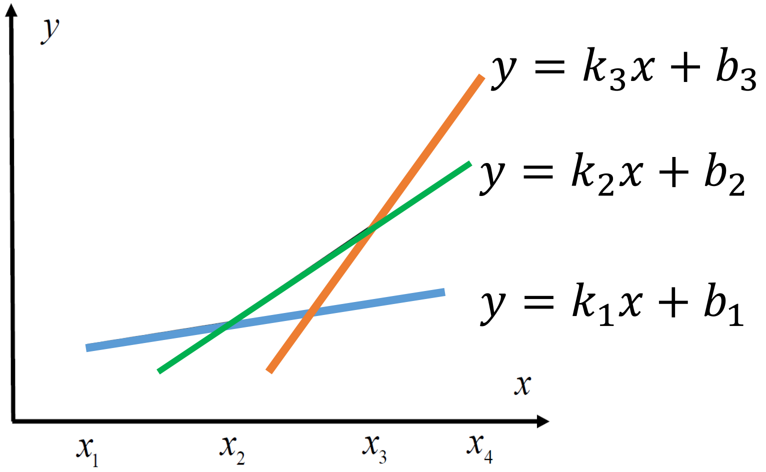
9. If we temporarily ignore the constraints, try to solve the problem using different methods.

Key points:

Method-3: Lecture-7 linearize piecewise linear function

- Linearize $2x_1^2, 4x_2^2, 3x_3^2$, respectively

$$\begin{aligned} \min_{x,\sigma}\ & \sigma \\ \text{s.t. } & \sigma \geq k_1 x + b_1 \\ & \sigma \geq k_2 x + b_2 \\ & \sigma \geq k_3 x + b_3 \\ & x_1 \leq x \leq x_4 \end{aligned}$$



$$\begin{aligned} \min_{x,y,\lambda}\ & y \\ \text{s.t. } & x = \sum_{n=1}^{N} \lambda_n x_n \\ & y = \sum_{n=1}^{N} \lambda_n f(x_n) \\ & 0 \leq \lambda_n \leq 1, \forall n = 1, ..., N \\ & \sum_{n=1}^{N} \lambda_n = 1 \end{aligned}$$

Mid-term exam

Time: **Oct 24, 2022 16:30-18:00**

Location: **LPN LT (in class)**

- **Closed-book, No calculator, No cheat sheet**.
- Formal answer books and scratch papers will be provided.
- **No discussion or communication** with others during the exam.
- Coverage: **Lecture 1-7**

Good luck!