香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #1

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Consider the following system.

$$\dot{x}(t) = -x(t) + x^2(t), \quad x(0) = x_0$$

(a) Find the equilibrium points for the system.
(b) Verify the solution of the system is given by

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t}, \quad 0 \le t < T$$

for some $T > 0$.
(c) Show that $T = \ln \frac{x_0}{x_0 - 1}$ when $x_0 > 1$.

Solution:

(a)

In this nonlinear dynamical system,

$$f(x) = -x(t) + x^2(t) \tag{1}$$

Letting $f(x) = 0$ gives

$$-x(t) + x^2(t) = 0 \Rightarrow x^* = \{0, 1\} \tag{2}$$

(b)

For

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t} \tag{3}$$

The left side of the equation is equal to

$$\dot{x}(t) = \left[\frac{x_0}{x_0 + (1 - x_0)e^t}\right]' = \frac{-x_0(1 - x_0)e^t}{[x_0 + (1 - x_0)e^t]^2} \tag{4}$$

The right side of the equation is equal to

$$\begin{aligned} -x(t) + x^2(t) &= -\frac{x_0}{x_0 + (1 - x_0)e^t} + \left[\frac{x_0}{x_0 + (1 - x_0)e^t}\right]^2 = \frac{-x_0\left[x_0 + (1 - x_0)e^t\right] + x_0^2}{[x_0 + (1 - x_0)e^t]^2} \\ &= \frac{-x_0(1 - x_0)e^t}{[x_0 + (1 - x_0)e^t]^2} = \text{left side} \quad \blacksquare \end{aligned} \tag{5}$$

Therefore, $x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t}$ is the solution of the system.

(c)

Equation (3) is not defined for all $t \ge 0$. In fact, it can be seen that when $x_0 > 1$, there exists a finite $t > 0$ such that

$$x_0 + (1 - x_0)e^t = 0 \quad \text{or} \quad t = \ln\frac{x_0}{x_0 - 1} \tag{6}$$

Therefore, Equation (3) is not defined at $t = \ln\frac{x_0}{x_0 - 1}$, which means

$$T = \ln\frac{x_0}{x_0 - 1} \tag{7}$$

Problem 2

It is known that the following Van del Pol equation has a limit cycle.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 0.2(x_1^2 - 1)x_2\end{aligned}$$

Write a MATLAB program to generate $(x_1(t), x_2(t))$, $0 \le t \le 100$ for $(x_1(0), x_2(0)) = (2.3, -2)$ and $(x_1(0), x_2(0)) = (0.2, 0.3)$. Plot the phase portraits in the same Figure (that is, $x_2(t)$ vs. $x_1(t)$ for $0 \le t \le 100$).
Hint: You can try the following Matlab 6 program

```
x01 = [2.3; -2];
x02 = [0.2; 0.3];
t0 = 0; tf = 100;
tspan=[t0 tf];
[t,x1] = ode23('limit', tspan , x01);
[t,x2] = ode23('limit', tspan , x02);
plot(x1(:,1), x1(:,2), x2(:,1), x2(:,2))
```

where **limit** is the following matlab function named as *limit.m*.

```
function xdot = limit(t,x)
xdot(1) = x(2);
xdot(2) = -x(1) -0.2*(x(1)*x(1)-1)*x(2);
xdot = xdot(:);
```

However, make sure you understand the program.

Solution:

The MATLAB code in main file is shown below:

```
clc; clf; clear all;
hold on;
x01 = [2.3; -2];
x02 = [0.2; 0.3];
t0 = 0; tf = 100;
tspan = [t0 tf];
[t,x1] = ode23('limit', tspan , x01);
[t,x2] = ode23('limit', tspan , x02);
plot(x1(:,1), x1(:,2),'color',[0.667 0.667 1],'LineWidth',2.5);
plot(x2(:,1), x2(:,2),'color',[1 0.5 0],'LineWidth',2.5);
xlabel('$x_1 \left(t\right)$','interpreter','latex');
ylabel('$x_2 \left(t\right)$','interpreter','latex');
legend(['$\left(x_1\left(0\right),x_2\left(0\right)\right)' ...
    '=\left(2.3,-2\right)$'],['$\left(x_1\left(0\right),' ...
    'x_2\left(0\right)\right)=\left(0.2,0.3\right)$'], ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "Q1_2_Code"+".pdf";
saveas(gcf,filename);
```

The MATLAB code in function that defines the Van del Pol equation is shown below:

```
function xdot = limit(t,x)
xdot(1) = x(2);
xdot(2) = -x(1) -0.2*(x(1)*x(1)-1)*x(2);
xdot = xdot(:);
```

And the results for phase portraits are plotted in Figure 1.

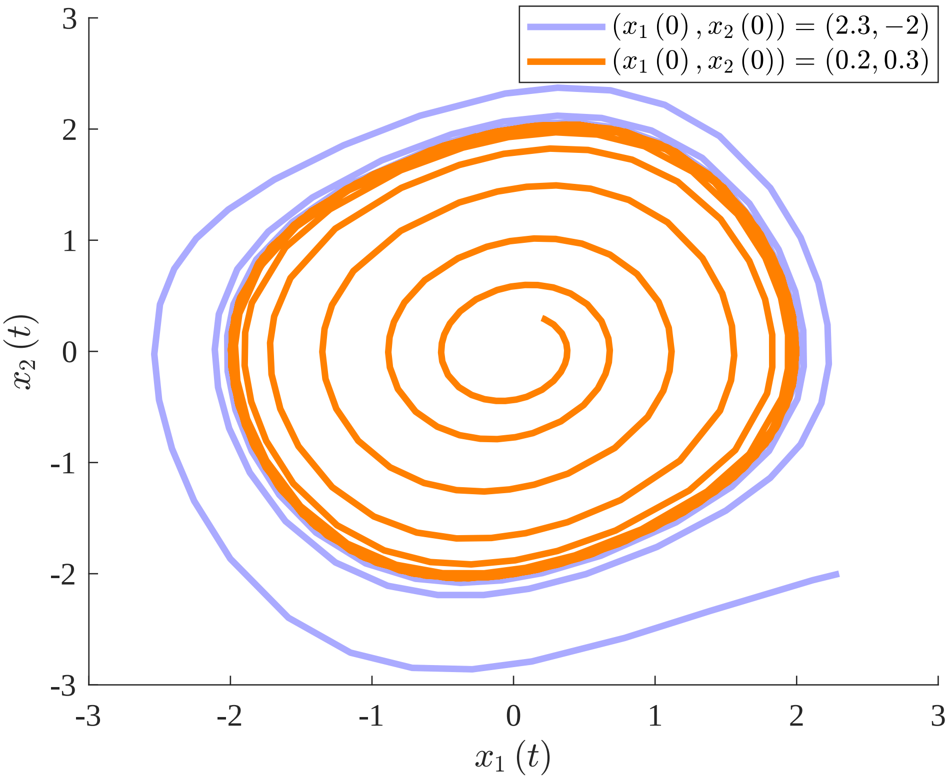


Figure 1: Phase Portraits for Given Nonlinear System.

Problem 3

It is known that the following system displays chaotic behavior.

$$\ddot{y} + 0.05\dot{y} + y^3 = 7.5\cos t$$

(a) Give a state space realization for this system.
(b) Write a MATLAB program to generate $y(t)$, $0 \leq t \leq 50$ for $(y(0), \dot{y}(0)) = (3, 4)$ and $(y(0), \dot{y}(0)) = (3.01, 4.01)$. Plot them in the same Figure. (The curve should be similar to Figure 1.6, page 11 of the text-book).

Solution:

(a)

First, we need to select the state variables. Choosing the state variables as successive derivatives, we get

$$x_1 = y \tag{8a}$$

$$x_2 = \dot{y} \tag{8b}$$

Differentiating both sides and making use of Equation (8) to find $\dot{x}_1$, and equation in the question to find $\tilde{y} = \dot{x}_2$, we obtain the state equations (Nise, 2020). The combined state and output equations are

$$\dot{x}_1 = x_2 \tag{9a}$$

$$\dot{x}_2 = -x_1^3 - 0.05x_2 + 7.5\cos t \tag{9b}$$

In vector-matrix form,

$$\dot{x}\left(t\right) = f\left(x\left(t\right), t\right) \tag{10}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f\left(x\left(t\right), t\right) = \begin{bmatrix} x_2 \\ -x_1^3 - 0.05x_2 + 7.5\cos t \end{bmatrix} \tag{11}$$

(b)

The MATLAB code in main file is shown below:

```
clc; clf; clear all;
hold on;
x01 = [3; 4];
x02 = [3.01; 4.01];
t0 = 0; tf = 50;
tspan = [t0 tf];
[t,x1] = ode23('limit', tspan , x01);
plot(t,x1(:,1),'color',[0.667 0.667 1],'LineWidth',2.5);
[t,x2] = ode23('limit', tspan , x02);
plot(t,x2(:,1),'color',[1 0.5 0],'LineWidth',2.5);
grid on;
```

```
xlabel('$t$','interpreter','latex');
ylabel('$y \left(t\right)$','interpreter','latex');
legend(['$\left(y\left(0\right),\dot{y}\left(0\right)\right)' ...
    '=\left(3,4\right)$'],['$\left(y\left(0\right),' ...
    '\dot{y}\left(0\right)\right)=\left(3.01,4.01\right)$'], ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "Q1_3_Code"+".pdf";
saveas(gcf,filename);
```

The MATLAB code in function that defines the Van del Pol equation is shown below:

```
function xdot = limit(t,x)
xdot(1) = x(2);
xdot(2) = -x(1)*x(1)*x(1)-0.05*x(2)+7.5*cos(t);
xdot = xdot(:);
```

And the results for $y\left(t\right)$ are plotted in Figure 2.

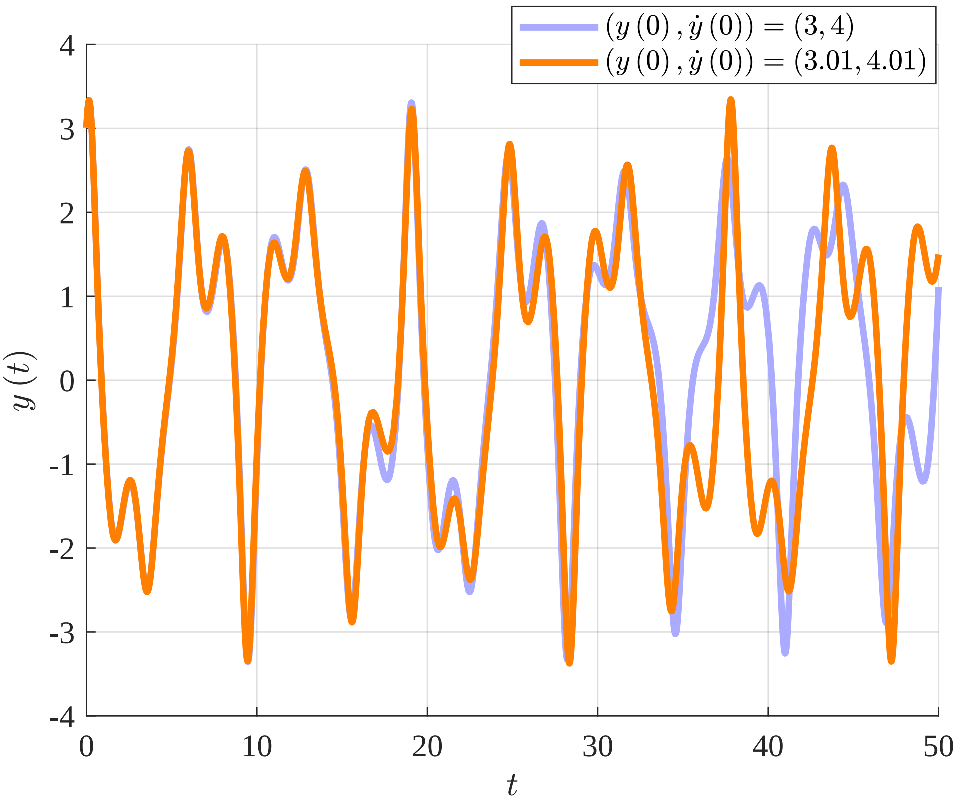


Figure 2: Results for Chaotic Behavior with Different Initial Conditions.

References

Nise, N. S. (2020). *Control systems engineering*. John Wiley & Sons.

香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #2

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium.

(a)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{x_1^3}{4} - x_2\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2(1 + x_1) \\ \dot{x}_2 &= -x_1(1 + x_1)\end{aligned}$$

Solution:

(a)

The equilibrium points $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ satisfy

$$\begin{cases} x_2^* = 0 \\ -x_1^* + \frac{{x_1^*}^3}{4} - x_2^* = 0 \end{cases} \Rightarrow x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} \tag{1}$$

Taking the Jacobian of the appropriate function yields that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{3}{4}x_1^2 & -1 \end{bmatrix} \tag{2}$$

For $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{3}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2} \tag{4}$$

Therefore, the equilibrium point $(0, 0)$ is a stable focus.

For $x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \tag{5}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{6}$$

Therefore, the equilibrium point $(2, 0)$ is a saddle point.

For $x^* = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \tag{7}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{8}$$

Therefore, the equilibrium point $(-2, 0)$ is a saddle point.

(b)

The equilibrium points $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ satisfy

$$\begin{cases} -2x_1^* + x_2^*\left(1 + x_1^*\right) = 0 \\ -x_1^*\left(1 + x_1^*\right) = 0 \end{cases} \Rightarrow x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

Taking the Jacobian of the appropriate function yields that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -2 + x_2 & 1 + x_1 \\ -1 + 2x_1 & 0 \end{bmatrix} \tag{10}$$

For $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \tag{11}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{12}$$

Therefore, the equilibrium point $(0, 0)$ is a stable node.

Problem 2

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)^2, x_1(0) = x_{10} \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)^2, x_2(0) = x_{20}\end{aligned}$$

(a) Show that the system has a limit cycle.
(b) Determine the stability of the limit cycle.
(*Hint:*) Use polar coordinates.

Solution:

(a)

The polar coordinates are introduced as follows:

$$r = \left(x_1^2 + x_2^2\right)^{1/2} \tag{13}$$

and

$$\theta = \tan^{-1}(x_2/x_1) \tag{14}$$

Rearranging Equation (13) gives

$$r^2 = x_1^2 + x_2^2 \tag{15}$$

Taking the derivative of Equation (15) yields

$$2r\frac{dr}{dt} = 2x_1\frac{dx_1}{dt} + 2x_2\frac{dx_2}{dt} \tag{16}$$

Substituting the conditions in the question into Equation (16) gets

$$2r\frac{dr}{dt} = 2x_1\left[x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)^2\right] + 2x_2\left[-x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)^2\right] \tag{17}$$

Simplifying Equation (17) leads to

$$\frac{dr}{dt} = -r\left(r^2 - 1\right)^2 \tag{18}$$

Rearranging Equation (14) gives

$$\tan\theta = \frac{x_2}{x_1} \tag{19}$$

Differentiating Equation (19) yields

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{x_1}\frac{dx_2}{dt} - x_2\frac{1}{x_1^2}\frac{dx_1}{dt} \tag{20}$$

Substituting the conditions in the question into Equation (20) gets

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{x_1}\left[-x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)^2\right] - x_2\frac{1}{x_1^2}\left[x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)^2\right] \tag{21}$$

Simplifying Equation (21) leads to

$$\frac{d\theta}{dt} = -1 \tag{22}$$

When the state starts on the unit circle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from the outside. When $r < 1$, then $\dot{r} < 0$. This implies that the state tends to diverge from it. Therefore, the unit circle is a semi-stable limit cycle (Slotine et al., 1991).

(b)

When the state starts on the unit circle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from the outside. When $r < 1$, then $\dot{r} < 0$. This implies that the state tends to diverge from it. Therefore, the limit cycle is semi-stable.

Problem 3

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= ax_1 + bx_2 - x_1^2 x_2 - x_1^3\end{aligned}$$

Show that there can be no limit cycle if $b < 0$.

Solution:

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = b - x_1^2 < 0 \tag{23}$$

By Bendixson's criterion, there are no periodic orbits. Therefore, there can be no limit cycle if $b < 0$.

References

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.

香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #3

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Does the system have any limit cycle?

$$\begin{aligned}\dot{x}_1 &= 2x_2^2 \sin x_2 \\ \dot{x}_2 &= 1 - \cos x_1 + 2x_2\end{aligned}$$

Solution:

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2 > 0 \tag{1}$$

By Bendixson's criterion, there are no periodic orbits. Therefore, there can be no limit cycle.

Problem 2

Consider the following nonlinear equation.

$$
\begin{aligned}
\dot{x}_1 &= x_2 - x_1^2 + 3x_2^3 \\
\dot{x}_2 &= x_3 - 2x_2x_1 - x_1x_3 \\
\dot{x}_3 &= 3x_1 + 2x_3x_2 - 2x_3 + u
\end{aligned}
$$

(a) Find the Jacobian linearization of the system at the origin.
(b) Using the Lyapunov's linearization method to determine the stability property of the closed-loop system under the state feedback control law $u = -Kx$ for $K = [-4 - 3 - 1]$.

Solution:

(a)

The Jacobian matrix of the nonlinear equation is (Close et al., 2001)

$$
\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \frac{\partial f_1(x)}{\partial x_3} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} \\ \frac{\partial f_3(x)}{\partial x_1} & \frac{\partial f_3(x)}{\partial x_2} & \frac{\partial f_3(x)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -2x_1 & 9x_2^2+1 & 0 \\ -2x_2 - x_3 & -2x_1 & -x_1+1 \\ 3 & 2x_3 & 2x_2 - 2 \end{bmatrix} \tag{2}
$$

For the system at the origin,

$$
A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix} \tag{3}
$$

$$
B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}
$$

Therefore, the Jacobian linearization of the system at the origin is

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{5}
$$

(b)

$$
A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -4 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & 3 & -1 \end{bmatrix} \tag{6}
$$

The eigenvalues of $A - BK$ are

$$
\begin{aligned}
\lambda_1 &= -1.5370 + 1.0064i \\
\lambda_2 &= -1.5370 - 1.0064i \\
\lambda_3 &= 2.0739
\end{aligned} \tag{7}
$$

Because the real part of one of the eigenvalues of $A - BK$–λ_3–is positive, the closed-loop system under the state feedback control law $u = -Kx$ for $K = \begin{bmatrix} -4 & -3 & -1 \end{bmatrix}$ is unstable.

Problem 3

The motion of the ball and beam system can be described by

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= Bx_1(t)x_4^2(t) - BGsin(x_3(t)) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= u(t) \\
y(t) &= x_1(t)
\end{aligned}$$

where x_1 is the position of the ball, u is the torque applied to the beam, $G = 9.81\ m/s^2$ is the acceleration of gravity, and $B = 0.7134$ is a constant.

(a) Show that the Jacobian linearization of (1) at the origin is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + bu$$

(b) Verify that the pair (A, b) is controllable, i.e., the matrix $[b\ Ab\ \cdots\ A^{n-1}b]$ is nonsingular.

(c) Using Arkerman's Formula to find K so that the eigenvalues of $A - bK$ are $\{-1, -2, -3, -24\}$.

(d) Simulate the closed-loop system composed of (1) and $u = -Kx$ with $x(0) = \alpha[1, 1, 1, 1]$ for $\alpha = 1, 20$ from $t = 0$ to $t = 20$. Is the equilibrium point $x = 0$ of the closed-loop system globally asymptotically stable?

Arkerman's Formula

Let $A \in R^{n\times n}$ and $b \in R^n$. Assume the pair (A, b) is controllable, i.e., the matrix $[b\ Ab\ \cdots\ A^{n-1}b]$ is nonsingular. Let $q(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$ which is called the desired polynomial.

Let $q(F) = F^n + \alpha_1 F^{n-1} + \cdots + \alpha_{n-1}F + \alpha_n I_n$

Then

$$K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times n} [b\ Ab\ \cdots\ A^{n-1}b]^{-1} q(F)$$

is such that

$$\det(sI - (A - BK)) = q(s)$$

Solution:

(a)

The Jacobian matrix of the nonlinear equation is

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \frac{\partial f_1(x)}{\partial x_3} & \frac{\partial f_1(x)}{\partial x_4} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} & \frac{\partial f_2(x)}{\partial x_4} \\ \frac{\partial f_3(x)}{\partial x_1} & \frac{\partial f_3(x)}{\partial x_2} & \frac{\partial f_3(x)}{\partial x_3} & \frac{\partial f_3(x)}{\partial x_4} \\ \frac{\partial f_4(x)}{\partial x_1} & \frac{\partial f_4(x)}{\partial x_2} & \frac{\partial f_4(x)}{\partial x_3} & \frac{\partial f_4(x)}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bx_4^2(t) & 0 & -BG\cos(x_3(t)) & 2Bx_1(t)x_4(t) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

For the system at the origin,

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

Therefore, the Jacobian linearization of the system at the origin is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{11}$$

(b)

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -BG \\ 0 & 0 & -BG & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = 4 \tag{13}$$

Therefore, the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$ is nonsingular, which indicates that the pair (A, b) is controllable.

(c)

According to Arkerman's Formula,

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -BG \\ 0 & 0 & -BG & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} q(A) \tag{14}$$

Because the eigenvalues of $A - bK$ are $\{-1, -2, -3, -24\}$ and $\det(sI - (A - BK)) = q(s)$, the desired polynomial is

$$q(s) = (s+1)(s+2)(s+3)(s+24) \tag{15}$$

Therefore,

$$\begin{aligned} q(A) &= (A+I)\cdot(A+2I)\cdot(A+3I)\cdot(A+24I) \\ &= \begin{bmatrix} 144 & 270 & 1084.8 & -210 \\ 0 & 144 & -1889.6 & -1084.8 \\ 0 & 0 & 144 & 270 \\ 0 & 0 & 0 & 144 \end{bmatrix} \end{aligned} \tag{16}$$

where I is the identity matrix. Then, we can get

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -6.9985 \\ 0 & 0 & -6.9985 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 144 & 270 & 1084.8 & -210 \\ 0 & 144 & -1889.6 & -1084.8 \\ 0 & 0 & 144 & 270 \\ 0 & 0 & 0 & 144 \end{bmatrix}$$
$$= \begin{bmatrix} -20.5760 & -38.5799 & 155.0000 & 30.0000 \end{bmatrix} \tag{17}$$

Instead of Arkerman's Formula, I also use another way to find out the answer, whose MATLAB codes are shown below:

```
clc; clear all;
B = 0.7134;
G = 9.81;
BG = B*G;
syms k1 k2 k3 k4
sp = sym('sp',[4 4 4]);
lambda = [-1 -2 -3 -24];
detsp = sym('detsp',[1 4]);
for i = 1:4
    sp(:,:,i) = [-lambda(i) 1 0 0;
        0 -lambda(i) -BG 0;
        0 0 -lambda(i) 1;
        -k1 -k2 -k3 -k4-lambda(i)];
    detsp(i) = det(sp(:,:,i));
end
[k1,k2,k3,k4] = solve(detsp==0);
```

(d)

I use the Simulink model as shown in Figure 1 to simulate the closed-loop system composed of (1) and $u = -Kx$ with $x(0) = \alpha \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ for $\alpha = 1$ and 20 from $t = 0$ to $t = 20$ s.

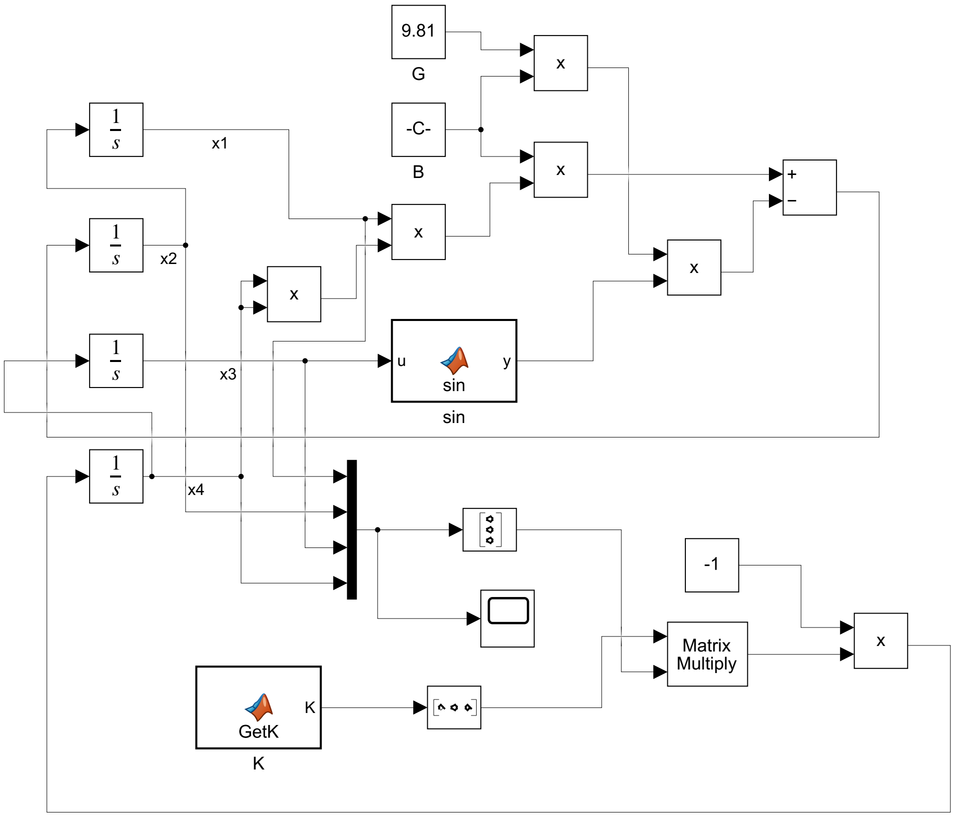


Figure 1: Simulink model to simulate the closed-loop system composed of (1) and $u = -Kx$ with $x(0) = \alpha \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ for $\alpha = 1$ and 20 from $t = 0$ to $t = 20$ s.

The results I get from the simulation is shown in Figure 2 and 3 for $\alpha = 1$ and 20, respectively.

From Figure 3, we can know that for the initial conditions of $x(0) = 20 \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, the system is unstable. Therefore, the equilibrium point $x = 0$ of the closed-loop system is not globally asymptotically stable.

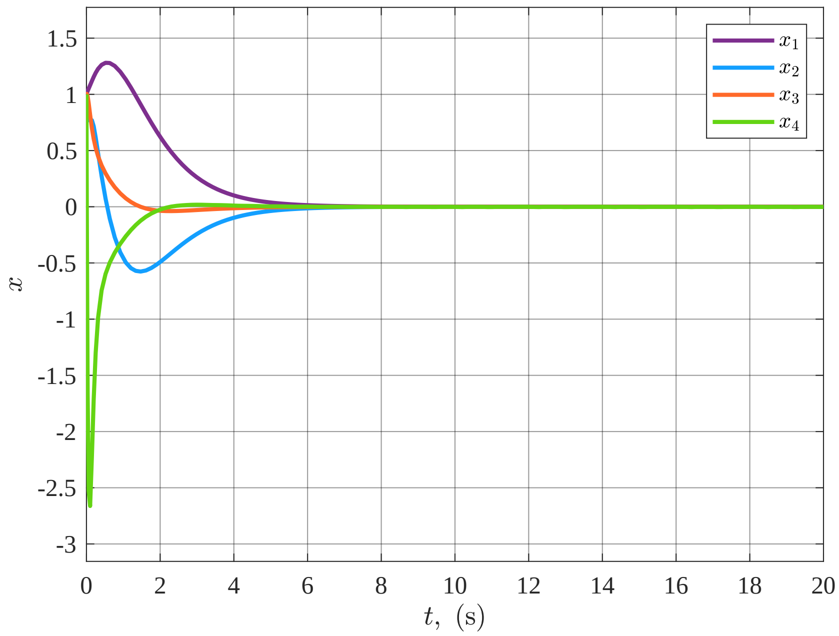


Figure 2: Simulation results for the system with $\alpha = 1$.

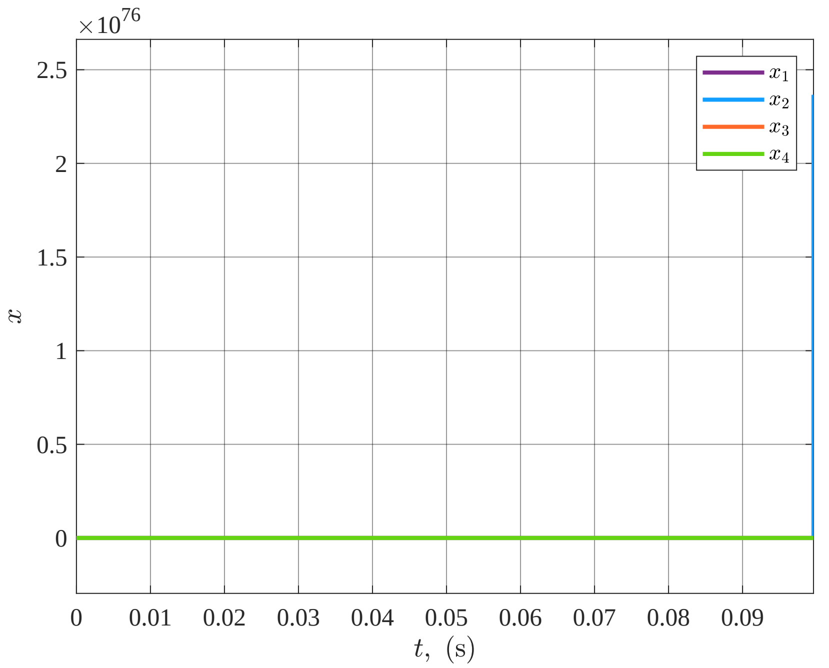


Figure 3: Simulation results for the system with $\alpha = 20$.

References

Close, C. M., Frederick, D. K., & Newell, J. C. (2001). *Modeling and analysis of dynamic systems*. John Wiley & Sons.

香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #4

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

For the following systems, find the equilibrium points and determine their stability. Indicate whether the stability is asymptotic, and whether it is global.

(a)

$$\dot{x} = -x^3 + \sin^4 x \tag{1}$$

(b)

$$\dot{x} = (5 - x)^5 \tag{2}$$

Solution:

(a) The equilibrium points x^* satisfy $-x^{*3} + \sin^4 x^* = 0$. Obviously, $x^* = 0$ is one solution. Next, I will prove that $x^* = 0$ is the only solution. Let $f(x) = -x^3 + \sin^4 x$.

$$f'(x) = 4\sin^3 x \cos x - 3x^2 \tag{3}$$

Because $\sin x \cos x \leq \frac{\sin^2 x + \cos^2 x}{2} = \frac{1}{2}$ (Cauchy–Schwarz inequality),

$$f'(x) = 4\sin^2 x \sin x \cos x - 3x^2 \leq 2\sin^2 x - 3x^2 \leq 2x^2 - 3x^2 = -x^2 \leq 0 \tag{4}$$

Hence, $f(x)$ is a monotonically decreasing function, which indicates that $x^* = 0$ is the only solution. Therefore, the equilibrium point for this system is $x^* = 0$. Then, the following Lyapunov function is selected as the candidate:

$$V(x) = \frac{1}{2}x^2 \tag{5}$$

$V(x)$ is positive definite for $\forall x \in \mathbf{R} - \{0\}$. Taking the derivative of Equation (5) yields that

$$\dot{V}(x) = x\dot{x} = x\left(-x^3 + \sin^4 x\right) \tag{6}$$

From the analysis above for $f(x)$, it can be concluded that $f(x) > 0$ when $x < 0$ and $f(x) < 0$ when $x > 0$. Combing this conclusion with Equation (6) obtains $\dot{V}(x) < 0$ for $\forall x \in \mathbf{R} - \{0\}$, indicating that $\dot{V}(x)$ is negative definite. Therefore, the system is globally asymptotically stable.

(b) The equilibrium points x^* satisfy $(5 - x^*)^5 = 0$. Obviously, $x^* = 5$ is one solution and $(5 - x)^5$ is monotonically-decreasing. Therefore, the equilibrium point for this system is $x^* = 5$. Then, the following Lyapunov function is selected as the candidate:

$$V(x) = \frac{1}{2}(5 - x)^2 \tag{7}$$

$V(x)$ is positive definite for $\forall x \in \mathbf{R} - \{5\}$. Taking the derivative of Equation (7) yields that

$$\dot{V}(x) = -(5 - x)\dot{x} = -(5 - x)^6 < 0 \text{ for } \forall x \in \mathbf{R} - \{5\} \tag{8}$$

Therefore, $\dot{V}(x)$ is negative definite, which means that the system is globally asymptotically stable.

Problem 2

Consider the following pendulum equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_1 \sin x_1 - a_2 x_2 \end{aligned} \tag{9}$$

where $a_1 > 0$ and $a_2 > 0$.

(a) Show the equilibrium point $x = 0$ is stable using the Lyapunov function candidate $V(x) = a_1(1 - \cos x_1) + \frac{1}{2}x_2^2$. Can you conclude the asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$?

(b) Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2\right) + a_1(1 - \cos x_1) \tag{10}$$

where $p_{22} = 1$ and $p_{11} = a_2 p_{12}$. Can you find appropriate value for p_{12} to conclude the asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$?

Solution:

(a)

$$V(x) = a_1(1 - \cos x_1) + \frac{1}{2}x_2^2 > a_1(1 - \cos x_1) \geq a_1(1 - 1) = 0 \tag{11}$$

$$\dot{V}(x) = a_1 \sin x_1 \dot{x}_1 + x_2\dot{x}_2 = a_1 x_2 \sin x_1 + x_2(-a_1 \sin x_1 - a_2 x_2) = -a_2 x_2^2 \leq 0 \tag{12}$$

Therefore, $V(x)$ is positive definite, and $\dot{V}(x)$ is negative semi-definite, which means I can not conclude the asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$.

(b)

$$\begin{aligned} V(x) &= \frac{1}{2}\left(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2\right) + a_1(1 - \cos x_1) \\ &= \frac{1}{2}\left(a_2 p_{12}x_1^2 + 2p_{12}x_1x_2 + x_2^2\right) + a_1(1 - \cos x_1) \\ &> a_1(1 - \cos x_1) \geq a_1(1 - 1) = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{V}(x) &= p_{11}x_1\dot{x}_1 + p_{12}\dot{x}_1 x_2 + p_{12}x_1\dot{x}_2 + p_{22}x_2\dot{x}_2 + a_1 \sin x_1 \dot{x}_1 \\ &= p_{11}x_1x_2 + p_{12}x_2^2 + p_{12}x_1(-a_1 \sin x_1 - a_2 x_2) \\ &\quad + p_{22}x_2(-a_1 \sin x_1 - a_2 x_2) + a_1 x_2 \sin x_1 \\ &= a_2 p_{12}x_1x_2 + p_{12}x_2^2 + p_{12}x_1(-a_1 \sin x_1 - a_2 x_2) \\ &\quad + x_2(-a_1 \sin x_1 - a_2 x_2) + a_1 x_2 \sin x_1 \\ &= (p_{12} - a_2)x_2^2 - a_1 p_{12} x_1 \sin x_1 \end{aligned} \tag{14}$$

For $x_1 \in (-\pi, \pi)$ and $x_2 \in \mathbf{R}$, $0 < p_{12} < a_2$ is selected to make $\dot{V}(x)$ ND. Therefore, the appropriate value for p_{12}, i.e. $0 < p_{12} < a_2$, can be selected to conclude that the equilibrium point $x = 0$ is locally asymptotic stable.

However, I can not find appropriate value for p_{12} to conclude the globally asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$. Because for $\forall p_{12} \in \mathbf{R}^+$, $\exists k \in \mathbf{Z}$, which satisfies

$$k > -\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4} \tag{15}$$

so that when $x_1 = 2k\pi + \frac{3}{2}\pi$,

$$\begin{aligned}
\dot{V}(x) &= (p_{12} - a_2)\, x_2^2 - a_1 p_{12} x_1 \sin x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left(2k\pi + \frac{3}{2}\pi\right) \\
&> (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left[2\pi \left(-\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4}\right) + \frac{3}{2}\pi\right] = 0
\end{aligned} \tag{16}$$

and for $\forall p_{12} \in \mathbf{R}^-$, $\exists k \in \mathbf{Z}$, which satisfies

$$k < -\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4} \tag{17}$$

so that when $x_1 = 2k\pi + \frac{3}{2}\pi$,

$$\begin{aligned}
\dot{V}(x) &= (p_{12} - a_2)\, x_2^2 - a_1 p_{12} x_1 \sin x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left(2k\pi + \frac{3}{2}\pi\right) \\
&> (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left[2\pi \left(-\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4}\right) + \frac{3}{2}\pi\right] = 0
\end{aligned} \tag{18}$$

In addition, for $p_{12} = 0$, this situation has been discussed in (a). Therefore, the appropriate value for p_{12} to conclude the globally asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$ can not be found.

Problem 3

Show that if symmetric p.d. matrices P and Q exists such that

$$A^T P + PA + 2\lambda P = -Q \tag{19}$$

then all the eigenvalues of A have a real part strictly less than $-\lambda$.

Solution:

Consider the linear homogeneous continuous-time system

$$\dot{x}(t) = (A + \lambda I)\, x(t) \tag{20}$$

Let us associate with this system and the equilibrium point $x^* = 0$ the quadratic function

$$V(x) = x^T P x \tag{21}$$

where P is symmetric and positive definite. This V is continuous and has continuous first partial derivatives. Furthermore, since P is positive definite, the origin is the unique minimum point of V. Thus in terms of general characteristics, such a positive definite quadratic form is a suitable candidate for a Lyapunov function. It remains, of course, to determine how $\dot{V}(x)$ is influenced by the dynamics of the system.

We have

$$\begin{aligned}
\dot{V}(x) &= \frac{d}{dt} x^T P x \\
&= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T (A + \lambda I) P x + x^T P (A + \lambda I) x \\
&= x^T \left(A^T P + PA + 2\lambda P \right) x \\
&= -x^T Q x
\end{aligned} \tag{22}$$

Because matrix Q is symmetric p.d., $\dot{V}(x) < 0$ for $\forall x \in \mathbf{R} - \{0\}$, indicating that $\dot{V}(x)$ is ND. Therefore, the system is globally asymptotically stable. To ensure the system is global asymptotically stable, the real parts of the eigenvalues of $(A + \lambda I)$ need to be always negative, which means all the eigenvalues of A have a real part strictly less than $-\lambda$ (Luenberger, 1979).

Problem 4

For the linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -6x_1 - 5x_2 \end{aligned} \tag{23}$$

(a) what can you say about its stability and asymptotic stability from the candidate Lyapunov functions

$$\begin{aligned} V_1(x) &= 6x_1^2 + x_2^2 \\ V_2(x) &= x_1^2 + x_2^2 - x_1x_2 \end{aligned} \tag{24}$$

(b) For $Q = I$, solve the Lyapunov equation for a symmetric p.d. matrix P.

$$A^T P + PA = -Q \tag{25}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \tag{26}$$

Solution:

(a) For the candidate V_1,

$$V_1(x) = 6x_1^2 + x_2^2 > 0 \text{ for } \forall x \in \mathbf{R}^2 - \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \tag{27}$$

Therefore, $V_1(x)$ is positive definite. Taking the derivative of Equation (27) yields that

$$\dot{V}_1(x) = 12x_1\dot{x}_1 + 2x_2\dot{x}_2 = 12x_1x_2 + 2x_2(-6x_1 - 5x_2) = -10x_2^2 \leq 0 \tag{28}$$

Therefore, $\dot{V}(x)$ is negative semi-definite, which means the system is stable at the equilibrium point $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

P.S. If Invariant Set Theorem is used in this question, asymptotic stability can be concluded. Because let $R = \mathbf{R}^2$, if $\dot{V}_1(x) = 0$, $x_2 = 0$, so $\dot{x}_2 = 0$. Substituting $x_2 = 0$ and $\dot{x}_2 = 0$ into Equation (23), we can get $x_1 = 0$. Therefore, $\dot{V}_1(x) = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Therefore, the system is asymptotically stable.

For the candidate V_2,

$$V_2(x) = x_1^2 + x_2^2 - x_1x_2 = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0 \text{ for } \forall x \in \mathbf{R}^2 - \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \tag{29}$$

Therefore, $V_2(x)$ is positive definite. Taking the derivative of Equation (29) yields that

$$\begin{aligned}\dot{V}_2(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 - \dot{x}_1x_2 - x_1\dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-6x_1 - 5x_2) - x_2^2 - x_1(-6x_1 - 5x_2) \\ &= 6x_1^2 - 11x_2^2 - 5x_1x_2 \\ &= (6x_1 - 11x_2)(x_1 + x_2)\end{aligned} \tag{30}$$

The sign of $\dot{V}_2(x)$ can not be told from Equation (30). Therefore, this candidate Lyapunov functions can not conclude the stability of the equilibrium point $x = 0$.

(b) Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, where $p_{12} = p_{21}$.

$$\begin{aligned} A^TP + PA = -Q &\Leftrightarrow \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} -6p_{21} & -6p_{22} \\ p_{11} - 5p_{21} & p_{12} - 5p_{22} \end{bmatrix} + \begin{bmatrix} -6p_{12} & p_{11} - 5p_{12} \\ -6p_{22} & p_{21} - 5p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} -6p_{21} - 6p_{12} & -6p_{22} + p_{11} - 5p_{12} \\ p_{11} - 5p_{21} - 6p_{22} & p_{12} - 5p_{22} + p_{21} - 5p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{31}$$

Therefore, we can know that

$$\begin{cases} -6p_{21} - 6p_{12} = -1 \\ -6p_{22} + p_{11} - 5p_{12} = 0 \\ p_{11} - 5p_{21} - 6p_{22} = 0 \\ p_{12} - 5p_{22} + p_{21} - 5p_{22} = -1 \end{cases} \tag{32}$$

Solving Equation (32) yields that

$$P = \begin{bmatrix} \frac{67}{60} & \frac{1}{12} \\ \frac{1}{12} & \frac{7}{60} \end{bmatrix} \tag{33}$$

References

Luenberger, D. G. (1979). *Introduction to dynamic systems: theory, models, and applications*, volume 1. Wiley New York.

香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #5

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Determine the stability of the following system at the origin. Indicate whether the stability is asymptotic, and whether it is global.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + \sin^4 x_1 - x_2^7 \end{aligned} \tag{1}$$

Solution:

Here we use Lemma 2 of Invariant Set Theorem (Slotine et al., 1991). For the system (1), $c(x_1) = x_1^3 - \sin^4 x_1$ and $b(x_2) = x_2^7$. Therefore,

$$yb(y) = y^8 > 0, y \neq 0 \tag{2}$$

and

$$yc(y) = y\left(y^3 - \sin^4 y\right) \tag{3}$$

Let $f(x) = x^3 - \sin^4 x$.

$$f'(x) = -4\sin^3 x \cos x + 3x^2 \tag{4}$$

Because $\sin x \cos x \leq \frac{\sin^2 x + \cos^2 x}{2} = \frac{1}{2}$ (Cauchy–Schwarz inequality),

$$f'(x) = -4\sin^2 x \sin x \cos x + 3x^2 \geq -\sin^2 x + 3x^2 \geq -2x^2 = 3x^2 = x^2 \geq 0 \tag{5}$$

Hence, $f(x)$ is a monotonically increase function, And we note that $f(0) = 0$. Therefore, $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x < 0$. Therefore, we can conclude that

$$yc(y) = y\left(y^3 - \sin^4 y\right) > 0 \tag{6}$$

Hence, the system is asymptotically stable at the equilibrium point $x = 0$.

Moreover,

$$\lim_{|y|\to\infty} \int_0^y c(r)\,dr = \lim_{|y|\to\infty} \int_0^y \left(r^3 - \sin^4 r\right) dr > \lim_{|y|\to\infty} \int_0^y r^3 dr = \lim_{|y|\to\infty} \frac{1}{4} y^4 = \infty \tag{7}$$

Hence, the system is globally asymptotically stable at the equilibrium point $x = 0$.

Problem 2

Consider Lienard's equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 \sin^2 x_1 - b(x_1)\end{aligned} \tag{8}$$

where $b(y)$ is continuous function over $y \in \mathbf{R}$ and satisfies $yb(y) > 0$, $y \neq 0$.

(a) Show the following function

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} b(y)\, dy \tag{9}$$

is a Lyapunov function for the system.

(b) Show that the origin is locally asymptotically stable.

(c) Can you conclude global asymptotic stability of the origin based on this Lyapunov function? Why?

Solution:

(a) Because $yb(y) > 0$, $b(y) > 0$ for $y > 0$ and $b(y) < 0$ for $y < 0$. Hence, $\int_0^{x_1} b(y)\, dy > 0$ for $x_1 \neq 0$. Therefore,

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} b(y)\, dy > 0 \tag{10}$$

Taking the derivative of Equation (9) gets

$$\begin{aligned}\dot{V}(x_1, x_2) &= \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 \\ &= b(x_1) x_2 + x_2 \left(-x_2 \sin^2 x_1 - b(x_1)\right) \\ &= x_2^2 \sin^2 x_1 \leq 0\end{aligned} \tag{11}$$

Therefore, $V(x)$ is positive definite, and $\dot{V}(x)$ is negative-semi definite, from which we can conclude that

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} b(y)\, dy \tag{12}$$

is a Lyapunov function for the system.

(b) Invariant Set Theorem is used in this question. For $x_1 \in (-\pi, \pi) - \{0\}$, if $\dot{V}(x_1, x_2) = 0$, $x_2 = 0$, so $\dot{x}_2 = 0$. Substituting $x_2 = 0$ and $\dot{x}_2 = 0$ into Equation (8), we can get $x_1 = 0$. Therefore, $\dot{V}_1(x) = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\{(x_1, x_2) \mid -\pi < x_1 < \pi, x_2 \in \mathbf{R}\}$. Therefore, the system is locally asymptotically stable.

(c) Global asymptotic stability of the origin based on this Lyapunov function can not be concluded because $\dot{V}(x_1, x_2) = 0$ when $\{(x_1, x_2) \mid x_1 = k\pi, k \in \mathbf{Z}, x_2 \in \mathbf{R}\}$ so we cannot find the invariant set.

Problem 3

Using the Krasovskii Theorem to show the global asymptotic stability of the equilibrium point at the origin of the following system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= x_1 - 3x_2 - x_2^5 \end{aligned} \tag{13}$$

Solution:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 - 5x_2^4 \end{bmatrix} \tag{14}$$

$$F(x) = \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} = \begin{bmatrix} -6 & 2 \\ 2 & -6 - 10x_2^4 \end{bmatrix} < 0, \forall x \tag{15}$$

Thus, the equilibrium point is asymptotic stable.

Moreover,

$$V(x) = f^T(x) f(x) = (-3x_1 + x_2)^2 + \left(x_1 - 3x_2 - x_2^5\right)^2 \tag{16}$$

is radially unbounded. Thus, the equilibrium point is global asymptotic stable.

Problem 4

Consider the following system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c\left(x_1\right) - b\left(x_2\right)\end{aligned} \tag{17}$$

where the functions c and b are continuous satisfying the sign condition. Using the variable gradient method to derive a Lyapunov function for Equation (17) as follows:

$$V\left(x_1, x_2\right) = \int_0^{x_1} c\left(y\right) dy + \frac{1}{2}x_2^3 \tag{18}$$

Solution:

Let $\nabla V = \left[a_1 c\left(x_1\right), a_2 x_2\right]$. Because

$$\frac{\partial \nabla V_1}{x_2} = \frac{\partial a_1 c\left(x_1\right)}{x_2} = 0 = \frac{\partial a_2 x_2}{x_1} = \frac{\partial \nabla V_2}{x_1} \tag{19}$$

we can make $a_1 = a_2 = 1$. Then, $\dot{V}$ can be computed as

$$\dot{V} = \nabla V \dot{x} = c\left(x_1\right) x_2 + x_2\left(-c\left(x_1\right) - b\left(x_2\right)\right) = -x_2 b\left(x_2\right) \tag{20}$$

Bcause b is continuous satisfying the sign condition, $\dot{V} < 0$, which means $\dot{V}$ is negative definite. Therefore, the Lyapunov function can be expressed as

$$\begin{aligned}V\left(x\right) &= \int_0^{x_1} \nabla V_1\left(x_1, 0\right) dx_1 + \int_0^{x_2} \nabla V_2\left(x_1, x_2\right) dx_2 \\ &= \int_0^{x_1} c\left(x_1\right) dx_1 + \int_0^{x_2} x_2 dx_2 \\ &= \int_0^{x_1} c\left(y\right) dy + \frac{1}{2}x_2^2\end{aligned} \tag{21}$$

References

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.

香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #6

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Show that the one-dimensional system $\dot{x} = -a(t)x$ where a(t) is continuous and nonnegative over $t \geq 0$ is exponentially stable if there exist a $T > 0$ such that, for any $t > 0$, $\int_t^{t+T} a(r)\,dr \geq \gamma$ for some $\gamma > 0$.

Hint: For any $t \geq t_0$, $e^{-\int_t^{t+T} a(r)dr} \leq e^{-\gamma} < 1$.

Solution:

For the system, $\dot{x} = -a(t)x$, the solutioin for $x(t)$ is

$$x(t) = x(t_0)\,e^{-\int_{t_0}^{t} a(x)dx} \tag{1}$$

If for any $t > 0$, $\int_t^{t+T} a(r)\,dr \geq \gamma$ for some $\gamma > 0$,

$$\int_{t_0}^{t} a(\tau)\,d\tau = \int_{t_0}^{t+T} a(\tau)\,d\tau + \int_{t_0+T}^{t_0+2T} a(\tau)\,d\tau + \cdots + \int_{t-T}^{t} a(\tau)\,d\tau \geq \frac{t-t_0}{T}\gamma \tag{2}$$

Therefore,

$$e^{-\int_{t_0}^{t} a(x)dx} \leq e^{-\frac{t-t_0}{T}\gamma} \tag{3}$$

Hence,

$$x(t) = x(t_0)\,e^{-\int_{t_0}^{t} a(x)dx} \leq x(t_0)\,e^{\frac{t_0}{T}\gamma}e^{-\frac{\gamma}{T}t} \tag{4}$$

We can conclude that $\dot{x} = -a(t)x$ is exponentially stable.

Problem 2

Condition (4.19) on the eigenvalues of $\boldsymbol{A}(t) + \boldsymbol{A}^T(t)$ is only, of course, a sufficient condition. For instance, show that the linear time-varying system associated with the matrix

$$\boldsymbol{A}(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix} \tag{5}$$

is globally asymptotically stable.

Solution:

$$\boldsymbol{A}(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + e^{\frac{t}{2}} x_2 \\ -x_2 \end{bmatrix} \tag{6}$$

we can obtain the solution for $x_2(t)$:

$$x_2(t) = x_2(t_0)\, e^{-(t-t_0)} \tag{7}$$

Because $\dot{x}_1 = -x_1 + e^{\frac{t}{2}} x_2$

$$\begin{aligned} x_1(t) &= x_1(t_0)\, e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-\tau)} e^{\frac{\tau}{2}} x_2(\tau)\, d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-\tau)} e^{\frac{\tau}{2}} x_2(t_0)\, e^{-(\tau-t_0)} d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + x_2(t_0)\, e^{-(t-t_0)} \int_{t_0}^{t} e^{\frac{\tau}{2}} d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + 2x_2(t_0)\, e^{-(t-t_0)} \left(e^{\frac{t}{2}} - e^{\frac{t_0}{2}} \right) \\ &= x_1(t_0)\, e^{-(t-t_0)} + 2x_2(t_0)\, e^{-\left(\frac{t}{2}-t_0\right)} - 2x_2(t_0)\, e^{-\left(t-\frac{3}{2}t_0\right)} \end{aligned} \tag{8}$$

We can conclude that $\dot{x} = -A(t)\, x$ is globally asymptotically stable since $\lim_{t\to\infty} x_i(t) = 0, i = 1, 2$.

Problem 3

Determine whether the following systems have a stable equilibrium. Indicate whether the stability is asymptotic, and whether it is global.

(a)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-10 & e^{3t}\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{9}$$

(b)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 2\sin t\\ 0 & -(t+1)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{10}$$

(c)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & e^{2t}\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{11}$$

Solution:

(a)

$$A(t)=\begin{bmatrix}-10 & e^{3t}\\ 0 & -2\end{bmatrix}\Longrightarrow\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-10x_1+e^{3t}x_2\\ -2x_2\end{bmatrix} \tag{12}$$

we can obtain the solution for $x_2(t)$:

$$x_2(t)=x_2(t_0)\,e^{-2(t-t_0)} \tag{13}$$

Because $\dot{x}_1=-10x_1+e^{3t}x_2$

$$\begin{aligned}
x_1(t)&=x_1(t_0)\,e^{-10(t-t_0)}+\int_{t_0}^{t}e^{-10(t-\tau)}e^{3\tau}x_2(\tau)\,d\tau\\
&=x_1(t_0)\,e^{-(t-t_0)}+\int_{t_0}^{t}e^{-10(t-\tau)}e^{3\tau}x_2(t_0)\,e^{-2(\tau-t_0)}d\tau\\
&=x_1(t_0)\,e^{-(t-t_0)}+x_2(t_0)\,e^{-(10t-2t_0)}\int_{t_0}^{t}e^{11\tau}d\tau\\
&=x_1(t_0)\,e^{-(t-t_0)}+\frac{1}{11}x_2(t_0)\,e^{-(t-2t_0)}\left(e^{11t}-e^{11t_0}\right)\\
&=x_1(t_0)\,e^{-(t-t_0)}+\frac{1}{11}x_2(t_0)\,e^{t+3t_0}-\frac{1}{11}x_2(t_0)\,e^{-(10t-13t_0)}
\end{aligned} \tag{14}$$

We can conclude that $\dot{x}=-A(t)\,x$ is unstable since $\lim_{t\to\infty}x_1(t)=\infty$.

(b)

$$A(t)+A^T(t)=\begin{bmatrix}-2 & 2\sin t\\ 2\sin t & -2(t+1)\end{bmatrix}\Longrightarrow-\left(A(t)+A^T(t)\right)=\begin{bmatrix}2 & -2\sin t\\ -2\sin t & 2(t+1)\end{bmatrix} \tag{15}$$

The determinant of $-\left(A\left(t\right)+A^{T}\left(t\right)\right)$ is equal to

$$\det\left[-\left(A\left(t\right)+A^{T}\left(t\right)\right)\right]=4\left(t+1\right)-4\sin^{2}t\geq4\left(t+1\right)-4t>0 \tag{16}$$

Therefore, $-\left(A\left(t\right)+A^{T}\left(t\right)\right)$ is positive definite, which means $A\left(t\right)+A^{T}\left(t\right)$ is negative definite. Hence, $\lambda_i\left(A\left(t\right)+A^{T}\left(t\right)\right)<-\lambda$ for some $\lambda>0$. We can conclude that the system is globally asymptotically stable.

(c)

$$\boldsymbol{A}\left(t\right)=\begin{bmatrix}-1 & e^{2t}\\ 0 & -2\end{bmatrix}\Longrightarrow\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1x_1+e^{2t}x_2\\ -2x_2\end{bmatrix} \tag{17}$$

we can obtain the solution for $x_2\left(t\right)$:

$$x_2\left(t\right)=x_2\left(t_0\right)e^{-2\left(t-t_0\right)} \tag{18}$$

Because $\dot{x}_1=-10x_1+e^{3t}x_2$

$$\begin{aligned}x_1\left(t\right)&=x_1\left(t_0\right)e^{-\left(t-t_0\right)}+\int_{t_0}^{t}e^{-\left(t-\tau\right)}e^{3\tau}x_2\left(\tau\right)d\tau\\&=x_1\left(t_0\right)e^{-\left(t-t_0\right)}+\int_{t_0}^{t}e^{-\left(t-\tau\right)}e^{2\tau}x_2\left(t_0\right)e^{-2\left(\tau-t_0\right)}d\tau\\&=x_1\left(t_0\right)e^{-\left(t-t_0\right)}+x_2\left(t_0\right)e^{-\left(t-2t_0\right)}\int_{t_0}^{t}e^{\tau}d\tau\\&=x_1\left(t_0\right)e^{-\left(t-t_0\right)}+x_2\left(t_0\right)e^{-\left(t-2t_0\right)}\left(e^{t}-e^{t_0}\right)\\&=x_1\left(t_0\right)e^{-\left(t-t_0\right)}+x_2\left(t_0\right)e^{t_0}-x_2\left(t_0\right)e^{-\left(t-3t_0\right)}\end{aligned} \tag{19}$$

Since we can find constant $r\left(R,t_0\right)=\frac{R}{4}e^{-3t_0},\forall R>0$, such that $\|x\left(t_0\right)\|<r\Rightarrow\|x\left(t\right)\|<R,\ t\geq t_0$, so, the equilibrium point at origin is stable.

However, $\lim_{t\to\infty}x_1\left(t\right)=x_2\left(t_0\right)e^{t_0}$, so it is not asymptotically stable.

Problem 4

Show that the following system is globally exponentially stable with a detailed argument.

$$\begin{aligned} \dot{x}_1 &= -\left(5 + x_2^5 + x_3^8\right) x_1 \\ \dot{x}_2 &= -x_2 + 4x_3^2 \\ \dot{x}_3 &= -\left(2 + \sin t\right) x_3 \end{aligned} \tag{20}$$

Solution:

Let $a(t) = 2 + \sin t$. Then

$$x_3(t) = x_3(t_0)\, e^{-\int_{t_0}^{t} (2+\sin\tau) d\tau} \Longrightarrow \|x_3\| e^{-(t-t_0)} \tag{21}$$

Therefore,

$$x_2(t) = e^{-(t-t_0)} x_2(t_0) + \int_{t_0}^{t} e^{-(t-\tau)} 4x_3^2(\tau)\, d\tau \tag{22}$$

Thus, it is ready to see that the system is globally exponentially stable upon using Proposition 1 on the x_1 subsystem (Slotine et al., 1991).

Problem 5

(i) For the autonomous system $\dot{x} = f(x), x \in R^n$, show that, if, in a certain neighborhood Ω of the origin, there exists a continuously differentiable scalar function $V(x)$ such that

- $V(0) = 0 \quad \forall t \geq 0$
- $V(x)$ can assume strictly positive values arbitrarily close to the origin.
- $\dot{V}(x)$ is positive definite (locally in Ω)

then the equilibrium point 0 is unstable.

(ii) Show that the E.P. of $\dot{x} = c(x)$ is unstable where $c(x)$ is continuous and satisfies $xc(x) > 0, x \neq 0$.

Hint: Let $R > 0$ be such that $\dot{V}$ is P.D. on $B_R = \left\{x | \|x\|^2 \leq R^2\right\}$ and $B_R \subset \Omega$, and let

$$M = \max_{x \in B_R} V(x) \tag{23}$$

V is continuous & B_R compact $\Longrightarrow$ M exists. Also, $M > 0$ since $V(x)$ can assume strictly positive values arbitrarily close to the origin. For any $R > r > 0$, there exists $x(0)$ such that $0 < \|x(0)\| < r$, and $V(x(0)) = a > 0$. Since $\dot{V}(x)$ is positive definite (locally in Ω), $V(x(t)) > V(x(0)) > 0$ for all $t \geq 0$. Let $U = \{x | x \in B_R \text{ and } V(x)\} \geq a$. Then U is compact. Thus there exists $L > 0$ such that

$$L = \min_{x \in U} \left\{\dot{V}\right\} \tag{24}$$

If $\|x(t, x_0)\| < R$ for all $t \geq 0$, then

$$\begin{aligned} V(x(t, x_0)) - V_0(x_0) = \int_0^t \dot{V}(x(t, x_0))\, dt \geq \int_0^t L dt = Lt \\ \Longrightarrow V(x(t, x_0)) \geq V_0(x_0) + Lt > M \end{aligned} \tag{25}$$

when $t > \frac{M - V(x_0)}{L}$ which contradicts Equation (23). Thus, the E.P. is unstable.

Solution:

(i) Let $R > 0$ be such that $\dot{V}$ is P.D. on $B_R = \left\{x | \|x\|^2 \leq R^2\right\}$ and $B_R \subset \Omega$, and let

$$M = \max_{x \in B_R} V(x) \tag{26}$$

V is continuous & B_R compact $\Longrightarrow$ M exists. Also, $M > 0$ since $V(x)$ can assume strictly positive values arbitrarily close to the origin. For any $R > r > 0$, there exists $x(0)$ such that $0 < \|x(0)\| < r$, and $V(x(0)) = a > 0$. Since $\dot{V}(x)$ is positive definite (locally in Ω), $V(x(t)) > V(x(0)) > 0$ for all $t \geq 0$. Let $U = \{x | x \in B_R \text{ and } V(x)\} \geq a$. Then U is compact. Thus there exists $L > 0$ such that

$$L = \min_{x \in U} \left\{\dot{V}\right\} \tag{27}$$

If $\|x(t, x_0)\| < R$ for all $t \geq 0$, then

$$\begin{aligned} V(x(t, x_0)) - V_0(x_0) = \int_0^t \dot{V}(x(t, x_0))\, dt \geq \int_0^t L dt = Lt \\ \Longrightarrow V(x(t, x_0)) \geq V_0(x_0) + Lt > M \end{aligned} \tag{28}$$

when $t > \frac{M - V(x_0)}{L}$ which contradicts Equation (23). Thus, the E.P. is unstable.

(ii) If we take $V(x) = x^2$, we can see that $V(0) = 0 \quad \forall t \geq 0$. And $\dot{V}(x) = 2x\dot{x} = 2xc(x) > 0, \forall x \neq 0$, so $\dot{V}$ is globally positive definite.

Therefore, the equilibrium point of $\dot{x} = c(x)$ is unstable.

References

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.

香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #7

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Show that if a function $x : [0, \infty) \to R^n$ is uniformly continuous, and there exists a positive definite quadratic function $V(x)$ such that

$$\int_0^\infty V(x(t))\, dt < 0 \tag{1}$$

then $x(t)$ tends to zero as $t \to \infty$.

Solution:

Because $V(x)$ is a positive definite quadratic function, we can express $V(x(t))$ as follows

$$V(x(t)) = x^T P x \tag{2}$$

where P is a positive definite matrix.

Define $f(t) = \int_0^t x(\tau)\, d\tau$. We claim that $f(t)$ has a finite limit as $t \to \infty$. Otherwise, if

$$\lim_{t\to\infty} f(t) = \int_0^\infty x(\tau)\, d\tau = \infty \tag{3}$$

we will have

$$\int_0^\infty \|x(\tau)\|^2 d\tau = \infty \tag{4}$$

In addition,

$$\int_0^\infty \|x(\tau)\|^2 d\tau \le \lambda_{max}(P) \int_0^\infty x^T P x dt = \lambda_{max}(P) \int_0^\infty V(x(t))\, dt \tag{5}$$

Hence,

$$\int_0^\infty V(x(t))\, dt = \infty \tag{6}$$

which is contradicted to Equation (1). Therefore, $f(t)$ has a finite limit as $t \to \infty$.

Besides, because $x : [0, \infty) \to R^n$ is uniformly continuous, that is $\dot{f}(t)$ is uniformly continuous, by Barbalat's Lemma 4.2, we can conclude that $\dot{f}(t) \to 0$ as $t \to 0$, that is $x(t)$ tends to zero as $t \to \infty$.

Problem 2

Consider the following one dimensional single-input nonlinear control system

$$\dot{x} = \theta g(x, t) + u \tag{7}$$

where θ is some constant parameter, and $g(x, t)$ is some bounded smooth function defined for all t and x.

(a) Assuming θ is known, show that, under the following state feedback nonlinear controller

$$u = -\theta g(x, t) - kx \tag{8}$$

where $k > 0$, the equilibrium point of the closed-loop system is globally asymptotically stable.

(b) If θ is unknown, the feedback controller $u = -\theta g(x, t) - kx$ is not implementable. One can use adaptive control to control the system. Show that, under the following adaptive controller

$$\begin{aligned} u &= -\hat{\theta} g(x, t) - kx \\ \dot{\hat{\theta}} &= g(x, t)\, x \end{aligned} \tag{9}$$

the closed-loop system takes the following form

$$\begin{aligned} \dot{x} &= \phi g(x, t) - kx \\ \dot{\hat{\theta}} &= g(x, t)\, x \end{aligned} \tag{10}$$

where $\phi = \theta - \hat{\theta}$ (you can interpret $\hat{\theta}$ as an estimation of θ).

(c) Using a Lyapunov-like function $V = x^2 + \phi^2$ to show that both x and $\hat{\theta}$ are bounded and $\lim_{t\to\infty} x(t) = 0$.

(d) For $g(x, t) = cos(x)\, sin(t)$ and $k = 2$, do the simulation for the closed-loop system using MATLAB with $x(0) = 0$ and $\hat{\theta}(0) = 1$. Plot $x(t)$; $\hat{\theta}(t)$; $\phi(t)$; $u(t)$ for $0 < t < 40$ seconds.

Solution:

(a) Substituting the controller in Equation (8) into the system in Equation (7) yields that

$$\dot{x} = \theta g(x, t) + u = \theta g(x, t) - \theta g(x, t) - kx = -kx \tag{11}$$

Because $k > 0$, according to linear system stability theory, $-k$ is Hurwitz. Therefore, the equilibrium point of the closed-loop system is globally asymptotically stable.

(b) Substituting the controller in Equation (9) into the system in Equation (7) yields that

$$\begin{aligned} \dot{x} &= \theta g(x,t) + u \\ u &= -\hat{\theta} g(x,t) - kx \\ \dot{\hat{\theta}} &= g(x,t)\, x \end{aligned} \implies \begin{aligned} \dot{x} &= \phi g(x,t) - kx \\ \dot{\hat{\theta}} &= g(x,t)\, x \end{aligned} \tag{12}$$

where $\phi = \theta - \hat{\theta}$ (you can interpret $\hat{\theta}$ as an estimation of θ).

(c) $V = x^2 + \phi^2$ is lower bounded obviously. And its derivative

$$\begin{aligned} \dot{V} &= 2x\dot{x} + 2\phi\dot{\phi} \\ &= 2x(\phi g(x,t) - kx) + 2\phi\left(\dot{\theta} - \dot{\hat{\theta}}\right) \\ &= 2x(\phi g(x,t) - kx) + 2\phi(0 - g(x,t)\,x) \\ &= -2kx^2 \leq 0 \end{aligned} \tag{13}$$

This implies that $V(x(t)) \leq V(x(0)), \forall t > 0$, which indicate that x and ϕ should all be bounded. Taking the derivative to $\dot{V}$ yields that

$$\ddot{V} = -4kx\dot{x} = -4kx(\phi g(x,t) - kx) \tag{14}$$

Here, x, ϕ, and $g(x,t)$ are all bounded. Therefore, $\ddot{V}$ is bounded. According to Barbalat's Lemma 4.1, $\dot{V}$ is uniformly continuous. Again, using Barbalat's Lemma 4.3, $\dot{V}(x,t) \to 0$ as $t \to \infty$, which means $\lim_{t\to\infty} x(t) = 0$.

(d) We set $\theta = \pi$ and use following Simulink to get the results:

And we use the following code to plot the results:

```
clear all; clc;
figg1 = openfig('x.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "x"+".pdf";
saveas(gcf,filename);
close(figg1);
figg2 = openfig('theta.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\hat{\theta}$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
```

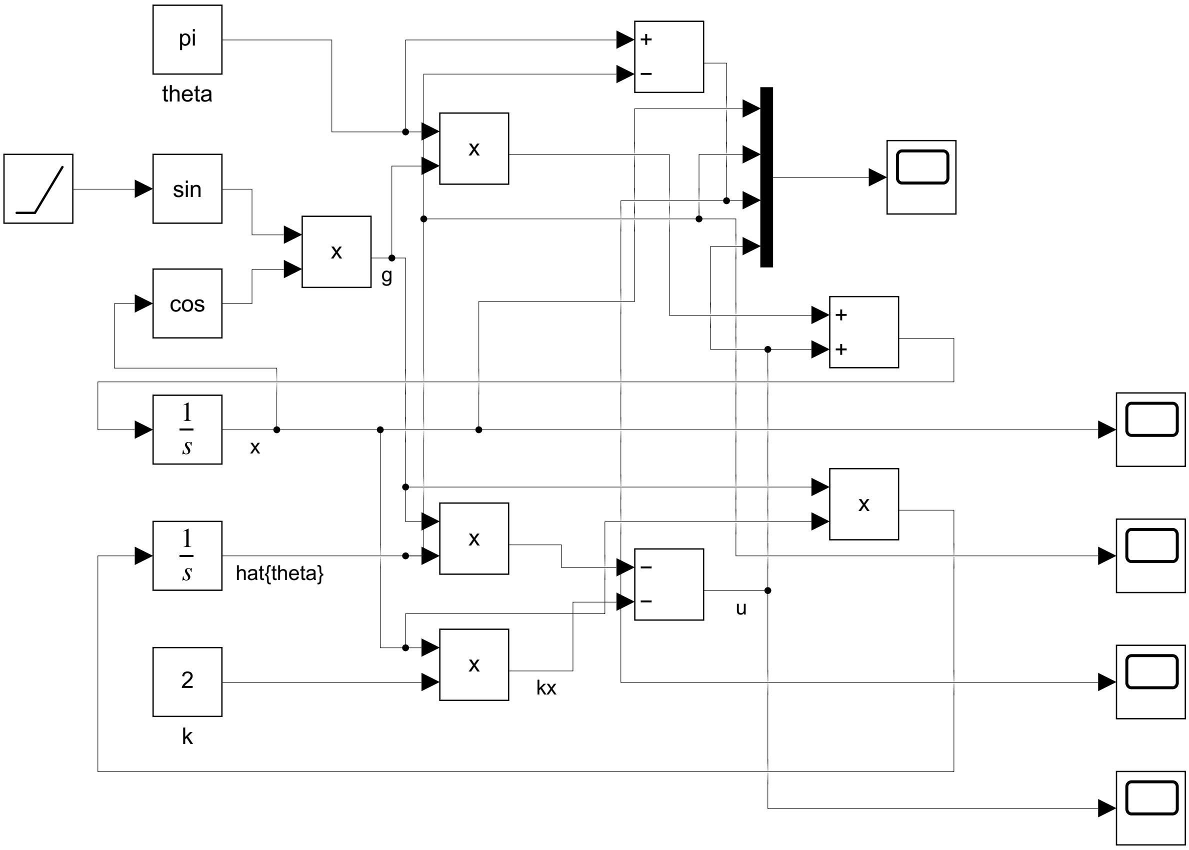


Figure 1: Block diagram for the system.

```
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "theta"+".pdf";
saveas(gcf,filename);
close(figg2);
figg3 = openfig('phi.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\phi$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "phi"+".pdf";
saveas(gcf,filename);
close(figg3);
figg4 = openfig('u.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$u$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
```

```
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "u"+".pdf";
saveas(gcf,filename);
close(figg4);
```

The results are shown as follows:

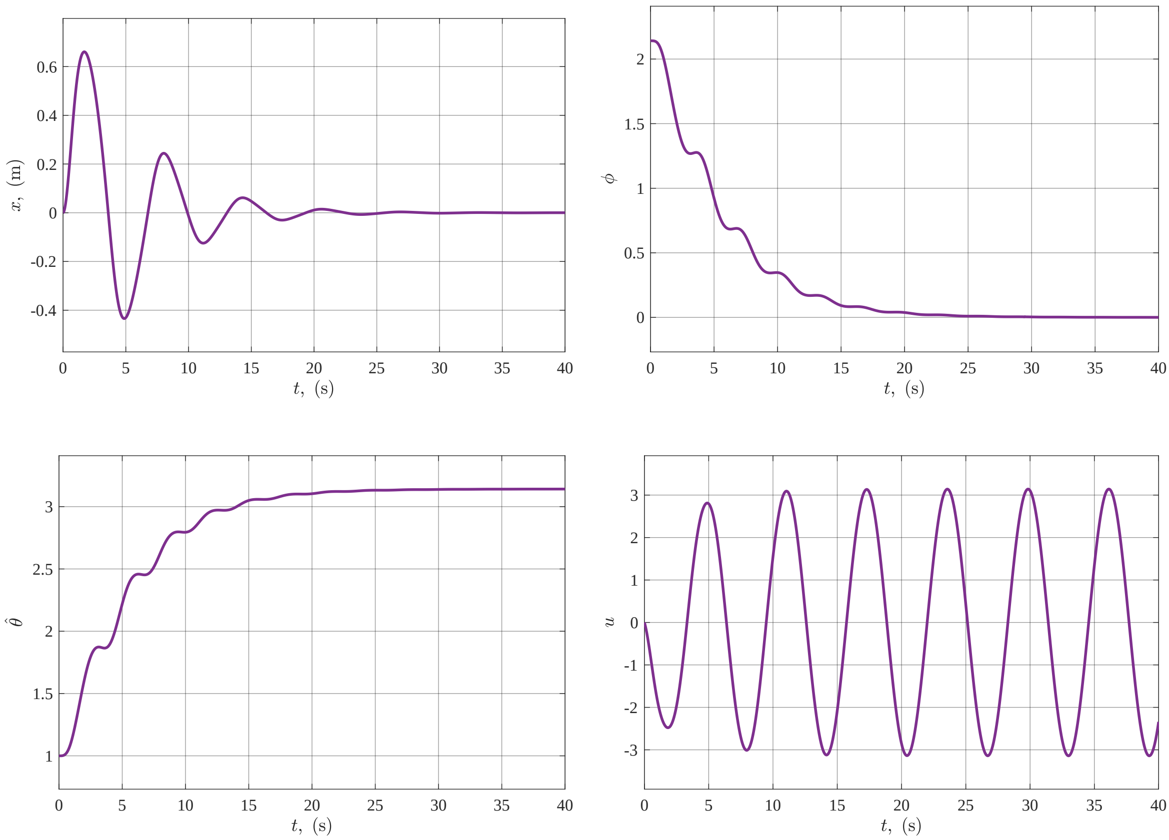


Figure 2: Simulation results.

Problem 3

Consider the system $\dot{x} = f(x)$ where $x \in R^n$, and f is continuously differentiable with $f(0) = 0$. Let J be the Jacobian matrix of f at the origin. It is known from the Lyapunov's linearization method that the equilibrium at $x = 0$ of this system is unstable if at least one of the eigenvalues of J has positive real part. Prove a special case of this result by assuming that $n = 2$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Hint: Let $V(x) = x_1^2 - x_2^2$.

Solution:

Because $J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, the system can be described as

$$\begin{cases} \dot{x}_1 = x_1 + g_1(x_1, x_2) \\ \dot{x}_2 = -2x_2 + g_2(x_1, x_2) \end{cases} \tag{15}$$

where $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are higher order terms.

Define $V(x) = x_1^2 - x_2^2$ satisfies $V(0) = 0$ and we can assume positive values arbitrarily near the origin. Take the time derivative to $V(x)$:

$$\begin{aligned} \dot{V}(x) &= 2x_1\dot{x}_1 - 2x_2\dot{x}_2 \\ &= 2x_1(x_1 + g_1(x_1, x_2)) - 2x_2(-2x_2 + g_2(x_1, x_2)) \\ &= 2x_1^2 + 4x_2^2 + 2x_1g_1(x_1, x_2) - 2x_2g_2(x_1, x_2) \end{aligned} \tag{16}$$

Because $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are higher order terms, $\forall \alpha_{1,2} > 0$, $\exists r > 0$ such that $\|g_i(x_1, x_2)\| < \alpha_i \|x_i\|, i = 1, 2, \forall x \in B_r$. Therefore,

$$\begin{aligned} \dot{V}(x) &= 2x_1^2 + 4x_2^2 + 2x_1g_1(x_1, x_2) - 2x_2g_2(x_1, x_2) \\ &\geq 2\|x_1\|^2 + 4\|x_2\|^2 - 2\alpha_1\|x_1\|^2 - 2\alpha_2\|x_2\|^2 \\ &= (2 - 2\alpha_1)\|x_1\|^2 + (4 - 2\alpha_2)\|x_2\|^2 \end{aligned} \tag{17}$$

as long as we select $0 < \alpha_1 < 1$ and $0 < \alpha_2 < \frac{1}{2}$, $\dot{V}(x)$ can defined as positive definite in B_r.

Problem 4

Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_1 + u \end{aligned} \tag{18}$$

using backstepping to design a state feedback controller to globally stabilize the origin.

Solution:

Letting $u = u_a - x_1$ gives

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= u_a \end{aligned} \tag{19}$$

In this form, $\eta = x_1$ and $\zeta = x_2$.

$$f(\eta) = 0 \tag{20}$$

and

$$g(\eta) = \eta \tag{21}$$

Therefore,

$$\phi(x_1) = \frac{-\alpha\eta - f(\eta)}{g(\eta)} = -\alpha \tag{22}$$

Here, $\alpha > 0$. The closed-loop system has a control Lyapunov function

$$V(x_1) = \frac{1}{2}x_1^2 \tag{23}$$

By Lemma 1, we can have a control Lyapunov function

$$\begin{aligned} V_\alpha(x_1, x_2) &= V(x_1) + \frac{1}{2}(x_2 - \phi(x_1))^2 \\ &= \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \alpha)^2 \end{aligned} \tag{24}$$

with respect to

$$\begin{aligned} u_a &= \phi_\alpha(x_1, x_2) \\ &= \frac{\partial\phi(\eta)}{\partial\eta}[f(\eta) + g(\eta)\zeta] - \frac{\partial V(\eta)}{\partial\eta}g(\eta) - k(\zeta - \phi(\eta)) = -x_1^2 - k(x_2 + \alpha) \end{aligned} \tag{25}$$

where $k > 0$. Therefore,

$$u = u_a - x_1 = -x_1^2 - x_1 - k(x_2 + \alpha) \tag{26}$$

Problem 5

(a) Using the Lyapunov function candidate $V(x) = x^2$ to determine the stability of the origin of the following system

$$\dot{x} = -x^3 + x^2 \sin^2 x \tag{27}$$

(b) Using backstepping to design a state feedback controller to globally stabilize the origin of the following system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 \sin^2 x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_3 + \exp(x_2) u \end{aligned} \tag{28}$$

Solution:

(a) $V(x) = x^2$ is positive definite. Taking the time derivative to $V(x)$ yields

$$\dot{V}(x) = 2x\dot{x} = 2x\left(-x^3 + x^2 \sin^2 x\right) = -2\left(x^4 - x^3 \sin^2 x\right) < -2\left(x^4 - x^3 x\right) = 0 \tag{29}$$

Hence, $\dot{V}(x)$ is negative definite. Therefore, the system is globally asymptotically stable.

(b) Letting $u = \exp(-x_2)(u_a - x_3)$ gives

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 \sin^2 x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_a \end{aligned} \tag{30}$$

In this form, $\eta = x_1$ and $\zeta = x_2$.

$$f(\eta) = -x_1^3 \tag{31}$$

and

$$g(\eta) = x_1 \sin^2 x_1 \tag{32}$$

From part (a), we can have a control Lyapunov function

$$V(x_1) = \frac{1}{2} x_1^2 \tag{33}$$

with respect to

$$\phi(x_1) = x_1 \tag{34}$$

By Lemma 1, we can have a control Lyapunov function

$$\begin{aligned} V_\alpha(x_1, x_2) &= V(x_1) + \frac{1}{2}\left(x_2 - \phi(x_1)\right)^2 \\ &= \frac{1}{2} x_1^2 + \frac{1}{2}\left(x_2 - x_1\right)^2 \end{aligned} \tag{35}$$

with respect to

$$\begin{aligned}\phi_\alpha(x_1,x_2)&\\&=\frac{\partial\phi(\eta)}{\partial\eta}\left[f(\eta)+g(\eta)\zeta\right]-\frac{\partial V(\eta)}{\partial\eta}g(\eta)-k(\zeta-\phi(\eta))\\&=-x_1^3+x_1x_2\sin^2x_1-x_1^2\sin^2x_1-k_1(x_2-x_1)\end{aligned}\tag{36}$$

where $k_1>0$. Applying the extension of Lemma 1 to the whole system yields

$$\begin{aligned}V(x_1,x_2,x_3)&=V_\alpha(x_1,x_2)+\frac{1}{2}(x_3-\phi(x_1,x_2))^2\\&=\frac{1}{2}x_1^2+\frac{1}{2}(x_2-x_1)^2\\&\quad+\frac{1}{2}\left(x_3+x_1^3-x_1x_2\sin^2x_1+x_1^2\sin^2x_1+k_1(x_2-x_1)\right)^2\end{aligned}\tag{37}$$

with respect to

$$\begin{aligned}u_a&=\phi_\alpha(x_1,x_2,x_3)\\&=\frac{\partial\phi_\alpha(x_1,x_2)}{\partial(x_1,x_2)}\left[f(x_1,x_2)+g(x_1,x_2)x_3\right]-\frac{\partial V_\alpha(x_1,x_2)}{\partial(x_1,x_2)}g(x_1,x_2)-k(x_3-\phi_\alpha(x_1,x_2))\\&=\left(-3x_1^2+x_2\sin^2x_1+2x_1x_2\sin x_1\cos x_1-2x_1\sin^2x_1-2x_1^2\sin x_1\cos x_1+k_1\right)\left(-x_1^3+x_1x_2\sin^2x_1\right)\\&\quad+\left(x_1\sin^2x_1+k_1\right)x_3-(x_2-x_1)-k_2\left(x_3+x_1^3-x_1x_2\sin^2x_1+x_1^2\sin^2x_1+k_1(x_2-x_1)\right)\end{aligned}\tag{38}$$

Therefore,

$$\begin{aligned}u&=\exp(-x_2)(u_a-x_3)\\&=\exp(-x_2)\left(-3x_1^2+x_2\sin^2x_1+2x_1x_2\sin x_1\cos x_1-2x_1\sin^2x_1-2x_1^2\sin x_1\cos x_1+k_1\right)\\&\quad\cdot\left(-x_1^3+x_1x_2\sin^2x_1\right)+\exp(-x_2)\left(x_1\sin^2x_1+k_1\right)x_3-\exp(-x_2)(x_2-x_1)\\&\quad-k_2\exp(-x_2)\left(x_3+x_1^3-x_1x_2\sin^2x_1+x_1^2\sin^2x_1+k_1(x_2-x_1)\right)\end{aligned}\tag{39}$$

香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #8

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Consider the controlled van del Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon \left(1 - x_1^2\right) x_2 + u, \epsilon > 0 \\ y &= x_1 \end{aligned} \tag{1}$$

(a) Calculate the relative degree of the system.

(b) Find a state feedback control law so that the equilibrium point at the origin of the closed-loop is globally asymptotically stable.

Solution:

(a)

$$\dot{y} = \dot{x}_1 = x_2 \tag{2}$$

$$\ddot{y} = \dot{x}_2 = -x_1 + \epsilon \left(1 - x_1^2\right) x_2 + u = \alpha\left(x\right) + \beta\left(x\right) u \tag{3}$$

Therefore, the relative degree of the system is 2.

(b) The state feedback control law so that the equilibrium point at the origin of the closed-loop is globally asymptotically stable is shown as follows:

$$\begin{aligned} u &= \frac{y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)} - \alpha\left(x\right)}{\beta\left(x\right)} \\ &= -\alpha_1 \dot{e} - \alpha_2 e - \alpha\left(x\right) \\ &= -\alpha_1 \dot{y} - \alpha_2 y - \alpha\left(x\right) \\ &= -\alpha_1 x_2 - \alpha_2 x_1 + x_1 - \epsilon \left(1 - x_1^2\right) x_2 \end{aligned} \tag{4}$$

Because $\rho = n$, the closed-loop system can always be made an asymptotically stable linear system.

Choosing $\alpha_1 = 1$ and $\alpha_2 = 2$, the control law becomes

$$u = -x_2 - x_1 \tag{5}$$

Problem 2

The motion equation of a single-link robot manipulator is given by

$$J\ddot{\theta} + MgL\sin\theta = u \tag{6}$$

(a) Give the state space equation of (6) with $x_1 = \theta$, $x_2 = \dot{\theta}$, and $y = x_1$

(b) Assume $J = 5$, $gL = 1$, and $M = 10$. Let $y_d(t)$ be a sufficiently smooth time function over $t \in [0, \infty)$. Let $e(t) = y(t) - y_d(t)$. Design a state feedback control law so that $e(t)$ satisfies $\ddot{e}(t) + 2\dot{e}(t) + e(t) = 0$.

(c) Check your design in simple simulation for $y_d(t)$ to be a unit step input, and a sinusoidal function $\sin t$, respectively.

Solution:

(a) The state space equation of (6) with $x_1 = \theta$, $x_2 = \dot{\theta}$, and $y = x_1$ is shown as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{MgL\sin x_1}{J} + \tfrac{1}{J}u \\ y &= x_1 \end{aligned} \tag{7}$$

(b) Because $J = 5$, $gL = 1$, and $M = 10$, the state space equation of (6) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\sin x_1 + 0.2u \\ y &= x_1 \end{aligned} \tag{8}$$

Then, we will find the relative degree ρ:

$$\dot{y} = \dot{x}_1 = x_2 \tag{9}$$

$$\ddot{y} = \dot{x}_2 = -2\sin x_1 + 0.2u = \alpha(x) + \beta(x)u \tag{10}$$

Therefore, the relative degree of the system is 2. state feedback control law so that the system is globally asymptotically stable

$$\begin{aligned} u &= \frac{y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)} - \alpha(x)}{\beta(x)} \\ &= \frac{\ddot{y}_d(t) - \alpha_1\dot{e}(t) - \alpha_2 e(t) - \alpha(x)}{\beta(x)} \\ &= 5\left(\ddot{y}_d(t) - \alpha_1\dot{e}(t) - \alpha_2 e(t) + 2\sin x_1\right) \end{aligned} \tag{11}$$

Substituting Equation (11) into Equation (10) obtains

$$\ddot{y}(t) = \ddot{y}_d(t) - \alpha_1\dot{e}(t) - \alpha_2 e(t) \tag{12}$$

That is

$$\ddot{e}(t) + \alpha_1 \dot{e}(t) + \alpha_2 e(t) = 0 \tag{13}$$

Therefore, $\alpha_1 = 2$ and $\alpha_2 = 1$ can satisfy the requirements of $\ddot{e}(t) + 2\dot{e}(t) + e(t) = 0$. Hence, the state feedback control law is

$$u = 5\left(\ddot{y}_d(t) - 2(x_2 - \dot{y}_d) - (x_1 - y_d) + 2\sin x_1\right) \tag{14}$$

(c) We use the following Simulink to get the results:

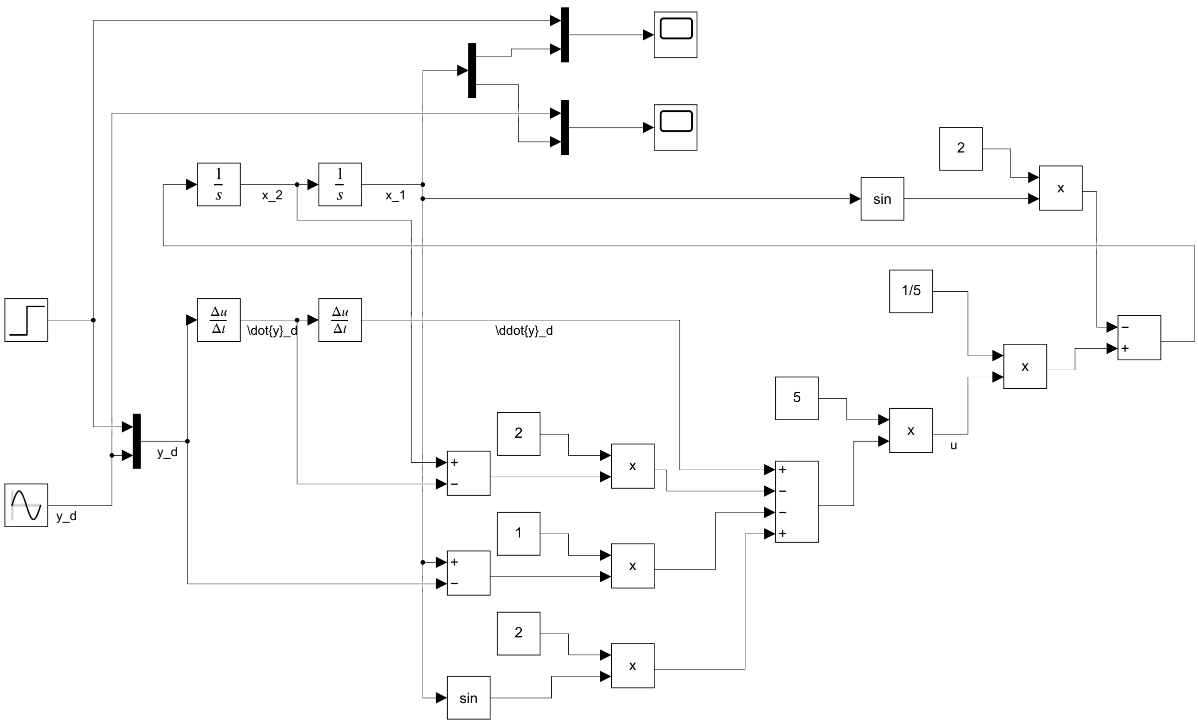


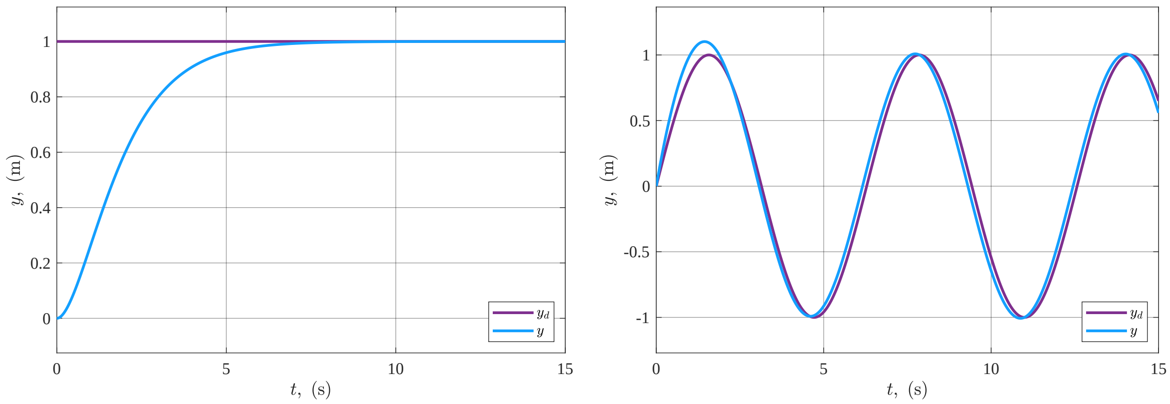
Figure 1: Block diagram for the system.

And we use the following code to plot the results:

```
clear all; clc;
figg1 = openfig('Q2Step.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$y, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
legend('$y_d$', '$y$', 'interpreter','latex','Location','southeast');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2Step"+".pdf";
saveas(gcf,filename);
close(figg1);
figg2 = openfig('Q2Sine.fig','reuse');
```

```
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$y, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
legend('$y_d$', '$y$', 'interpreter','latex','Location','southeast');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2Sine"+".pdf";
saveas(gcf,filename);
close(figg2);
```

The results are shown as follows:



(a) Step input. **(b)** Sinusoidal input.

Figure 2: Simulation results.

Problem 3

Another way to achieve asymptotic tracking: Consider

$$y^{(n)} = \alpha(x) + \beta(x) u \tag{15}$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \alpha(x) + \beta(x) u \\ y &= x_1 \end{aligned} \tag{16}$$

where $x = \begin{bmatrix} y & \dot{y} & \cdots & y^{(n-1)} \end{bmatrix}$, $\alpha(x)$ and $\beta(x)$ are known and $\beta(x) \neq 0$ for all x. Given $y_d(t)$, let $e(t) = y(t) - y_d(t)$ and define

$$s = e^{(n-1)} + \alpha_1 e^{(n-2)} + \ldots + \alpha_{n-1} e \tag{17}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are such that

$$\lambda^{n-1} + \alpha_1 \lambda^{n-2} + \ldots + \alpha_{n-2}\lambda + \alpha_{n-1} \tag{18}$$

is a stable polynomial.

(a) Design a control law such that

$$\dot{s} + ks = 0 \tag{19}$$

where $k > 0$.

(b) Show that the control law achieves $\lim_{t\to\infty} e(t) = 0$.

(c) Show that, when $y_d = 0$, the closed-loop system is globally asymptotically stable.

Solution:

(a) To achieve asymptotic tracking for Equation (16), note that using an input transformation

$$\alpha(x) + \beta(x) u = u_a \tag{20}$$

or

$$u = \frac{u_a - \alpha(x)}{\beta(x)} \tag{21}$$

gives

$$y^{(n)} = u_a \tag{22}$$

which is in the chain integrator form. In order to achieve Equation (19) with $\rho = n$, i.e.,

$$\begin{aligned}&\left(y^{(n)} - y_d^{(n)}\right) + (\alpha_1 + k)\left(y^{(n-1)} - y_d^{(n-1)}\right) + (\alpha_2 + k\alpha_1)\left(y^{(n-2)} - y_d^{(n-2)}\right)\\&+ \ldots + (\alpha_{n-1} + k\alpha_{n-2})\left(y^{(1)} - y_d^{(1)}\right) + k\alpha_{n-1}(y - y_d) = 0\end{aligned} \tag{23}$$

Substituting Equation (22) into (23) gives

$$\begin{aligned}&\left(u_a - y_d^{(n)}\right) + (\alpha_1 + k)\left(y^{(n-1)} - y_d^{(n-1)}\right) + (\alpha_2 + k\alpha_1)\left(y^{(n-2)} - y_d^{(n-2)}\right)\\&+ \ldots + (\alpha_{n-1} + k\alpha_{n-2})\left(y^{(1)} - y_d^{(1)}\right) + k\alpha_{n-1}(y - y_d) = 0\\\Longrightarrow & u_a = y_d^{(n)} - \sum_{i=1}^{n}(\alpha_i + k\alpha_{i-1})\, e^{n-i}\end{aligned} \tag{24}$$

Here $\alpha_0 = 1$ and $\alpha_n = 0$. Thus,

$$u = \frac{u_a - \alpha(x)}{\beta(x)} = \frac{y_d^{(n)} - \sum_{i=1}^{n}(\alpha_i + k\alpha_{i-1})\, e^{n-i} - \alpha(x)}{\beta(x)} \tag{25}$$

(b) Because

$$s = e^{(n-1)} + \alpha_1 e^{(n-2)} + \ldots + \alpha_{n-1} e \tag{26}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are such that

$$\lambda^{n-1} + \alpha_1\lambda^{n-2} + \ldots + \alpha_{n-2}\lambda + \alpha_{n-1} \tag{27}$$

is a stable polynomial, for

$$\dot{s} = e^{(n)} + \alpha_1 e^{(n-1)} + \ldots + \alpha_{n-1} e^{(1)} \tag{28}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are such that

$$\lambda^{n} + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-2}\lambda^2 + \alpha_{n-1}\lambda \tag{29}$$

is also a stable polynomial by Routh-Hurwitz stability criterion. Therefore,

$$\begin{aligned}\dot{s} + ks = e^{(n)} &+ (\alpha_1 + k)\, e^{(n-1)} + (\alpha_2 + k\alpha_1)\, e^{(n-2)}\\&+ \ldots + (\alpha_{n-1} + k\alpha_{n-2})\, e^{(1)} + k\alpha_{n-1} e = 0\end{aligned} \tag{30}$$

where $\alpha_i + k\alpha_{i-1}$, $i = 1, \ldots, n$ are such that

$$\lambda^{n} + (\alpha_1 + k)\,\lambda^{n-1} + \ldots + (\alpha_{n-1} + k\alpha_{n-2})\,\lambda + k\alpha_{n-1} \tag{31}$$

is also a stable polynomial. As a result, e satisfies $\lim_{t\to\infty} e(t) = 0$.

(c) When $y_d = 0$, the closed-loop system is

$$\dot{x} = Ax + Bk(x, 0, \ldots, 0) = (A - B[k\alpha_{n-1}, (\alpha_{n-1} + k\alpha_{n-2}), \ldots, (\alpha_1 + k)])\, x$$
$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -k\alpha_{n-1} & -(\alpha_{n-1} + k\alpha_{n-2}) & -(\alpha_{n-2} + k\alpha_{n-3}) & \ldots & -(\alpha_1 + k) \end{bmatrix} x \tag{32}$$

Clearly, $(A - B[k\alpha_{n-1}, (\alpha_{n-1} + k\alpha_{n-2}), \ldots, (\alpha_1 + k)])$ is a companion matrix with its characteristic polynomial being

$$\lambda^n + (\alpha_1 + k)\lambda^{n-1} + \ldots + (\alpha_{n-1} + k\alpha_{n-2})\lambda + k\alpha_{n-1} \tag{33}$$

Thus, when $y_d = 0$, the closed-loop system is globally asymptotically stable.

香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #9

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= x_3 + \cos(x_1)u \\ \dot{x}_3 &= x_1 + x_2^2 + \lambda x_3 \\ y &= x_1\end{aligned} \tag{1}$$

(a) For what values of λ is the system minimum phase? nonminimum phase?

(b) Assume a state feedback control law $u = k(x)$ is such that $\ddot{y}(t) + 4\dot{y}(t) + 2y(t) = 0$. Is the equilibrium point at the origin of the closed-loop system (locally) asymptotically stable for all $\lambda \in R$? Why or Why not?

Solution:

(a) The Jacobian linearization of the system at the origin is given by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned} \tag{2}$$

with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{3}$$

The transfer function of the above linear system is then given by

$$\begin{aligned} H(s) &= C(sI - A)^{-1} B \\ &= \frac{1}{s(s-1)(s-\lambda)+1} C \begin{bmatrix} s(s-\lambda) & s-\lambda & 1 \\ 1 & (s-1)(s-\lambda) & s-1 \\ s & 1 & s(s-1) \end{bmatrix} B \\ &= \frac{s-\lambda}{s(s-1)(s-\lambda)+1} \end{aligned} \tag{4}$$

The system has a zero at $s = \lambda$. Thus, it is minimum phase for all $\lambda < 0$ and it is nonminimum phase for all $\lambda \geq 0$.

(b) No. Since when $\lambda \geq 0$, the system is nonminimum phase at the origin, the equilibrium cannot be made locally asymptotically stable.

Problem 2

The motion equation of a single-link robot rigid-joint manipulator is given by

$$\ddot{y} + a\dot{y}\sin(y) = \beta(x)u \tag{5}$$

(a) Give the state space equation of Equation (5) with $x_1 = y$ and $x_2 = \dot{y}$.

(b) Assume $1 \le a \le 2$ and $0.5 < \beta(x) < 2.5$, using $\hat{a} = 1.5$ to design a sliding mode control law u such that

$$\frac{ds^2}{dt} \le -|s| \tag{6}$$

where $s = \dot{e} + 2e$ with $e = y - y_d$.

(c) Assume y_d is a unit step function, simulate your design on the closed-loop system consisting of the plant Equation (5) with $a = 2$ and the sliding mode control law with $\hat{a} = 1.5$. Illustrate the performance of your control law by plotting $y(t)$ and $y_d(t)$ in the same figure for $0 \le t \le 20$. Also, plot $u(t)$ for $0 \le t \le 20$.

(d) In the control law designed in part (b), replace $sgn(s)$ by $sat(s/0.2)$, and repeat part (c).

Hint: In simulation, you can let $F(x) = |\alpha(x) - \hat{\alpha}(x)|$ and $\beta(x) = 1$.

Solution:

(a) Let $x_1 = y$ and $x_2 = \dot{y}$, the state-space equation is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_2\sin x_1 + \beta(x)u \\ y = x_1 \end{cases} \tag{7}$$

(b) Define the estimate $\hat{\beta}$ of $\beta(x)$ by $\hat{\beta} = (b_{min}b_{max})^{1/2} = \frac{\sqrt{5}}{2}$ and let $b = (b_{max}/b_{min})^{1/2} = \sqrt{5}$.

Using $\hat{a} = 1.5$ yields that

$$\hat{\alpha} = -1.5x_2\sin x_1 \tag{8}$$

$$\Delta\alpha = \alpha(x) - \hat{\alpha} \Longrightarrow |\Delta\alpha| = |\alpha(x) - \hat{\alpha}| \le 0.5|x_2\sin x_1| \Longrightarrow F(x) = 0.5|x_2\sin x_1| \tag{9}$$

Therefore,

$$\hat{u} = -\hat{\alpha}(x) + \ddot{y}_d - \alpha_1\dot{e} \tag{10}$$

Because $s = \dot{e} + 2e$, $\alpha_1 = 2$. Therefore,

$$\hat{u} = 1.5x_2\sin x_1 + \ddot{y}_d - 2\dot{e} \tag{11}$$

Because the sliding mode control law u should satisfy

$$\frac{ds^2}{dt} \le -|s| \tag{12}$$

we can know that $\eta = \frac{1}{2}$. Therefore, we can let $\phi(x)$ be

$$
\begin{aligned}
\phi(x) &= b\left(F(x)+\eta\right)+(b-1)\left|\hat{u}\right| \\
&= \sqrt{5}\left(0.5\left|x_2\sin x_1\right|+\frac{1}{2}\right)+\left(\sqrt{5}-1\right)\left|1.5x_2\sin x_1+\ddot{y}_d-2\dot{e}\right| \\
&= \sqrt{5}\left(0.5\left|x_2\sin x_1\right|+\frac{1}{2}\right)+\left(\sqrt{5}-1\right)\left|1.5x_2\sin x_1+\ddot{y}_d-2\left(\dot{y}-\dot{y}_d\right)\right| \\
&= \sqrt{5}\left(0.5\left|x_2\sin x_1\right|+\frac{1}{2}\right)+\left(\sqrt{5}-1\right)\left|1.5x_2\sin x_1+\ddot{y}_d-2\left(x_2-\dot{y}_d\right)\right|
\end{aligned}
\tag{13}
$$

Hence, we can design a sliding mode control law u as follows:

$$
\begin{aligned}
u &= \hat{\beta}^{-1}\left[\hat{u}-\phi(x)\,sgn(s)\right] \\
&= \frac{2}{\sqrt{5}}\left[1.5x_2\sin x_1+\ddot{y}_d-2\dot{e}-\phi(x)\,sgn(s)\right] \\
&= \frac{2}{\sqrt{5}}\left[1.5x_2\sin x_1+\ddot{y}_d-2\left(x_2-\dot{y}_d\right)-\phi(x)\,sgn(s)\right]
\end{aligned}
\tag{14}
$$

where

$$
\phi(x) = \sqrt{5}\left(0.5\left|x_2\sin x_1\right|+\frac{1}{2}\right)+\left(\sqrt{5}-1\right)\left|1.5x_2\sin x_1+\ddot{y}_d-2\left(x_2-\dot{y}_d\right)\right| \tag{15}
$$

$$
s = \dot{e}+2e = \left(\dot{y}-\dot{y}_d\right)+2\left(y-y_d\right) = \left(x_2-\dot{y}_d\right)+2\left(x_1-y_d\right) \tag{16}
$$

The MATLAB shown below is used to simulate the performance of the designed controller.

```
clc; clf; clear all;
%% sgn part
[t,x] = ode45('Q9_2_Systemsgn',[0,20],[0 0]);
phi = 0.5+0.5*abs(x(:,2).*sin(x(:,1)));
y_d = 1;
y_ddot = 0;
y_dddot = 0;
y = x(:,1);
ydot = x(:,2);
s = ydot-y_ddot+2*(y-y_d);
u = -phi.*sgn(s)+1.5*x(:,2).*sin(x(:,1))+y_dddot-2*(ydot-y_ddot);
figure(1);
hold on;
plot(t, y,'color',[0.667 0.667 1],'LineWidth',2.5);
plot(t, y_d+t*0,'color',[1 0.5 0],'LineWidth',2.5);
hold off;
grid on;
legend('$y\left(t\right)$','$y_d\left(t\right)$','interpreter','latex');
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$\theta, \ (\mathrm{rad})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
```

```
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-2-yyd-sgn"+".pdf";
saveas(gcf,filename);
figure(2);
plot(t, u,'color',[0.667 0.667 1],'LineWidth',2.5);
grid on;
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$u, \ (\mathrm{N\cdot m})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-2-u-sgn"+".pdf";
saveas(gcf,filename);
%% sat part
[t,x] = ode45('Q9_2_Systemsat',[0,20],[0 0]);
phi = 0.5+0.5*abs(x(:,2).*sin(x(:,1)));
y_d = 1;
y_ddot = 0;
y_dddot = 0;
y = x(:,1);
ydot = x(:,2);
s = ydot-y_ddot+2*(y-y_d);
u = -phi.*sgn(s)+1.5*x(:,2).*sin(x(:,1))+y_dddot-2*(ydot-y_ddot);
figure(3);
hold on;
plot(t, y,'color',[0.667 0.667 1],'LineWidth',2.5);
plot(t, y_d+t*0,'color',[1 0.5 0],'LineWidth',2.5);
hold off;
grid on;
legend('$y\left(t\right)$','$y_d\left(t\right)$','interpreter','latex');
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$\theta, \ (\mathrm{rad})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-2-yyd-sat"+".pdf";
saveas(gcf,filename);
figure(4);
plot(t, u,'color',[0.667 0.667 1],'LineWidth',2.5);
grid on;
```

```
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$u, \ (\mathrm{N\cdot m})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-2-u-sat"+".pdf";
saveas(gcf,filename);
```

where the codes for the system representation are shown below:

```
function xd = Q9_2_Systemsgn(t,x)
    xd(1) = x(2);
    phi = 0.5+0.5*abs(x(2)*sin(x(1)));
    y_d = 1;
    y_ddot = 0;
    y_dddot = 0;
    y = x(1);
    ydot = x(2);
    s = ydot-y_ddot+2*(y-y_d);
    u = -phi*sgn(s)+1.5*x(2)*sin(x(1))+y_dddot-2*(ydot-y_ddot);
    xd(2) = -2*x(2)*sin(x(1))+u;
    xd = xd';
end
```

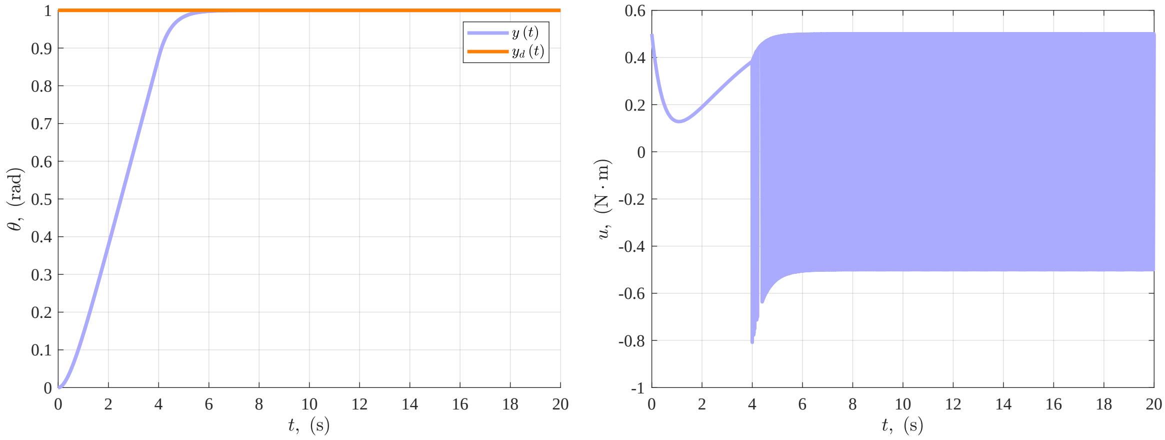
and the sgn function is designed as follows:

```
function y = sgn(s)
    if s == 0
        y = 0;
    else
        y = s./abs(s);
    end
end
```

The simulation results are shown in Figure 1.

(c) The codes for the controller are changed for sat function as shown below:

```
function xd = Q9_2_Systemsat(t,x)
    xd(1) = x(2);
    phi = 0.5+0.5*abs(x(2)*sin(x(1)));
    y_d = 0*t+1;
    y_ddot = 0;
    y_dddot = 0;
    y = x(1);
    ydot = x(2);
```



(a) $y(t)$ and $y_d(t)$ v.s. t. **(b)** u v.s. t.

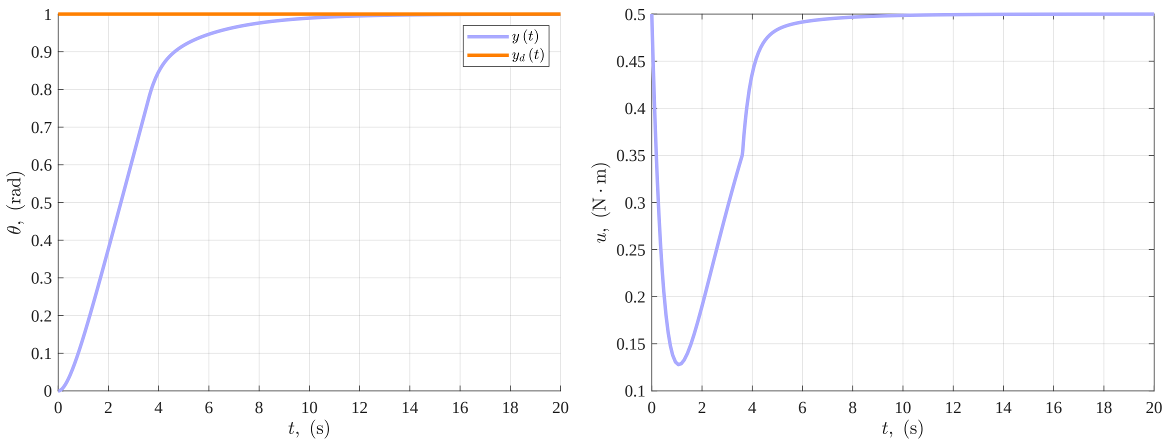
Figure 1: Simulation results for the controller with sgn.

```
    s = ydot-y_ddot+2*(y-y_d);
    u = -phi*sat(s)+1.5*x(2)*sin(x(1))+y_dddot-2*(ydot-y_ddot);
    xd(2) = -2*x(2)*sin(x(1))+u;
    xd = xd';
end
```

and the sat function is designed as follows:

```
function y = sat(s)
    if abs(s) < 0.2
        y = s;
    elseif s < 0
        y = -1;
    else
        y = 1;
    end
end
```

The simulation results are shown in Figure 2.



(a) $y(t)$ and $y_d(t)$ v.s. t.

(b) u v.s. t.

Figure 2: Simulation results for the controller with sat.

Problem 3

Consider the motion equation of a single-link robot rigid-joint manipulator given in Equation (5) where $\beta(x) = 1$ for all x.

(a) Assume $a = 1.5$, $y_d = 2\sin t$ and $s = \dot{e} + 3e$. Design a control law of the form (8.7) of the lecture note with $k = 2$ and simulate the performance of your control law by plotting $y(t)$ and $y_d(t)$ in the same figure for $0 \le t \le 20$. Also, plot $u(t)$ for $0 \le t \le 20$.

(b) Assume the actual value of $a = 2$. Use the same control law as the one in Part (i) to simulate the performance of your control law by plotting $y(t)$ and $y_d(t)$ in the same figure for $0 \le t \le 20$. Also, plot $u(t)$ for $0 \le t \le 20$.

(c) Assume a is unknown, put the system in the form (8.3) of the lecture note and identify a_0, a_1, a_2 and f_1, f_2.

(d) Design an adaptive control law of the form (8.10) and (8.12) of the lecture note with $\gamma_i = 3$. Assume the actual value of $a = 2.5$, respectively. Simulate the performance of your control law by plotting $y(t)$ and $y_d(t)$ in the same figure for $0 \le t \le 20$. Also, plot $u(t)$ and $\hat{a}_i(t)$ for $0 \le t \le 20$.

Hint: Note Part (iv) of Remark 8.1.

Solution:

(a) Because $\beta(x) = 1$, $a_0 = 1$. Consider the control law,

$$u = a_0 f_0(x,t) - ks + \sum_{i=1}^{m} a_i f_i(x,t) \tag{17}$$

where $k = 2$, and

$$f_0(x,t) = \ddot{y}_d - \alpha_1 \dot{e} \tag{18}$$

Because $s = \dot{e} + 3e$, $\alpha_1 = 3$. Therefore, the designed control law for the system is as follows:

$$\begin{aligned} u &= \ddot{y}_d - 3\dot{e} - 2s + 1.5\dot{y}\sin(y) \\ &= \ddot{y}_d - 3(x_2 - \dot{y}_d) - 2((x_2 - \dot{y}_d) + 3(x_1 - y_d)) + 1.5x_2 \sin x_1 \end{aligned} \tag{19}$$

The MATLAB shown below is used to simulate the performance of the designed controller.

```
clc; clf; clear all;
%% Q9-3-a
[t,x] = ode45('Q9_3_a_System',[0,20],[0 0]);
y_d = 2*sin(t);
y_ddot = 2*cos(t);
y_dddot = -2*sin(t);
```

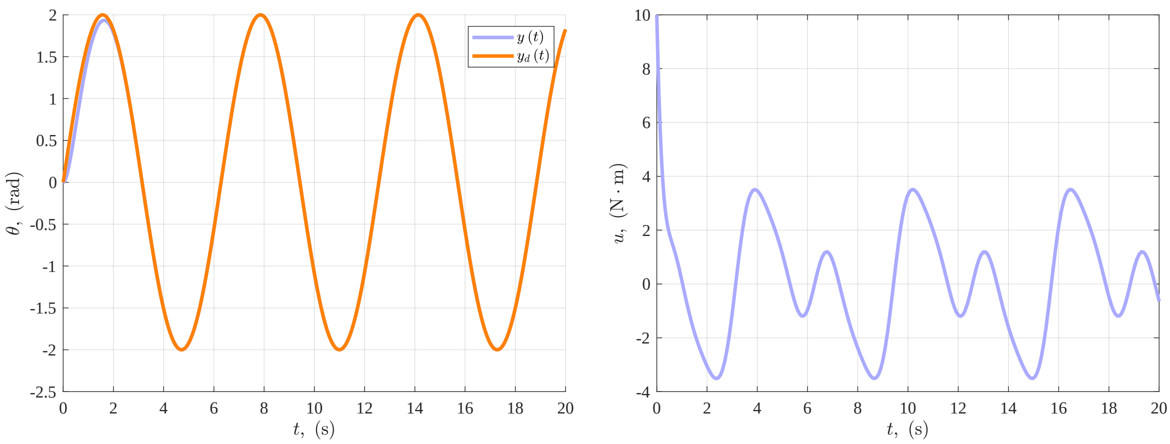
```
y = x(:,1);
ydot = x(:,2);
s = ydot-y_ddot+3*(y-y_d);
a0 = 1;
f0 = y_dddot-3*(ydot-y_ddot);
u = a0*f0-2*s+1.5*x(:,2).*sin(x(:,1));
figure(1);
hold on;
plot(t, y,'color',[0.667 0.667 1],'LineWidth',2.5);
plot(t, y_d+t*0,'color',[1 0.5 0],'LineWidth',2.5);
hold off;
grid on;
legend('$y\left(t\right)$','$y_d\left(t\right)$','interpreter','latex');
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$\theta, \ (\mathrm{rad})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-a-yyd"+".pdf";
saveas(gcf,filename);
figure(2);
plot(t, u,'color',[0.667 0.667 1],'LineWidth',2.5);
grid on;
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$u, \ (\mathrm{N\cdot m})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-a-u"+".pdf";
saveas(gcf,filename);
%% Q9-3-b
[t,x] = ode45('Q9_3_b_System',[0,20],[0 0]);
y_d = 2*sin(t);
y_ddot = 2*cos(t);
y_dddot = -2*sin(t);
y = x(:,1);
ydot = x(:,2);
s = ydot-y_ddot+3*(y-y_d);
a0 = 1;
f0 = y_dddot-3*(ydot-y_ddot);
u = a0*f0-2*s+1.5*x(:,2).*sin(x(:,1));
figure(3);
```

```
hold on;
plot(t, y,'color',[0.667 0.667 1],'LineWidth',2.5);
plot(t, y_d+t*0,'color',[1 0.5 0],'LineWidth',2.5);
hold off;
grid on;
legend('$y\left(t\right)$','$y_d\left(t\right)$','interpreter','latex');
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$\theta, \ (\mathrm{rad})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-b-yyd"+".pdf";
saveas(gcf,filename);
figure(4);
plot(t, u,'color',[0.667 0.667 1],'LineWidth',2.5);
grid on;
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$u, \ (\mathrm{N\cdot m})$','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-b-u"+".pdf";
saveas(gcf,filename);
```

where the codes for the system representation are shown below:

```
function xd = Q9_3_a_System(t,x)
    xd(1) = x(2);
%     phi = 0.5+0.5*abs(x(2)*sin(x(1)));
    y_d = 2*sin(t);
    y_ddot = 2*cos(t);
    y_dddot = -2*sin(t);
    y = x(1);
    ydot = x(2);
    s = ydot-y_ddot+3*(y-y_d);
    a0 = 1;
    f0 = y_dddot-3*(ydot-y_ddot);
    u = a0*f0-2*s+1.5*x(2)*sin(x(1));
    xd(2) = -1.5*x(2)*sin(x(1))+u;
    xd = xd';
end
```

The simulation results are shown in Figure 3.



(a) $y(t)$ and $y_d(t)$ v.s. t. **(b)** u v.s. t.

Figure 3: Simulation results for the controller with for the adaptive control with $k = 2$.

(b) The codes for the system representation are changed as shown below:

```
function xd = Q9_3_b_System(t,x)
    xd(1) = x(2);
%     phi = 0.5+0.5*abs(x(2)*sin(x(1)));
    y_d = 2*sin(t);
    y_ddot = 2*cos(t);
    y_dddot = -2*sin(t);
    y = x(1);
    ydot = x(2);
    s = ydot-y_ddot+3*(y-y_d);
    a0 = 1;
    f0 = y_dddot-3*(ydot-y_ddot);
    u = a0*f0-2*s+1.5*x(2)*sin(x(1));
    xd(2) = -2*x(2)*sin(x(1))+u;
    xd = xd';
end
```

The simulation results are shown in Figure 4.

(c)

$$a_0 y^{(n)} + \sum_{i=1}^{m} a_i f(x,t) = u \tag{20}$$

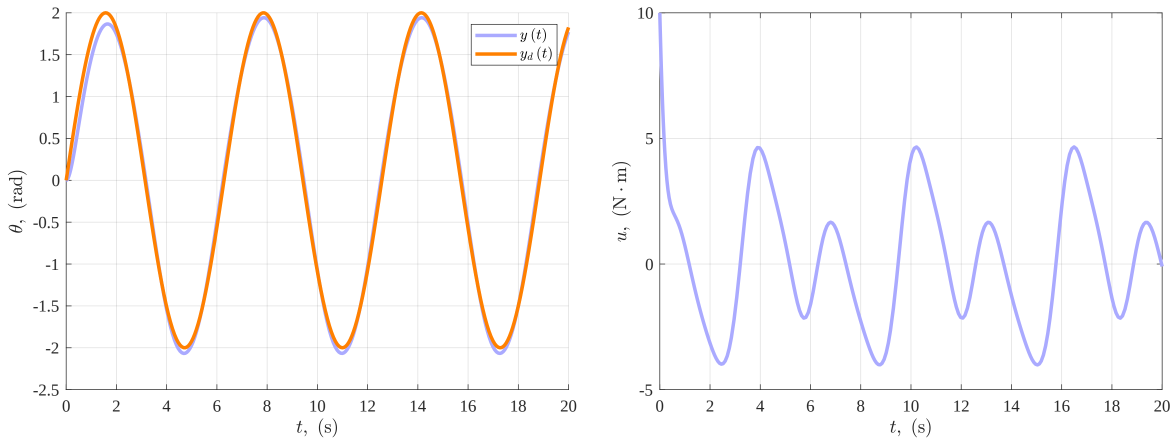
Here, $a_0 = 1$, $a_1 = a$, $a_2 = 0$, $f_1(x) = x_2 \sin x_1$, $f_2(x) = 0$

(d)

$$\begin{aligned} u &= a_0 f_0(x,t) - ks + \textstyle\sum_{i=1}^{m} \hat{a}_i f_i(x,t) \\ \dot{\hat{a}}_i &= -\gamma_i sgn(a_0)\, s f_i, i = 1, \cdots, m \end{aligned} \tag{21}$$

Because $\gamma_i = 3$ and $a_0 = 1$, Equation (21) can be simplified into

$$\begin{aligned} u &= (\ddot{y}_d - 3(x_2 - \dot{y}_d)) - 2((x_2 - \dot{y}_d) + 3(x_1 - y_d)) + \hat{a}_1 x_2 \sin x_1 \\ \dot{\hat{a}}_1 &= -3((x_2 - \dot{y}_d) + 3(x_1 - y_d)) x_2 \sin x_1 \end{aligned} \tag{22}$$



(a) $y(t)$ and $y_d(t)$ v.s. t. **(b)** u v.s. t.

Figure 4: Simulation results for the controller with for the adaptive control with $k = 2$ and the actual value of $a = 2$.

The Simulink as shown in Figure 5 is used to simulate the performance of the designed controller.

And we use the following code to plot the results:

```
clear all; clc;
figg1 = openfig('Q9-3-d-ahat.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\hat{a}$', 'interpreter','latex');
% legend('$y_d$', '$y$', 'interpreter','latex','Location','southeast');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-d-ahat"+".pdf";
saveas(gcf,filename);
close(figg1);
figg2 = openfig('Q9-3-d-u.fig','reuse');
grid on;
xlim([0.07 20]);
ylim([-5 6]);
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$u, \ (\mathrm{N\cdot m})$','interpreter','latex');
% legend('$y_d$', '$y$', 'interpreter','latex','Location','southeast');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
```

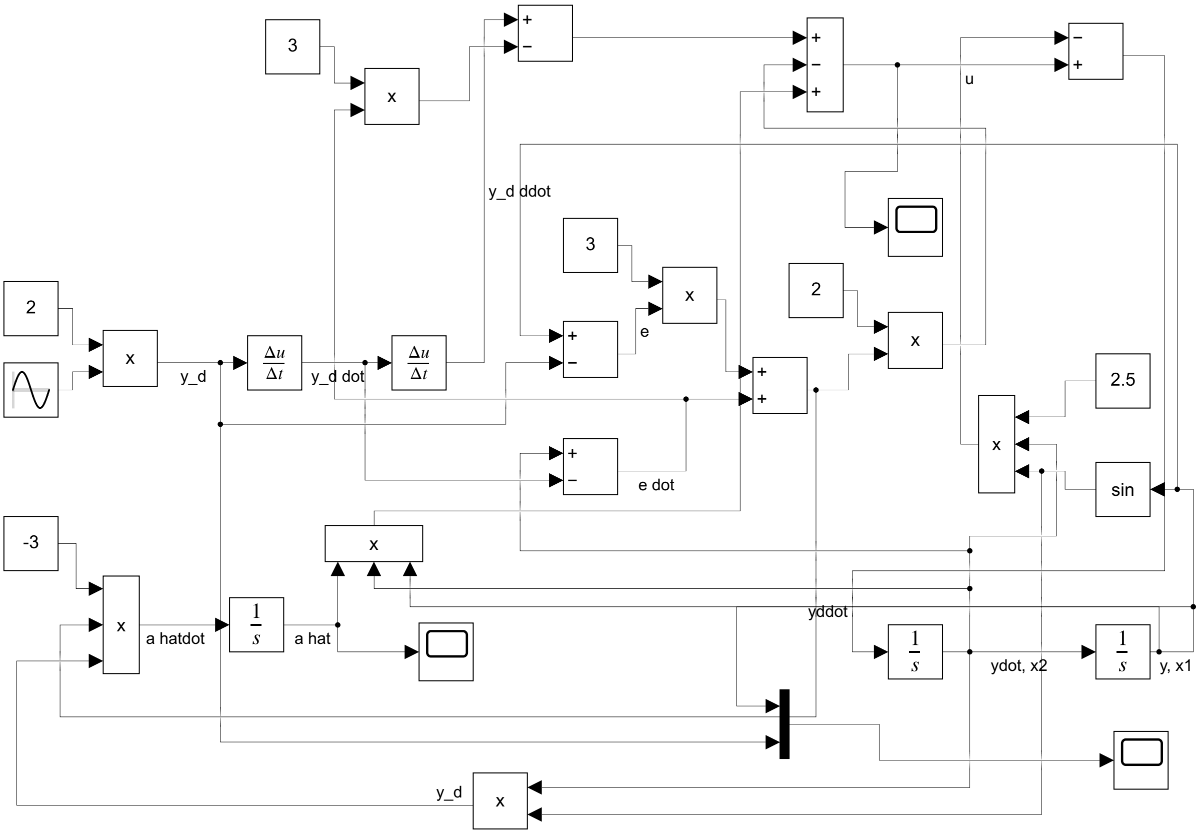
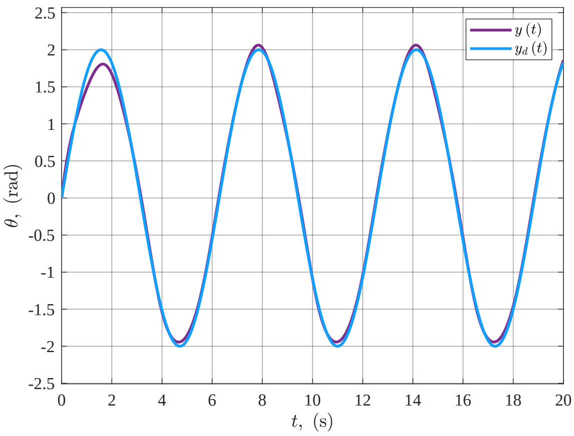


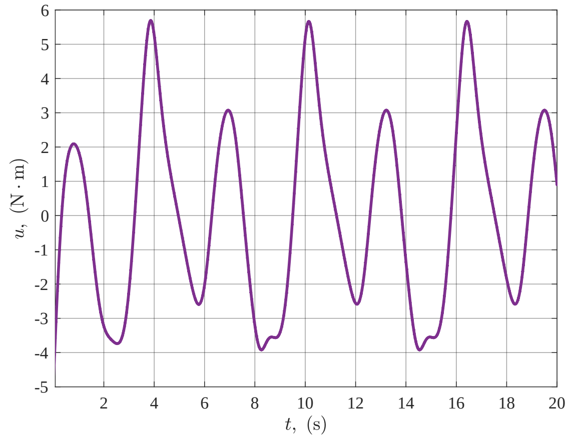
Figure 5: Block diagram for the system.

```
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-d-u"+".pdf";
saveas(gcf,filename);
close(figg2);
figg3 = openfig('Q9-3-d-yyd.fig','reuse');
grid on;
legend('$y\left(t\right)$','$y_d\left(t\right)$','interpreter','latex');
xlabel('$t, \ (\mathrm{s})$','interpreter','latex');
ylabel('$\theta, \ (\mathrm{rad})$','interpreter','latex');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q9-3-d-yyd"+".pdf";
saveas(gcf,filename);
close(figg3);
```

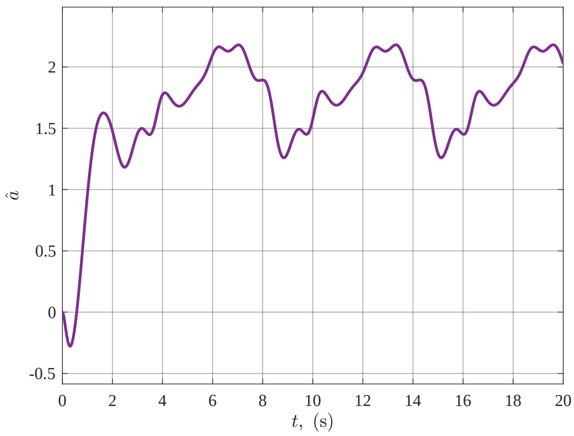
The simulation results are shown in Figure 6.



(a) $y\,(t)$ and $y_d\,(t)$ v.s. t.



(b) u v.s. t.



(c) $\hat{a}$ v.s. t.

Figure 6: Simulation results for the controller with for the adaptive control law of the form (8.10) and (8.12) of the lecture note.