Name:_______________ SID:________________

MAEG4070 Engineering Optimization

Mid-Term Exam

Date: Oct 21, 2021 1.5 hours

1. Are the following statements correct (√) or wrong (×)? (10 points)
   (1) A linear program always has an optimal solution.
   (2) For a point x^*, if we have $f'(x^*) = 0$, then x^* is an optimal point.
   (3) A stationary point is not always an optimal point.
   (4) Function $f(x) = |x + 2|$ is a linear function.
   (5) Newton method can always reach the optimal point.

Answer:

(1) __×__ (2) __×__ (3) __√__ (4) __×__ (5)__×__

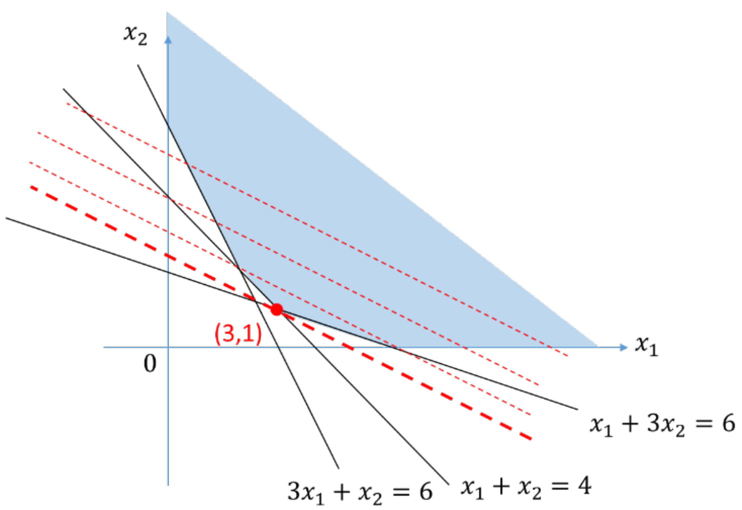
2. Consider the following linear program:

$$\begin{aligned} \min_{x_1,x_2}\ & x_1 + 2x_2 \\ \text{s.t. } & x_1 + 3x_2 \geq 6 \\ & x_1 + x_2 \geq 4 \\ & 3x_1 + x_2 \geq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

   (1) Determine the optimal solution using Graphical method. (20 points)
   (2) Write down its dual problem. (10 points)

Answer:

(1)



The optimal solution is $x^* = (3,1)$.

(You may continue writing your solution for Problem 2 on this page)

(2) the dual problem is

$$\begin{aligned} \max_{\lambda_1,\lambda_2,\lambda_3} \quad & 6\lambda_1 + 4\lambda_2 + 6\lambda_3 \\ \text{s.t.} \quad & \lambda_1 + \lambda_2 + 3\lambda_3 \le 1 \\ & 3\lambda_1 + \lambda_2 + \lambda_3 \le 2 \\ & \lambda_1, \lambda_2, \lambda_3 \ge 0 \end{aligned}$$

3. Prove that set $S = \{(x_1, x_2) | x_1 + 2x_2 \ge 1, 2x_1 - x_2 \ge 3\}$ is convex. (15 points)

Answer:

Given any two points $x = [x_1, x_2] \in S, y = [y_1, y_2] \in S$. For any $0 \le \lambda \le 1$, we have

$$\lambda x + (1-\lambda)y = [\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2]$$

Then we check whether $\lambda x + (1-\lambda)y$ is in S.

$$\begin{aligned} \lambda x_1 + (1-\lambda)y_1 + 2[\lambda x_2 + (1-\lambda)y_2] &= \lambda(x_1 + 2x_2) + (1-\lambda)(y_1 + 2y_2) \\ &\ge \lambda + (1-\lambda) = 1 \end{aligned}$$

$$\begin{aligned} 2[\lambda x_1 + (1-\lambda)y_1] - [\lambda x_2 + (1-\lambda)y_2] &= \lambda(2x_1 - x_2) + (1-\lambda)(2y_1 - y_2) \\ &\ge 3\lambda + 3(1-\lambda) = 3 \end{aligned}$$

Therefore, $\lambda x + (1-\lambda)y \in S$, so S is a convex set.

4. (1) Prove that function $f(x_1, x_2) = 12x_1^2 - 2x_1x_2 + 3x_2^2$ is convex. (10 points)

(2) Determine the optimal point of $f(x_1, x_2) = 12x_1^2 - 2x_1x_2 + 3x_2^2$. Is it a *global* optimum or a *local* optimum? Is it a *strict* optimum or a *relative* optimum? (15 points)

Answer:

(1) The gradient of $f(x_1, x_2) = 12x_1^2 - 2x_1x_2 + 3x_2^2$ is

$$\frac{\partial f}{\partial x_1} = 24x_1 - 2x_2, \qquad \frac{\partial f}{\partial x_2} = -2x_1 + 6x_2$$

The Hessian is

$$H(x) = \begin{pmatrix} 24 & -2 \\ -2 & 6 \end{pmatrix}$$

$H(x)$ is positive definite. Therefore, $f(x_1, x_2)$ is a (strictly) convex function.

(2) Let $\nabla f = 0$, we have $x^* = (0,0)$. Since the Hessian matrix is positive definite. According to the sufficient and necessary condition, x^* is a *global strict* minimum.

5. Given the primal and dual problems are follows:

Primal	Dual
$\min_{x_1,x_2,x_3,x_4} \; 4x_1+5x_2+3x_3+6x_4$ s.t. $x_1+2x_2+x_4 \geq 4$ $3x_1+x_2+x_3+x_4 \geq 9$ $x_3+x_4 \geq 2$ $x_1+x_3 \geq 1$ $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$	$\max_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \; 4\lambda_1+9\lambda_2+2\lambda_3+\lambda_4$ s.t. $\lambda_1+3\lambda_2+\lambda_4 \leq 4$ $2\lambda_1+\lambda_2 \leq 5$ $\lambda_2+\lambda_3+\lambda_4 \leq 3$ $\lambda_1+\lambda_2+\lambda_3 \leq 6$ $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0$

(1) One of the complementary and slackness conditions is $0 \leq (x_3+x_4-2) \perp \lambda_3 \geq 0$. Try to linearize it using the Big-M method. (5 points)

(2) If we know the optimal solution of the primal problem is $x^* = (2,1,2,0)$, try to determine the optimal solution of dual problem based on complementary and slackness condition. (15 points)

Answer:

(1) The condition $0 \leq (x_3+x_4-2) \perp \lambda_3 \geq 0$ is equivalent to

$$0 \leq x_3+x_4-2 \leq Mz$$
$$0 \leq \lambda_3 \leq M(1-z)$$
$$z \in \{0,1\}$$

(2) Since $x_1^* > 0, x_2^* > 0, x_3^* > 0$, we have

$$\lambda_1^*+3\lambda_2^*+\lambda_4^* = 4$$
$$2\lambda_1^*+\lambda_2^* = 5$$
$$\lambda_2^*+\lambda_3^*+\lambda_4^* = 3$$

Moreover, as $x_1^*+x_3^* > 1$, we have $\lambda_4^* = 0$. Therefore,

$$\lambda_1^*+3\lambda_2^* = 4$$
$$2\lambda_1^*+\lambda_2^* = 5$$
$$\lambda_2^*+\lambda_3^* = 3$$

This implies $\lambda_1^* = \frac{11}{5}, \lambda_2^* = \frac{3}{5}, \lambda_3^* = \frac{12}{5}, \lambda_4^* = 0$.