MAEG4070 Engineering Optimization

Summary of Lecture 8-10

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What have we learned?

Lecture 8:
- Constrained optimization with equality (necessary/sufficiency condition)
- Constrained optimization with inequality (necessary condition)

Lecture 9:
- Check if a problem is a convex optimization?
- Determine the global optimum of a convex optimization

Lecture 10:
- Write the dual problem of an optimization

Constrained Optimization with Equality

$$\min_x \; f(x)$$
$$\text{s.t. } h_i(x) = 0, i = 1, ..., m$$

where $f : \mathbb{R}^n \rightarrow R$, $h_i : \mathbb{R}^n \rightarrow R, \forall i = 1, ..., m.$

- We suppose both f and $h_i, \forall i$ are continuously differentiable functions
- Note that the theory also applies to case where f and $h_i, \forall i$ are continuously differentiable in a neighborhood of a local minimum.

Constrained Optimization with Equality

Necessary condition (local optimum → ?)

Define the Lagrangian function

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$$

Then, if x^* is a local minimum which is regular,
the Lagrange multiplier conditions are written

$$\nabla_x L(x^*, \lambda^*) = 0, \quad \nabla_\lambda L(x^*, \lambda^*) = 0$$

There are $n + m$ unknowns variables and $n + m$ equations.

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) = 0 \qquad h_i(x^*) = 0, i = 1, \ldots, m$$

Constrained Optimization with Equality

Sufficiency condition (? → local optimum)

When f and $h_i, \forall i$ are *twice* continuously differentiable.
If $x^* \in \mathbb{R}^n$ and $\lambda^* \in \mathbb{R}^m$ satisfy

$$\nabla_x L(x^*, \lambda^*) = 0, \;\; \nabla_\lambda L(x^*, \lambda^*) = 0$$

$$y^T \nabla^2_{xx} L(x^*, \lambda^*) y > 0, \forall y \neq 0, \nabla h(x^*)^T y = 0$$

Then x^* is a strict local minimum.

$$\left(\nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(x^*) \right)$$

Constrained Optimization with Inequality

General Form:

$$\min_x \ f(x)$$
$$\text{s.t. } h(x) = 0, g(x) \leq 0$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^r$ are continuously differentiable.
Here,

$$h = (h_1, ..., h_m)$$
$$g = (g_1, ..., g_r)$$

Constrained Optimization with Inequality

Necessary condition (local optimum → ?)

Assume that x^* is regular, similarly, we can write down the Lagrangian function

$$L(x, \lambda, \mu) = f(x^*) + \sum_{i=1}^{m} \lambda_i^* h_i(x^*) + \boxed{\sum_{j=1}^{r} \mu_j^* g_j(x^*)}$$

Let x^* be a local minimum and a regular point.
Then there exist unique Lagrange multiplier vectors $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$, $\mu^* = (\mu_1^*, ..., \mu_r^*)$

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$

$$h_i(x^*) = 0, \forall i = 1, ..., m$$

$$\boxed{0 \le -g_j(x^*) \perp \mu_j^* \ge 0, \forall j = 1, ..., r}$$

Complementary Slackness

Example

Find the KKT point of

$$\min_{x_1,x_2,x_3} \quad x_1^2 + x_2^2 + x_3^2$$
$$\text{s.t. } x_1 + x_2 + x_3 \geq 3$$
$$x_1 + x_2 - x_3 \geq 1$$

Solution: First, write the problem in the standard form

$$\min_{x_1,x_2,x_3} \quad x_1^2 + x_2^2 + x_3^2$$
$$\text{s.t. } -x_1 - x_2 - x_3 + 3 \leq 0$$
$$-x_1 - x_2 + x_3 + 1 \leq 0$$

The Lagrangian function is

$$L(x, \mu) = x_1^2 + x_2^2 + x_3^2 + \mu_1(-x_1 - x_2 - x_3 + 3) + \mu_2(-x_1 - x_2 + x_3 + 1)$$

Example

The KKT condition is

$$
\begin{aligned}
2x_1 - \mu_1 - \mu_2 &= 0 \\
2x_2 - \mu_1 - \mu_2 &= 0 \\
2x_3 - \mu_1 + \mu_2 &= 0 \\
\mu_1(-x_1 - x_2 - x_3 + 3) &= 0 \\
\mu_2(-x_1 - x_2 + x_3 + 1) &= 0 \\
\mu_1 &\geq 0 \\
\mu_2 &\geq 0 \\
-x_1 - x_2 - x_3 + 3 &\leq 0 \\
-x_1 - x_2 + x_3 + 1 &\leq 0
\end{aligned}
$$

Example

Start with $\mu_1(-x_1 - x_2 - x_3 + 3) = 0$, $\mu_2(-x_1 - x_2 + x_3 + 1) = 0$.

Case I: $\mu_1 = 0$

Case I-1: $\mu_2 = 0$

We have $x_1 = x_2 = x_3 = 0$, but $-x_1 - x_2 - x_3 + 3 = 3 > 0$, there is no solution.

Case I-2: $\mu_2 \neq 0$

Then we have $-x_1 - x_2 + x_3 + 1 = 0$, together with

$$2x_1 - \mu_2 = 0$$
$$2x_2 - \mu_2 = 0$$
$$2x_3 + \mu_2 = 0$$

We have $x_1 = x_2 = \frac{1}{3}$, $x_3 = -\frac{1}{3}$, but $-x_1 - x_2 - x_3 + 3 = \frac{8}{3} > 0$, there is no solution.

Example

Case II: $\mu_1 \neq 0$
Case II-1: $\mu_2 = 0$
Then we have $-x_1 - x_2 - x_3 + 3 = 0$, together with

$$
\begin{aligned}
2x_1 - \mu_1 &= 0 \\
2x_2 - \mu_1 &= 0 \\
2x_3 - \mu_1 &= 0
\end{aligned}
$$

We have $x_1 = x_2 = x_3 = 1$, $\mu_1 = 2 \geq 0$ and $-x_1 - x_2 + x_3 + 1 = 0 \leq 0$.
Therefore, $(x^*, \mu^*) = (1, 1, 1, 2, 0)$ is a KKT point.

Example

Case II: $\mu_1 \neq 0$

Case II-2: $\mu_2 \neq 0$ Then we have

$$
\begin{aligned}
-x_1 - x_2 - x_3 + 3 &= 0 \\
-x_1 - x_2 + x_3 + 1 &= 0 \\
2x_1 - \mu_1 - \mu_2 &= 0 \\
2x_2 - \mu_1 - \mu_2 &= 0 \\
2x_3 - \mu_1 + \mu_2 &= 0
\end{aligned}
$$

We have $x_1 = x_2 = x_3 = 1$, $\mu_1 = 2$, $\mu_2 = 0$, which is contradict with $\mu_2 \neq 0$. No solution.

Convex Optimization

Case 1

$$\min_x \; f(x)$$
$$\text{s.t. } a_i^T x - b_i = 0, \forall i = 1, ..., m$$
$$g_j(x) \leq 0, j = 1, ..., r$$

is convex optimization if $f(x)$ and $g_j(x), \forall j$ are all convex functions.

Case 2

$$\max_x \; f(x)$$
$$\text{s.t. } a_i^T x - b_i = 0, \forall i = 1, ..., m$$
$$g_j(x) \geq 0, j = 1, ..., r$$

is convex optimization if $f(x)$ and $g_j(x), \forall j$ are all concave functions.

Convex Optimization

Necessity condition

Let x^* be a local minimum and a regular point.
Then there exist unique Lagrange multiplier vectors $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$, $\mu^* = (\mu_1^*, ..., \mu_r^*)$

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$
$$h_i(x^*) = 0, \forall i = 1, ..., m$$
$$0 \leq -g_j(x^*) \perp \mu_j^* \geq 0, \forall j = 1, ..., r$$

Sufficiency condition

If the optimization is a convex optimization, and point x^* is a regular and KKT point, then x^* is a global optimum.

Example

Determine the optimal solution of

$$\begin{aligned} \min_{x_1, x_2} \; & (x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{s.t.} \; & x_1 + x_2 - 2 \leq 0 \\ & x_2 - x_1 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

The Lagrangian function is

$$L(x, \mu, \lambda) = (x_1 - 1)^2 + (x_2 - 1)^2 + \mu_1(x_1 + x_2 - 2) + \lambda(x_2 - x_1 - 1) - \mu_2 x_1 - \mu_3 x_2$$

Example

First, since $f(x)$ is a convex function, $g_i(x), \forall i = 1, 2, 3$ and $h(x)$ are linear functions. Therefore, this problem is a convex optimization.
The KKT condition is

$$\begin{aligned}
2(x_1 - 1) + \mu_1 - \lambda - \mu_2 &= 0 \\
2(x_2 - 1) + \mu_1 + \lambda - \mu_3 &= 0 \\
x_2 - x_1 &= 1 \\
\mu_1(x_1 + x_2 - 2) &= 0 \\
\mu_2 x_1 &= 0 \\
\mu_3 x_2 &= 0 \\
\mu_1, \mu_2, \mu_3 &\geq 0 \\
-x_1 - x_2 + 2 &\geq 0 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}$$

Example

Case I: $\mu_1 = 0$
Case I-1: $\mu_2 = 0$
Case I-1-1: $\mu_3 = 0$

$$2(x_1 - 1) - \lambda = 0$$
$$2(x_2 - 1) + \lambda = 0$$
$$x_2 - x_1 = 1$$

So $(x_1, x_2) = (\frac{1}{2}, \frac{3}{2})$. Easy to check other conditions are satisfied.
Case I-1-2: $\mu_3 \neq 0$
Then we have $x_2 = 0$, $x_1 = -1 < 0$. No solution.
Case I-2: $\mu_2 \neq 0$
Then we have $x_1 = 0$, $x_2 = 1$, $\mu_3 = 0$.
Since $2(x_2 - 1) + \mu_1 + \lambda - \mu_3 = 0 + 0 + \lambda - 0 = 0$, so $\lambda = 0$.
Since $2(x_1 - 1) + \mu_1 - \lambda - \mu_2 = -2 - \mu_2 = 0$, we have $\mu_2 = -2 < 0$. No solution.

Example

Case II: $\mu_1 \neq 0$
So $x_1 + x_2 = 2$. With $x_2 - x_1 = 1$, we have $x = (\frac{1}{2}, \frac{3}{2})$.
Therefore, $\mu_2 = \mu_3 = 0$.

$$
\begin{aligned}
2(x_1 - 1) + \mu_1 - \lambda = -1 + \mu_1 - \lambda = 0 \\
2(x_2 - 1) + \mu_1 + \lambda = 1 + \mu_1 + \lambda = 0
\end{aligned}
$$

$\lambda = -1, \mu_1 = 0$. No solution.
The optimization problem has a global minimum $x^* = (\frac{1}{2}, \frac{3}{2})$ with $f^* = \frac{1}{2}$.

Dual Optimization

The primal problem:

$$\min_{x} \; f(x)$$
$$\text{s.t. } h_i(x) = 0, \forall i = 1, ..., m$$
$$g_j(x) \leq 0, \forall j = 1, ..., r$$

The Lagrangian is

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x), \;\; \mu_j \geq 0, \forall j$$

We define the dual function as

$$Q(\lambda, \mu) = \min_{x} L(x, \lambda, \mu)$$

Then the dual optimization is

$$\max_{\lambda, \mu} \; Q(\lambda, \mu)$$
$$\text{s.t. } \mu \geq 0$$

Example

Write the dual problem of

$$\max_x \; -x^2$$
$$\text{s.t. } 2 \le x \le 4$$

First, rewrite it in the standard form:

$$\min_x \; x^2$$
$$\text{s.t. } x - 4 \le 0$$
$$-x + 2 \le 0$$

The Lagrangian function is

$$L(x, \mu) = x^2 + \mu_1(x - 4) + \mu_2(-x + 2)$$

Example

$$
\begin{aligned}
Q(\mu_1, \mu_2) = &\ \min_x L(x, \mu) \\
= &\ \min_x x^2 + \mu_1(x-4) + \mu_2(-x+2) \\
= &\ \min_x (x + \frac{\mu_1 - \mu_2}{2})^2 - (\frac{\mu_1 - \mu_2}{2})^2 - 4\mu_1 + 2\mu_2 \\
= &\ -(\frac{\mu_1 - \mu_2}{2})^2 - 4\mu_1 + 2\mu_2
\end{aligned}
$$

Therefore, the dual problem is

$$
\begin{aligned}
\max_{\mu_1, \mu_2} &\ -(\frac{\mu_1 - \mu_2}{2})^2 - 4\mu_1 + 2\mu_2 \\
\text{s.t. } &\ \mu_1 \geq 0, \mu_2 \geq 0
\end{aligned}
$$

Thanks！