Name:______________ SID:_______________

MAEG4070 Engineering Optimization

Mid-Term Exam

Date: Oct 24, 2022 1.5 hours

1. Are the following statements correct (√) or wrong (×)? Briefly explain why. (10 points)
   (1) The optimal solution of a linear program cannot be a point that is not a vertex of its feasible region.
   (2) If $f(x)$ is a convex function, then $-3f(x)$ is a concave function.
   (3) Gradient descent method can always reach the global optimal point.
   (4) Function $f(x) = 2|x| + |-2x|$ is a linear function.
   (5) Halfspace $\{x|a^T x \leq b, a \neq 0\}$ is an affine set.

Answer:

(1) __×__ (2) __√__ (3) __×__ (4) __×__ (5)__×__

Explanation:

(1) For the case with multiple optimal solutions, a point that is not a vertex can also be an optimal point.
(2) $f(x)$ is a convex function, so $\nabla^2 f(x) \geq 0$. Hence, $\nabla^2(-3f(x)) = -3\nabla^2 f(x) \leq 0$ and $-3f(x)$ is a concave function.
(3) It may reach a saddle point or a local optimal point.
(4) Let $x = 1$ and $y = -1$, then $f(x+y) = 0 \neq f(x) + f(y)$.
(5) Page 13, Lecture 4.

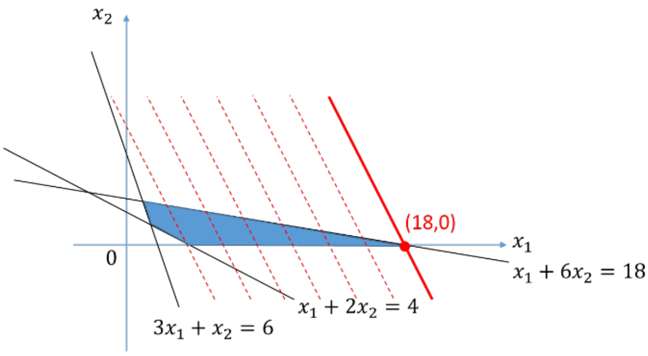
2. Consider the following linear program:

$$\max_{x_1,x_2} \ 2x_1 + x_2$$
$$\text{s.t. } x_1 + 6x_2 \leq 18$$
$$x_1 + 2x_2 \geq 4$$
$$3x_1 + x_2 \geq 6$$
$$x_1 \geq 0, x_2 \geq 0$$

   (1) Determine the optimal solution using graphical method. (20 points)
   (2) Suppose λ_1 is the dual variable corresponding to the first constraint, then one of the complementary slackness conditions is $0 \leq (-x_1 - 6x_2 + 18) \perp \lambda_1 \geq 0$. Please linearize it using the Big-M method. (5 points)

Answer:

(1)



The optimal solution is $x^* = (18,0)$.

(2) The complementary slackness condition $0 \leq (-x_1 - 6x_2 + 18) \perp \lambda_1 \geq 0$ can be linearized as

$$0 \leq (-x_1 - 6x_2 + 18) \leq Mz$$
$$0 \leq \lambda_1 \leq M(1-z)$$
$$z \in \{0,1\}$$

where M is a large enough constant.

3. Suppose $S_1 = \{(x_1, x_2) | 3x_1 + x_2 \geq 2\}$ and $S_2 = \{(x_1, x_2) | x_1 + 6x_2 \leq 12\}$, please prove that:

(1) S_1 and S_2 are all convex sets. (10 points)

(2) $S_1 \cap S_2$ is a convex set. (5 points)

Answer:

(1) Given any two points $x = [x_1, x_2] \in S_1, y = [y_1, y_2] \in S_1$. For any $0 \leq \lambda \leq 1$, we have

$$\lambda x + (1-\lambda)y = [\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2]$$

Since

$$3[\lambda x_1 + (1-\lambda)y_1] + [\lambda x_2 + (1-\lambda)y_2] = \lambda(3x_1 + x_2) + (1-\lambda)(3y_1 + y_2)$$
$$\geq 2\lambda + 2(1-\lambda) = 2$$

We have $\lambda x + (1-\lambda)y$ is in S_1, so S_1 is a convex set.

Given any two points $x = [x_1, x_2] \in S_2, y = [y_1, y_2] \in S_2$. For any $0 \leq \lambda \leq 1$, we have

$$\lambda x + (1-\lambda)y = [\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2]$$

Since

$$[\lambda x_1 + (1-\lambda)y_1] + 6[\lambda x_2 + (1-\lambda)y_2] = \lambda(x_1 + 6x_2) + (1-\lambda)(y_1 + 6y_2)$$
$$\leq 12\lambda + 12(1-\lambda) = 12$$

We have $\lambda x + (1-\lambda)y$ is in S_2, so S_2 is a convex set.

(3) Let $S \coloneqq S_1 \cap S_2$. Given any two points $x \in S, y \in S$. We have $x \in S_1, y \in S_1$ and $x \in S_2, y \in S_2$. For any $0 \leq \lambda \leq 1$, we have

$$\lambda x + (1-\lambda)y \in S_1 \text{ and } \lambda x + (1-\lambda)y \in S_2,$$

which is due to the convexity of S_1 and S_2 proved in (1).

Hence, $\lambda x + (1-\lambda)y \in S$, showing that S is a convex set.

4. (1) Determine the optimal solution of $\min f(x_1, x_2) = 9x_1^2 - 2x_1x_2 + 8x_2^2$. Is it a global or local optimum? Is it a strict or relative optimum? (20 points)

(2) Given an initial point $x^{(0)} = (1,1)^T$ and step size $\alpha = 0.1$, apply the gradient descent method to solve $\min_{x_1,x_2} f(x_1, x_2)$ for one iteration. (5 points)

Answer:

(1) The gradient of $f(x_1, x_2) = 9x_1^2 - 2x_1x_2 + 8x_2^2$ is

$$\frac{\partial f}{\partial x_1} = 18x_1 - 2x_2, \qquad \frac{\partial f}{\partial x_2} = -2x_1 + 16x_2$$

The Hessian is

$$H(x) = \begin{pmatrix} 18 & -2 \\ -2 & 16 \end{pmatrix}$$

$H(x)$ is positive definite. Therefore, $f(x_1, x_2)$ is a convex function.

Let $\nabla f = 0$, we have $x^* = (0,0)$. Since the Hessian matrix is positive definite, according to the sufficient and necessary condition, x^* is a *global strict* minimum.

(2)

$$\nabla f(x^{(0)}) = \begin{pmatrix} 16 \\ 14 \end{pmatrix}$$

$$x^{(1)} = x^{(0)} - \alpha \nabla f(x^{(0)}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 0.1 \times \begin{pmatrix} 16 \\ 14 \end{pmatrix} = \begin{pmatrix} -0.6 \\ -0.4 \end{pmatrix}$$

5. Suppose the primal problem is:

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 3x_1 + 4x_2 - x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 \leq 10 \\ & 2x_1 + 2x_2 - x_3 \leq 16 \\ & x_1 - 2x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \leq 0 \end{aligned}$$

(1) Please write down the dual problem. (15 points)

(2) If we know the optimal solution of the primal problem is $x^* = (6, 2, 0)$, try to determine the optimal solution of dual problem based on the complementary slackness condition. (10 points)

Answer:

(1) The dual problem is

$$\begin{aligned} \min_{\lambda_1,\lambda_2,\lambda_3} \quad & 10\lambda_1 + 16\lambda_2 + \lambda_3 \\ \text{s.t.} \quad & \lambda_1 + 2\lambda_2 + \lambda_3 \geq 3 \\ & 2\lambda_1 + 2\lambda_2 - 2\lambda_3 \geq 4 \\ & -\lambda_1 - \lambda_2 \leq -1 \\ & \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \leq 0 \end{aligned}$$

(2) Since $x_1^* > 0, x_2^* > 0$, we have

$$\lambda_1^* + 2\lambda_2^* + \lambda_3^* = 3$$
$$2\lambda_1^* + 2\lambda_2^* - 2\lambda_3^* = 4$$

Moreover, as $x_1^* - 2x_2^* > 1$, we have $\lambda_3^* = 0$. Therefore,

$$\lambda_1^* + 2\lambda_2^* = 3$$
$$2\lambda_1^* + 2\lambda_2^* = 4$$

This implies $\lambda_1^* = 1, \lambda_2^* = 1, \lambda_3^* = 0$.