MAEG4070 Engineering Optimization

Summary of Lecture 1-4

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What have we learned?

Lecture 1:

- The basic concept of optimization
- How to model an engineering problem

Lecture 2:

- Definition & Standard form of linear programming
- Graphical method for solving an LP

Lecture 3:

- The definition and interpretations of dual problems
- How to construct the dual problem of an LP

Lecture 4:

- Affine sets, convex sets, cones, polyhedron, etc
- Convex function & methods to prove a function is convex

Engineering problem → Optimization model

A company produces two products: A and B. The raw material consumptions, space needed for storage, production rates, and selling prices for these products are

	Product		Endowment
	A	B	
Storage space (ft^2/unit)	4	5	1500
Raw material (lb/unit)	5	3	1575
Production rate (units/hr)	60	30	7 hours to produce
Selling price ($/unit)	13	11	

Objective: maximize the selling income

Engineering problem → Optimization model

Variables: Denote x_1, x_2 be the production of products A, B, respectively.

Objective: How to maximize the selling income?

Constraints: do not violate the resource endowment

$$\begin{aligned} \max_{x_1, x_2} \ & 13x_1 + 11x_2 \\ \text{s.t.} \ & 4x_1 + 5x_2 \leq 1500 \\ & 5x_1 + 3x_2 \leq 1575 \\ & x_1 + 2x_2 \leq 420 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

$$\frac{x_1}{60} + \frac{x_2}{30} \leq 7$$

Non-standard form → Standard form (compact form)

$$\max_{x_1,x_2} \; 13x_1 + 11x_2$$
$$\text{s.t. } 4x_1 + 5x_2 \leq 1500$$
$$5x_1 + 3x_2 \leq 1575$$
$$x_1 + 2x_2 \leq 420$$
$$x_1 \geq 0, x_2 \geq 0$$

$$\min_{x_1,x_2} \; -13x_1 - 11x_2$$
$$\text{s.t. } 4x_1 + 5x_2 \leq 1500$$
$$5x_1 + 3x_2 \leq 1575$$
$$x_1 + 2x_2 \leq 420$$
$$x_1 \geq 0, x_2 \geq 0$$

$$\min_{x_1,x_2,y_1,y_2,y_3} \; -13x_1 - 11x_2$$
$$\text{s.t. } 4x_1 + 5x_2 + y_1 = 1500$$
$$5x_1 + 3x_2 + y_2 = 1575$$
$$x_1 + 2x_2 + y_3 = 420$$
$$x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

$$x = [x_1, x_2, y_1, y_2, y_3]^T$$
$$c = [-13, -11, 0, 0, 0]$$
$$b = [1500, 1575, 420]^T$$
$$A = \begin{bmatrix} 4 & 5 & 1 & 0 & 0 \\ 5 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

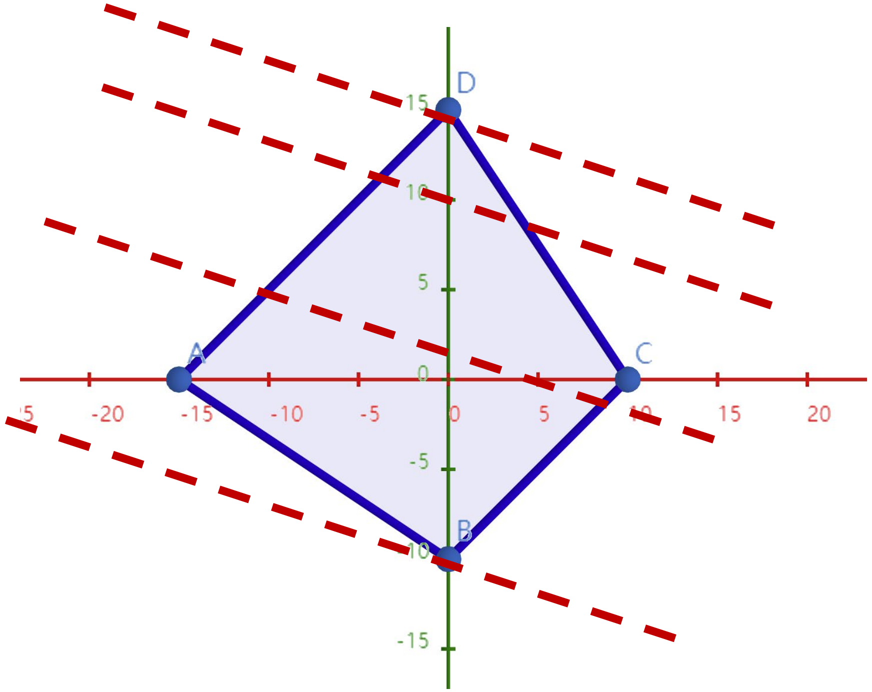
Graphical method for solving LP

Procedure:

Step 1: Draw the feasible region of the LP problem

Step 2: Draw the contours of the objective function

Step 3: Move the contour until it reaches the optimal point



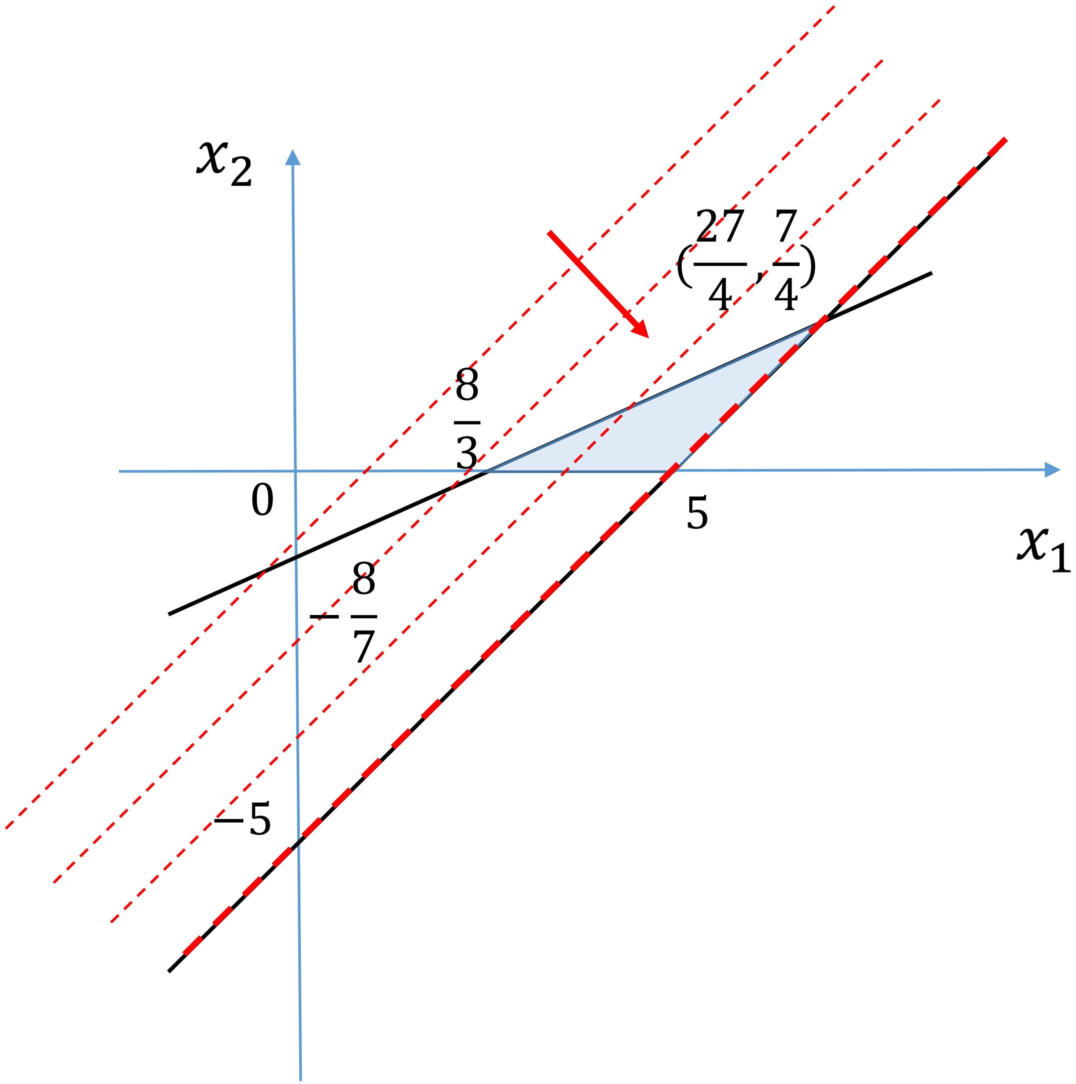
Example - 1

$$\min_{x_1,x_2} \quad -x_1 + x_2$$

$$\text{s.t. } 3x_1 - 7x_2 \geq 8$$

$$x_1 - x_2 \leq 5$$

$$x_1 \geq 0, x_2 \geq 0$$



Vertices are

$\left(\frac{8}{3}, 0\right), (5,0), \left(\frac{27}{4}, \frac{7}{4}\right)$

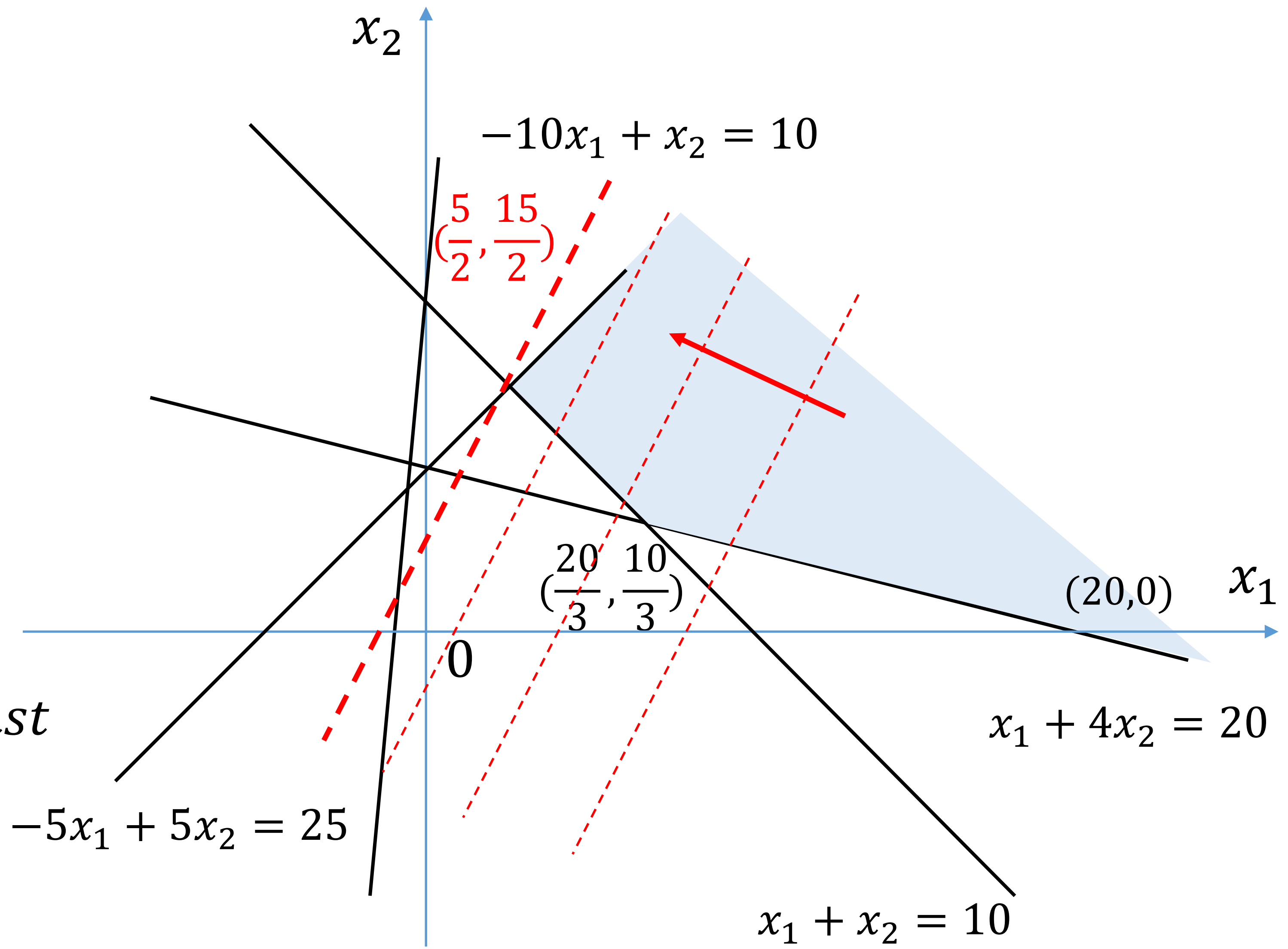
The contour is $-x_1 + x_2 = const$

There are multiple optimal solutions

For example, $x^* = \left(\frac{27}{4}, \frac{7}{4}\right), f^* = -5$

Example - 2

$$\max_{x_1,x_2} \; -20x_1 + 10x_2$$
$$\text{s.t. } x_1 + x_2 \geq 10$$
$$-10x_1 + x_2 \leq 10$$
$$-5x_1 + 5x_2 \leq 25$$
$$x_1 + 4x_2 \geq 20$$
$$x_1 \geq 0, x_2 \geq 0$$



Vertices are

$\left(\frac{20}{3}, \frac{10}{3}\right)$, (20,0), $\left(\frac{5}{2}, \frac{15}{2}\right)$

The contour is $-20x_1 + 10x_2 = const$

$x^* = \left(\frac{5}{2}, \frac{15}{2}\right)$, f^*=25

Principles for LP duality

$$\min_x \; c^T x \quad \text{s.t.} \; Ax \geq b, \; x \geq 0$$

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

$$\max_y \; b^T y \quad \text{s.t.} \; A^T y \leq c, \; y \geq 0$$

General Form

Primal problem

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. }\ & A_1 x \geq b_1 \ :\ \lambda_1 \\ & A_2 x \leq b_2 \ :\ \lambda_2 \\ & A_3 x = b_3 \ :\ \lambda_3 \\ & x \geq 0 \ :\ \lambda_4 \end{aligned}$$

Dual problem

$$\begin{aligned} \max\ & b_1^T \lambda_1 + b_2^T \lambda_2 + b_3^T \lambda_3 \\ \text{s.t. }\ & A_1^T \lambda_1 + A_2^T \lambda_2 + A_3^T \lambda_3 \leq c \\ & \lambda_1 \geq 0, \lambda_2 \leq 0, \lambda_3 \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. }\ & A_1 x \geq b_1 \ :\ \lambda_1 \\ & -A_2 x \geq -b_2 \ :\ \lambda_2 \\ & A_3 x \geq b_3 \ :\ \lambda_3 \\ & -A_3 x \geq -b_3 \ :\ \lambda_3' \\ & x \geq 0 \ :\ \lambda_4 \end{aligned}$$

$$\begin{aligned} \max\ & b_1^T \lambda_1 - b_2^T \lambda_2 + b_3^T (\lambda_3 - \lambda_3') + 0^T \lambda_4 \\ \text{s.t. }\ & A_1^T \lambda_1 - A_2^T \lambda_2 + A_3^T (\lambda_3 - \lambda_3') + I\lambda_4 = c \\ & \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_3' \geq 0, \lambda_4 \geq 0 \end{aligned}$$

Example - 1

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

$$\min_{x_1,x_2,x_3} \quad 5x_1 + 4x_2 + 3x_3$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 = 4$$
$$3x_1 + 2x_2 + x_3 = 5$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$\max_{\lambda_1,\lambda_2} \quad 4\lambda_1 + 5\lambda_2$$
$$\text{s.t.} \quad \lambda_1 + 3\lambda_2 \leq 5$$
$$\lambda_1 + 2\lambda_2 \leq 4$$
$$\lambda_1 + \lambda_2 \leq 3$$

Example - 2

Primal LP		Dual LP	
Objective: min **Objective coefficient: c^T** **Constraint coefficient: (A,b)**		**Objective: max** **Objective coefficient: b^T** **Constraint coefficient: (A^T,c)**	
Vars:	***n*-th variable** **≥0** **≤0** **free**	**Cons:**	***n*-th constraint** **≤** **≥** **=**
Cons:	***m*-th constraint** **≤** **≥** **=**	**Vars:**	***m*-th variable** **≤0** **≥0** **free**

$$
\begin{aligned}
\max_{x_1,x_2,x_3} \quad & -x_1 + x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 + 2x_3 \leq 25 \\
& -x_1 + 2x_2 - x_3 \geq 2 \\
& x_1 - x_2 + x_3 = 3 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{\lambda_1,\lambda_2,\lambda_3} \quad & 25\lambda_1 + 2\lambda_2 + 3\lambda_3 \\
\text{s.t.} \quad & \lambda_1 - \lambda_2 + \lambda_3 \geq -1 \\
& \lambda_1 + 2\lambda_2 - \lambda_3 \geq 1 \\
& 2\lambda_1 - \lambda_2 + \lambda_3 = 1 \\
& \lambda_1 \geq 0,\ \lambda_2 \leq 0
\end{aligned}
$$

Duality Theorems

Primal problem

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \max_{\lambda} \quad & b^T \lambda \\ & A^T \lambda \leq c \\ & \lambda \geq 0 \end{aligned}$$

Complementarity and slackness: Suppose x^*, λ^* are the primal and dual optimal solutions, respectively. Then, we have

$$a_n^T x^* > b \quad \Rightarrow \quad \lambda_n^* = 0$$
$$\lambda_n^* > 0 \quad \Rightarrow \quad a_n^T x^* = b$$

An Example

Primal problem

$$\begin{aligned} \min_{x_1,x_2,x_3,x_4} \quad & 8x_1 + 6x_2 + 3x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_4 \geq 3 \\ & 3x_1 + x_2 + x_3 + x_4 \geq 6 \\ & x_3 + x_4 \geq 2 \\ & x_1 + x_3 \geq 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \max_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \quad & 3\lambda_1 + 6\lambda_2 + 2\lambda_3 + 2\lambda_4 \\ \text{s.t.} \quad & \lambda_1 + 3\lambda_2 + \lambda_4 \leq 8 \\ & 2\lambda_1 + \lambda_2 \leq 6 \\ & \lambda_2 + \lambda_3 + \lambda_4 \leq 3 \\ & \lambda_1 + \lambda_2 + \lambda_3 \leq 6 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

If we know the optimal solution of the primal problem is $x^* = (1, 1, 2, 0)$, Try to determine the optimal solution of dual problem based on Complementarity and slackness.

An Example

Primal problem

$$\min_{x_1,x_2,x_3,x_4} \quad 8x_1 + 6x_2 + 3x_3 + 6x_4$$
$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 + x_4 \geq 3 && \boldsymbol{\lambda_1} \\ & 3x_1 + x_2 + x_3 + x_4 \geq 6 && \boldsymbol{\lambda_2} \\ & x_3 + x_4 \geq 2 && \boldsymbol{\lambda_3} \\ & x_1 + x_3 \geq 2 && \boldsymbol{\lambda_4} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Dual problem

$$\max_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \quad 3\lambda_1 + 6\lambda_2 + 2\lambda_3 + 2\lambda_4$$
$$\begin{aligned} \text{s.t.} \quad & \lambda_1 + 3\lambda_2 + \lambda_4 \leq 8 && \boldsymbol{x_1} \\ & 2\lambda_1 + \lambda_2 \leq 6 && \boldsymbol{x_2} \\ & \lambda_2 + \lambda_3 + \lambda_4 \leq 3 && \boldsymbol{x_3} \\ & \lambda_1 + \lambda_2 + \lambda_3 \leq 6 && \boldsymbol{x_4} \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

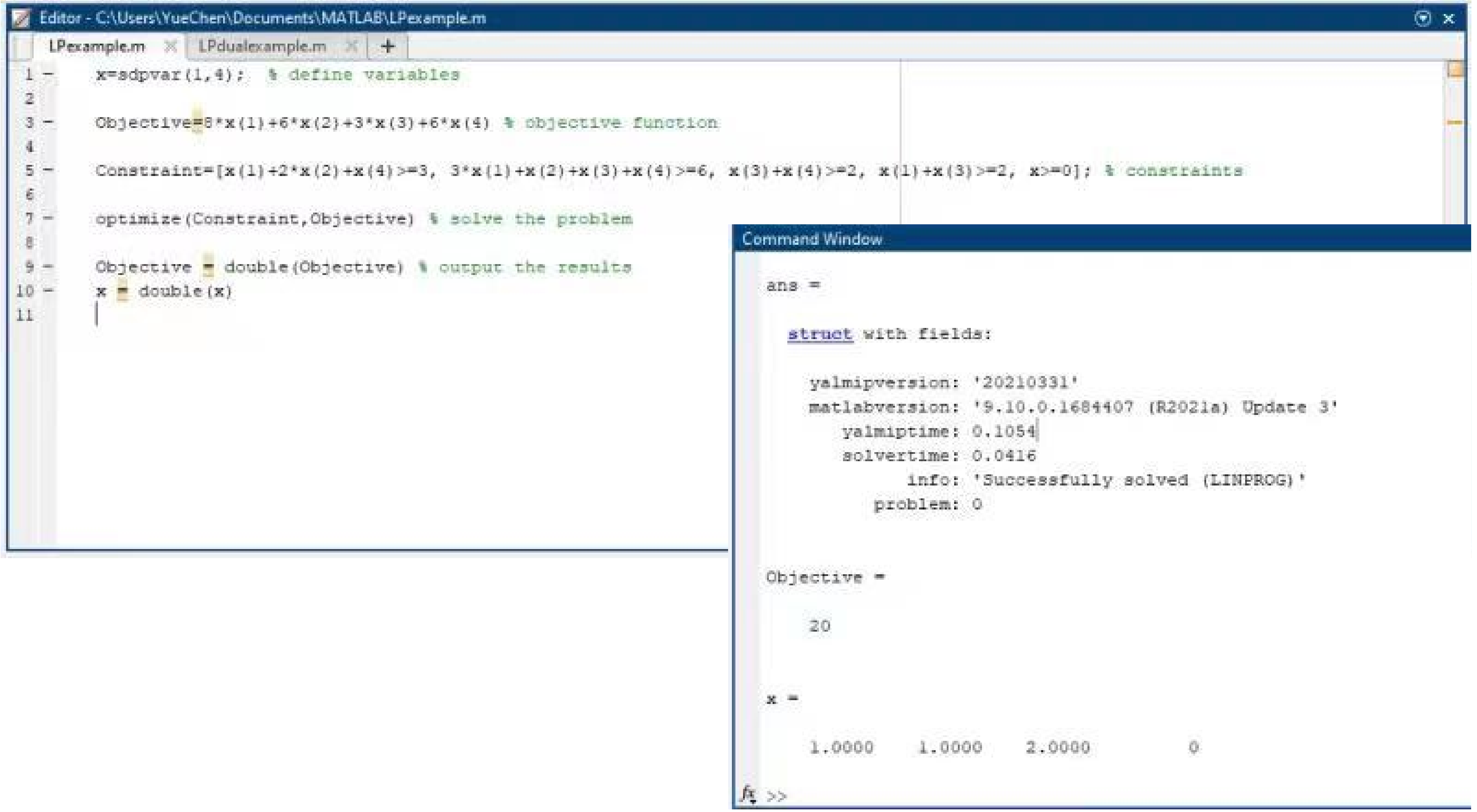
Since $x_1^* > 0, x_2^* > 0, x_3^* > 0$, we have

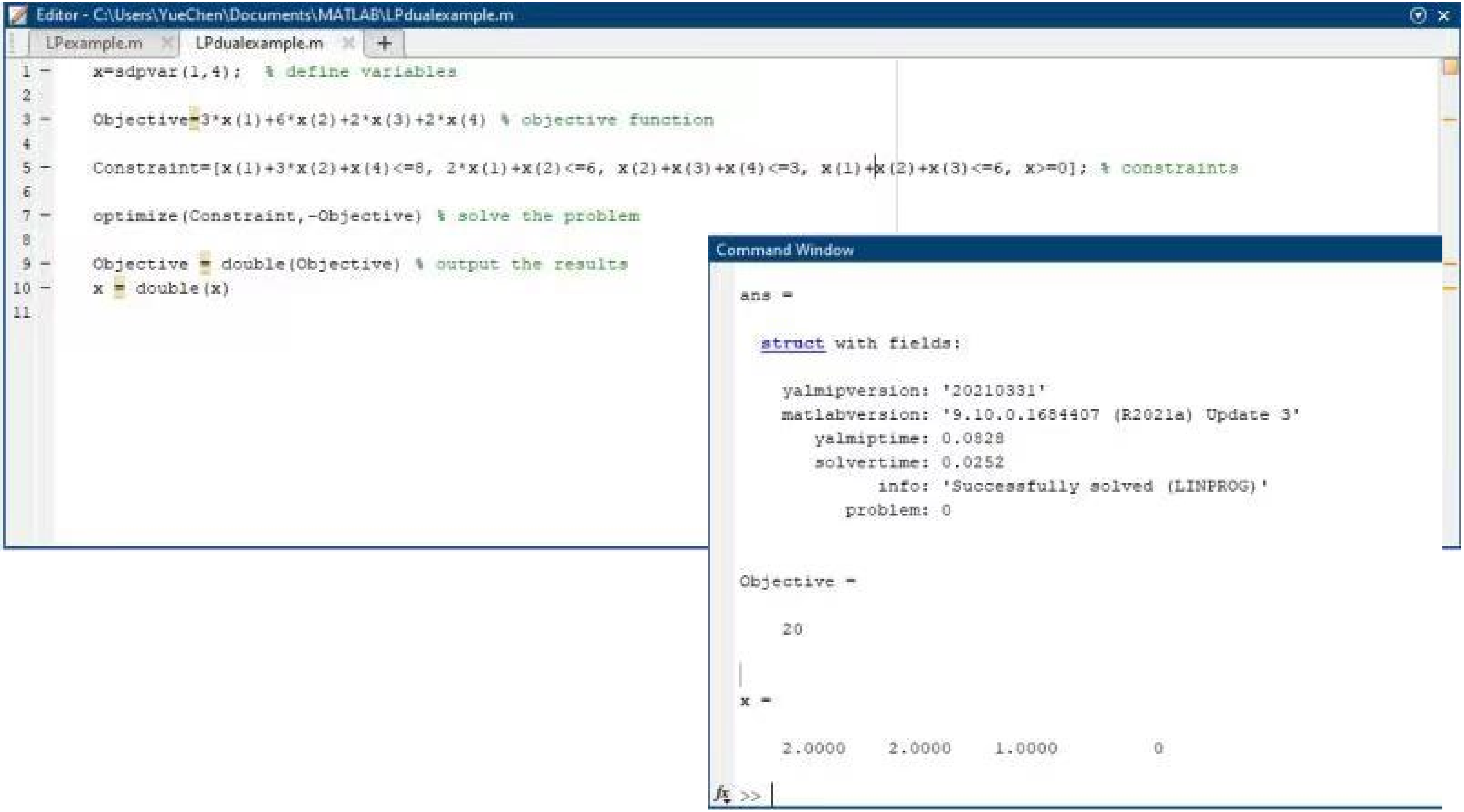
$$\begin{aligned} \lambda_1^* + 3\lambda_2^* + \lambda_4^* &= 8 \\ 2\lambda_1^* + \lambda_2^* &= 6 \\ \lambda_2^* + \lambda_3^* + \lambda_4^* &= 3 \end{aligned}$$

Moreover, as $x_1^* + x_3^* > 2$, we have $\lambda_4^* = 0$.
Therefore,

$$\begin{aligned} \lambda_1^* + 3\lambda_2^* &= 8 \\ 2\lambda_1^* + \lambda_2^* &= 6 \\ \lambda_2^* + \lambda_3^* &= 3 \end{aligned}$$

It indicates $\lambda_1^* = 2, \lambda_2^* = 2, \lambda_3^* = 1$. $\lambda^* = (2, 2, 1, 0)$.

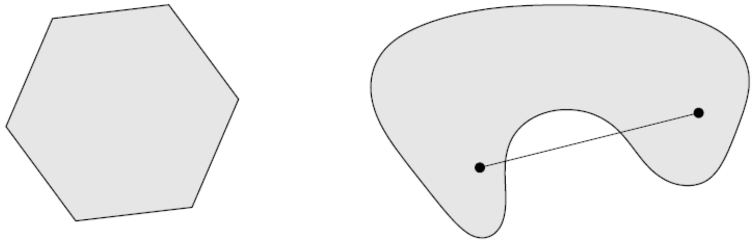

Editor - C:\Users\YueChen\Documents\MATLAB\LPexample.m
LPexample.m
LPdualexample.m
x=sdpvar(1,4); % define variables
Objective=8*x(1)+6*x(2)+3*x(3)+6*x(4) % objective function
Constraint=[x(1)+2*x(2)+x(4)>=3, 3*x(1)+x(2)+x(3)+x(4)>=6, x(3)+x(4)>=2, x(1)+x(3)>=2, x>=0]; % constraints
optimize(Constraint,Objective) % solve the problem
Objective = double(Objective) % output the results
x = double(x)
Command Window
ans =
struct with fields:
yalmipversion: '20210331'
matlabversion: '9.10.0.1684407 (R2021a) Update 3'
yalmiptime: 0.1054
solvertime: 0.0416
info: 'Successfully solved (LINPROG)'
problem: 0
Objective =
20
x =
1.0000 1.0000 2.0000 0
>>


Editor - C:\Users\YueChen\Documents\MATLAB\LPdualexample.m
LPexample.m
LPdualexample.m
x=sdpvar(1,4); % define variables
Objective=3*x(1)+6*x(2)+2*x(3)+2*x(4) % objective function
Constraint=[x(1)+3*x(2)+x(4)<=8, 2*x(1)+x(2)<=6, x(2)+x(3)+x(4)<=3, x(1)+x(2)+x(3)<=6, x>=0]; % constraints
optimize(Constraint,-Objective) % solve the problem
Objective = double(Objective) % output the results
x = double(x)
Command Window
ans =
struct with fields:
yalmipversion: '20210331'
matlabversion: '9.10.0.1684407 (R2021a) Update 3'
yalmiptime: 0.0828
solvertime: 0.0252
info: 'Successfully solved (LINPROG)'
problem: 0
Objective =
20
x =
2.0000 2.0000 1.0000 0
>>

Convex Sets

Convex set: the set that contains all line segment between any two distinct points in the set $\mathcal{C}$

$$\forall x_1, x_2 \in \mathcal{C}, \theta \in [0,1] \quad \Rightarrow \quad \theta x_1 + (1-\theta)x_2 \in \mathcal{C}$$



Procedure:

1. Given any two points $x_1, x_2 \in \mathcal{C}$ and $\theta \in [0,1]$
2. Try to prove that $\theta x_1 + (1-\theta)x_2$ satisfies the constraints of $\mathcal{C}$ with the knowledge that both x_1 and x_2 satisfy the constraints of $\mathcal{C}$

Example - 1

Suppose S_1 and S_2 are two convex sets, $\beta \in \mathbb{R}$.
Please prove the following sets are convex:

1. $\beta S_1 = \{\beta x \mid x \in S_1\}$
2. $S_1 \cap S_2$
3. $S_1 + S_2 = \{x^{(1)} + x^{(2)} \middle| x^{(1)} \in S_1, x^{(2)} \in S_2\}$

1. Denote $\mathcal{C} = \beta S_1$.
For any two points $y_1, y_2 \in \mathcal{C}$, there exists $x_1, x_2 \in S_1$ such that $y_1 = \beta x_1, y_2 = \beta x_2$.
Then for any $\theta \in [0, 1]$, since

$$\begin{aligned} & \theta y_1 + (1-\theta) y_2 \\ = \; & \theta \beta x_1 + (1-\theta)\beta x_2 \\ = \; & \beta \left[\theta x_1 + (1-\theta) x_2\right] \end{aligned}$$

As S_1 is a convex set and $x_1, x_2 \in S_1$, we have $\hat{x} := \theta x_1 + (1-\theta)x_2 \in S_1$.
Therefore, $\theta y_1 + (1-\theta) y_2 = \beta \hat{x}, \hat{x} \in S_1$.
$\theta y_1 + (1-\theta) y_2 \in \mathcal{C}$.

Example - 1

2. Denote $\mathcal{C} = S_1 \cap S_2$.
For any two points $y_1, y_2 \in \mathcal{C}$, then $y_1, y_2 \in S_1$ and $y_1, y_2 \in S_2$.
For any $\theta \in [0, 1]$:
Since S_1 and S_2 are convex sets, $\theta y_1 + (1-\theta) y_2 \in S_1$, $\theta y_1 + (1-\theta) y_2 \in S_2$.
Therefore, $\theta y_1 + (1-\theta) y_2 \in S_1 \cap S_2 = \mathcal{C}$.

3. Denote $\mathcal{C} = S_1 + S_2$.
For any two points $y_1, y_2 \in \mathcal{C}$, then y_1 can be represented as the sum of two points $x_1^{(1)} \in S_1, x_1^{(2)} \in S_2$; y_2 can be represented as the sum of two points $x_2^{(1)} \in S_1, x_2^{(2)} \in S_2$.
Given $\theta \in [0, 1]$:

$$\begin{aligned}
& \theta y_1 + (1-\theta) y_2 \\
= \; & \theta(x_1^{(1)} + x_1^{(2)}) + (1-\theta)(x_2^{(1)} + x_2^{(2)}) \\
= \; & \underbrace{\theta x_1^{(1)} + (1-\theta) x_2^{(1)}}_{\in S_1} + \underbrace{\theta x_1^{(2)} + (1-\theta) x_2^{(2)}}_{\in S_2}
\end{aligned}$$

Therefore, $\theta y_1 + (1-\theta) y_2 \in \mathcal{C}$.

Example - 2

Prove that $S = \{(x_1, x_2) | x_2 \geq |x_1|\}$ is a convex set.

For any $y_1 := (x_1^{(1)}, x_2^{(1)}) \in S$, $y_2 := (x_1^{(2)}, x_2^{(2)}) \in S$, and $\theta \in [0, 1]$:

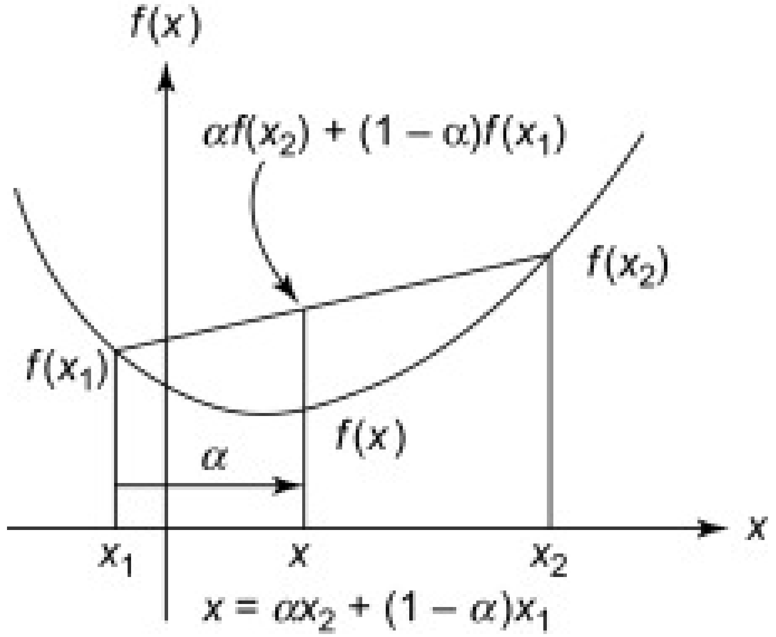
$$\begin{aligned} & |\theta x_1^{(1)} + (1-\theta) x_1^{(2)}| \\ \leq\ & |\theta x_1^{(1)}| + |(1-\theta) x_1^{(2)}| \\ \leq\ & \theta |x_1^{(1)}| + (1-\theta) |x_1^{(2)}| \\ \leq\ & \theta x_2^{(1)} + (1-\theta) x_2^{(2)} \end{aligned}$$

Therefore, $\theta y_1 + (1-\theta) y_2 \in S$.

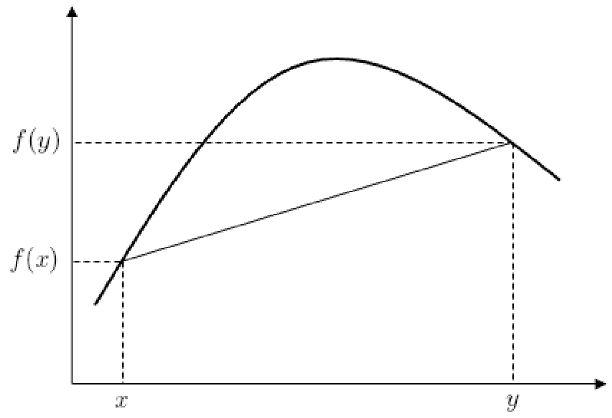
Convex function

Function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** if dom(f) is a convex set, and the following inequality holds

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2), \forall \theta \in [0,1], \forall x_1, x_2 \in dom(f)$$



If we change $\leq$ into $\geq$, then it is **concave**



Convex function

Function $f : \mathbb{R}^n \to \mathbb{R}$ is **strictly convex** if dom(f) is a convex set, and the following inequality holds

$$f(\theta x_1 + (1-\theta)x_2) < \theta f(x_1) + (1-\theta)f(x_2), \forall \theta \in (0,1), \forall x_1 \neq x_2 \in dom(f)$$

Function f is strongly convex if $\exists \alpha \geq 0$: $f(x) - \alpha \|x\|_2^2$ is convex

f is (strictly, strongly) concave if $-f$ is (strictly, strongly) convex

stronger

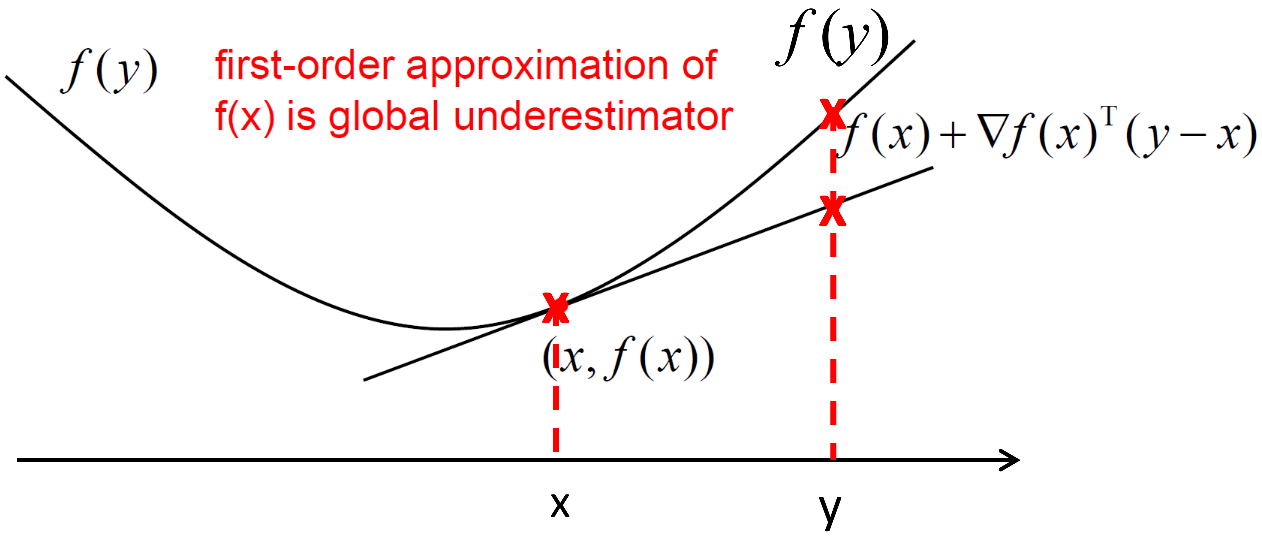
Convex function

Apart from proving the convexity by definition, in the following, we provide two conditions, i.e. first-order condition & second-order condition

Suppose f is differentiable and $\nabla f(x)$ exists at each $x \in dom(f)$

First-order condition f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y \in dom(y)$$



Pictures from Google image

Convex function

Suppose f is twice differentiable and the Hessian $H(x)$ exists at every $x \in dom(f)$.

Second-order condition function f with convex domain is

- convex iff

$$H(x) \succeq 0, \forall x \in dom(f)$$

- Strictly convex iff

$$H(x) \succ 0, \forall x \in dom(f)$$

- Strongly convex iff

$$H(x) - \alpha I \succeq 0, \forall x \in dom(f)$$

Example - 1

Is $f(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2 + x_1 + x_2$ a convex function?

Solution: The gradient and Hessian matrix of f are

$$\frac{\partial f}{\partial x_1} = 2x_1 - 4x_2 + 1, \frac{\partial f}{\partial x_2} = -4x_1 + 2x_2 + 1$$

$$H(x) = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$$

is indefinite matrix. Therefore, $f(x_1, x_2)$ is not a convex function.

Example - 2

Is $f(x_1, x_2) = (x_1 - x_2)^2 + 4x_1x_2 + e^{x_1+x_2}$ a convex function?

Solution: The gradient and Hessian matrix of f are

$$\frac{\partial f}{\partial x_1} = 2(x_1 - x_2) + 4x_2 + e^{x_1+x_2}$$

$$\frac{\partial f}{\partial x_2} = -2(x_1 - x_2) + 4x_1 + e^{x_1+x_2}$$

$$H(x) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (2 + e^{x_1+x_2})$$

is positive semi-definite matrix. Therefore, $f(x_1, x_2)$ is a convex function.

Thanks！