香港中文大學
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Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #1

by

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Problem 1

Consider the following system.

$$\dot{x}(t) = -x(t) + x^2(t), \quad x(0) = x_0$$

(a) Find the equilibrium points for the system.
(b) Verify the solution of the system is given by

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t}, \quad 0 \leq t < T$$

for some $T > 0$.
(c) Show that $T = \ln \frac{x_0}{x_0 - 1}$ when $x_0 > 1$.

Solution:

(a)

In this nonlinear dynamical system,

$$f(x) = -x(t) + x^2(t) \tag{1}$$

Letting $f(x) = 0$ gives

$$-x(t) + x^2(t) = 0 \Rightarrow x^* = \{0, 1\} \tag{2}$$

(b)

For

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t} \tag{3}$$

The left side of the equation is equal to

$$\dot{x}(t) = \left[\frac{x_0}{x_0 + (1 - x_0)e^t}\right]' = \frac{-x_0(1 - x_0)e^t}{[x_0 + (1 - x_0)e^t]^2} \tag{4}$$

The right side of the equation is equal to

$$\begin{aligned} -x(t) + x^2(t) &= -\frac{x_0}{x_0 + (1 - x_0)e^t} + \left[\frac{x_0}{x_0 + (1 - x_0)e^t}\right]^2 = \frac{-x_0\left[x_0 + (1 - x_0)e^t\right] + x_0^2}{[x_0 + (1 - x_0)e^t]^2} \\ &= \frac{-x_0(1 - x_0)e^t}{[x_0 + (1 - x_0)e^t]^2} = \text{left side} \quad \blacksquare \end{aligned} \tag{5}$$

Therefore, $x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t}$ is the solution of the system.

(c)

Equation (3) is not defined for all $t \geq 0$. In fact, it can be seen that when $x_0 > 1$, there exists a finite $t > 0$ such that

$$x_0 + (1 - x_0)e^t = 0 \quad \text{or} \quad t = \ln \frac{x_0}{x_0 - 1} \tag{6}$$

Therefore, Equation (3) is not defined at $t = \ln \frac{x_0}{x_0 - 1}$, which means

$$T = \ln \frac{x_0}{x_0 - 1} \tag{7}$$

Problem 2

It is known that the following Van del Pol equation has a limit cycle.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 0.2(x_1^2 - 1)x_2 \end{aligned}$$

Write a MATLAB program to generate $(x_1(t), x_2(t))$, $0 \le t \le 100$ for $(x_1(0), x_2(0)) = (2.3, -2)$ and $(x_1(0), x_2(0)) = (0.2, 0.3)$. Plot the phase portraits in the same Figure (that is, $x_2(t)$ vs. $x_1(t)$ for $0 \le t \le 100$).
Hint: You can try the following Matlab 6 program

```
x01 = [2.3; -2];
x02 = [0.2; 0.3];
t0 = 0; tf = 100;
tspan=[t0 tf];
[t,x1] = ode23('limit', tspan , x01);
[t,x2] = ode23('limit', tspan , x02);
plot(x1(:,1), x1(:,2), x2(:,1), x2(:,2))
```

where **limit** is the following matlab function named as *limit.m*.

```
function xdot = limit(t,x)
xdot(1) = x(2);
xdot(2) = -x(1) -0.2*(x(1)*x(1)-1)*x(2);
xdot = xdot(:);
```

However, make sure you understand the program.

Solution:

The MATLAB code in main file is shown below:

```
clc; clf; clear all;
hold on;
x01 = [2.3; -2];
x02 = [0.2; 0.3];
t0 = 0; tf = 100;
tspan = [t0 tf];
[t,x1] = ode23('limit', tspan , x01);
[t,x2] = ode23('limit', tspan , x02);
plot(x1(:,1), x1(:,2),'color',[0.667 0.667 1],'LineWidth',2.5);
plot(x2(:,1), x2(:,2),'color',[1 0.5 0],'LineWidth',2.5);
xlabel('$x_1 \left(t\right)$','interpreter','latex');
ylabel('$x_2 \left(t\right)$','interpreter','latex');
legend(['$\left(x_1\left(0\right),x_2\left(0\right)\right)' ...
    '=\left(2.3,-2\right)$'],['$\left(x_1\left(0\right),' ...
    'x_2\left(0\right)\right)=\left(0.2,0.3\right)$'], ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "Q1_2_Code"+".pdf";
saveas(gcf,filename);
```

The MATLAB code in function that defines the Van del Pol equation is shown below:

```
function xdot = limit(t,x)
xdot(1) = x(2);
xdot(2) = -x(1) -0.2*(x(1)*x(1)-1)*x(2);
xdot = xdot(:);
```

And the results for phase portraits are plotted in Figure 1.

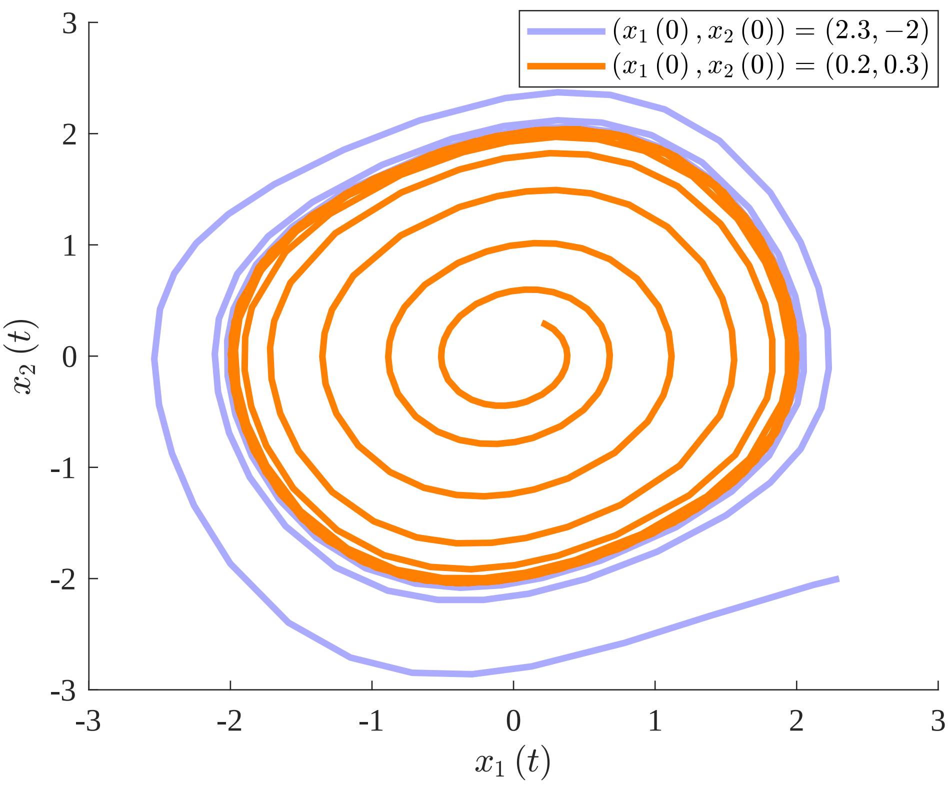

Figure 1: Phase Portraits for Given Nonlinear System.

Problem 3

It is known that the following system displays chaotic behavior.

$$\ddot{y} + 0.05\dot{y} + y^3 = 7.5\cos t$$

(a) Give a state space realization for this system.
(b) Write a MATLAB program to generate $y(t)$, $0 \leq t \leq 50$ for $(y(0), \dot{y}(0)) = (3, 4)$ and $(y(0), \dot{y}(0)) = (3.01, 4.01)$. Plot them in the same Figure. (The curve should be similar to Figure 1.6, page 11 of the textbook).

Solution:

(a)

First, we need to select the state variables. Choosing the state variables as successive derivatives, we get

$$x_1 = y \tag{8a}$$

$$x_2 = \dot{y} \tag{8b}$$

Differentiating both sides and making use of Equation (8) to find $\dot{x}_1$, and equation in the question to find $\tilde{y} = \dot{x}_2$, we obtain the state equations (Nise, 2020). The combined state and output equations are

$$\dot{x}_1 = x_2 \tag{9a}$$

$$\dot{x}_2 = -x_1^3 - 0.05x_2 + 7.5\cos t \tag{9b}$$

In vector-matrix form,

$$\dot{x}(t) = f(x(t), t) \tag{10}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(x(t), t) = \begin{bmatrix} x_2 \\ -x_1^3 - 0.05x_2 + 7.5\cos t \end{bmatrix} \tag{11}$$

(b)

The MATLAB code in main file is shown below:

```
clc; clf; clear all;
hold on;
x01 = [3; 4];
x02 = [3.01; 4.01];
t0 = 0; tf = 50;
tspan = [t0 tf];
[t,x1] = ode23('limit', tspan , x01);
plot(t,x1(:,1),'color',[0.667 0.667 1],'LineWidth',2.5);
[t,x2] = ode23('limit', tspan , x02);
plot(t,x2(:,1),'color',[1 0.5 0],'LineWidth',2.5);
grid on;
```

```
xlabel('$t$','interpreter','latex');
ylabel('$y \left(t\right)$','interpreter','latex');
legend(['$\left(y\left(0\right),\dot{y}\left(0\right)\right)' ...
    '=\left(3,4\right)$'],['$\left(y\left(0\right),' ...
    '\dot{y}\left(0\right)\right)=\left(3.01,4.01\right)$'], ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "Q1_3_Code"+".pdf";
saveas(gcf,filename);
```

The MATLAB code in function that defines the Van del Pol equation is shown below:

```
function xdot = limit(t,x)
xdot(1) = x(2);
xdot(2) = -x(1)*x(1)*x(1)-0.05*x(2)+7.5*cos(t);
xdot = xdot(:);
```

And the results for $y\left(t\right)$ are plotted in Figure 2.

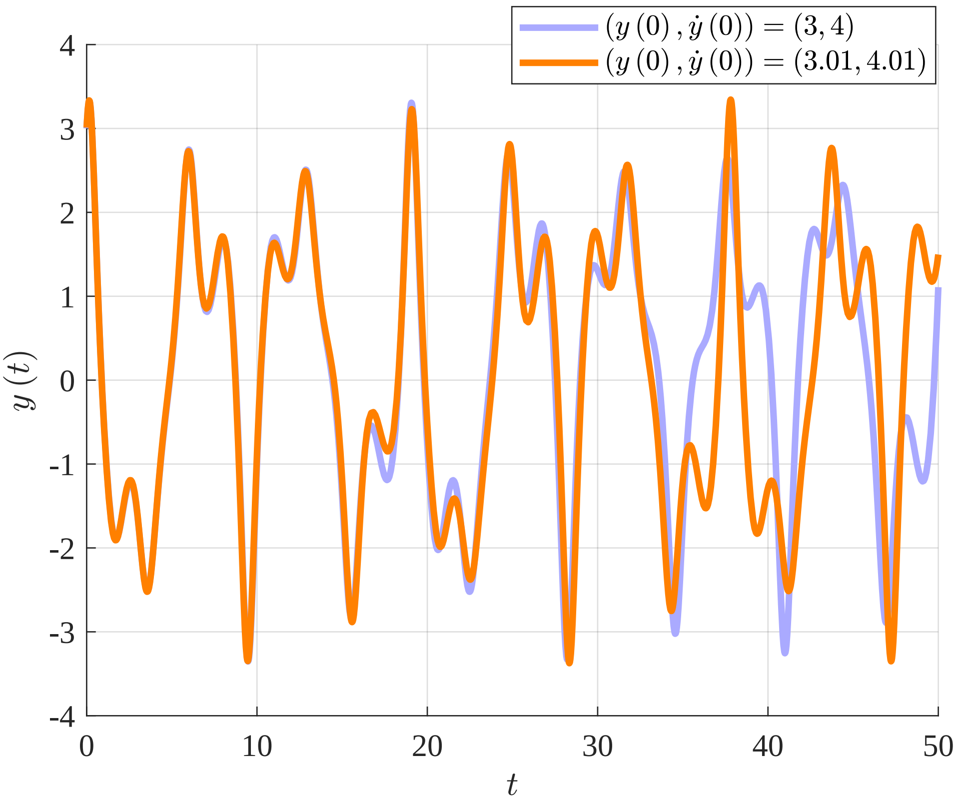

Figure 2: Results for Chaotic Behavior with Different Initial Conditions.

References

Nise, N. S. (2020). *Control systems engineering*. John Wiley & Sons.