香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #2

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium.
(a)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{x_1^3}{4} - x_2 \end{aligned}$$

(b)

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2(1 + x_1) \\ \dot{x}_2 &= -x_1(1 + x_1) \end{aligned}$$

Solution:

(a)

The equilibrium points $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ satisfy

$$\begin{cases} x_2^* = 0 \\ -x_1^* + \frac{{x_1^*}^3}{4} - x_2^* = 0 \end{cases} \Rightarrow x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} \tag{1}$$

Taking the Jacobian of the appropriate function yields that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{3}{4}x_1^2 & -1 \end{bmatrix} \tag{2}$$

For $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{3}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2} \tag{4}$$

Therefore, the equilibrium point $(0, 0)$ is a stable focus.

For $x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \tag{5}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{6}$$

Therefore, the equilibrium point $(2,0)$ is a saddle point.

For $x^* = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \tag{7}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{8}$$

Therefore, the equilibrium point $(-2,0)$ is a saddle point.

(b)

The equilibrium points $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ satisfy

$$\begin{cases} -2x_1^* + x_2^*\left(1 + x_1^*\right) = 0 \\ -x_1^*\left(1 + x_1^*\right) = 0 \end{cases} \Rightarrow x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

Taking the Jacobian of the appropriate function yields that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -2 + x_2 & 1 + x_1 \\ -1 + 2x_1 & 0 \end{bmatrix} \tag{10}$$

For $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$A_{x^*} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \tag{11}$$

The eigenvalues of A_{x^*} are

$$\lambda_{1,2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{12}$$

Therefore, the equilibrium point $(0,0)$ is a stable node.

Problem 2

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)^2, x_1(0) = x_{10} \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)^2, x_2(0) = x_{20} \end{aligned}$$

(a) Show that the system has a limit cycle.
(b) Determine the stability of the limit cycle.
(*Hint:*) Use polar coordinates.

Solution:

(a)

The polar coordinates are introduced as follows:

$$r = \left(x_1^2 + x_2^2\right)^{1/2} \tag{13}$$

and

$$\theta = \tan^{-1}\left(x_2/x_1\right) \tag{14}$$

Rearranging Equation (13) gives

$$r^2 = x_1^2 + x_2^2 \tag{15}$$

Taking the derivative of Equation (15) yields

$$2r\frac{dr}{dt} = 2x_1\frac{dx_1}{dt} + 2x_2\frac{dx_2}{dt} \tag{16}$$

Substituting the conditions in the question into Equation (16) gets

$$2r\frac{dr}{dt} = 2x_1\left[x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)^2\right] + 2x_2\left[-x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)^2\right] \tag{17}$$

Simplifying Equation (17) leads to

$$\frac{dr}{dt} = -r\left(r^2 - 1\right)^2 \tag{18}$$

Rearranging Equation (14) gives

$$\tan\theta = \frac{x_2}{x_1} \tag{19}$$

Differentiating Equation (19) yields

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{x_1}\frac{dx_2}{dt} - x_2\frac{1}{x_1^2}\frac{dx_1}{dt} \tag{20}$$

Substituting the conditions in the question into Equation (20) gets

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{x_1}\left[-x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)^2\right] - x_2\frac{1}{x_1^2}\left[x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)^2\right] \tag{21}$$

Simplifying Equation (21) leads to

$$\frac{d\theta}{dt} = -1 \tag{22}$$

When the state starts on the unit circle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from the outside. When $r < 1$, then $\dot{r} < 0$. This implies that the state tends to diverge from it. Therefore, the unit circle is a semi-stable limit cycle (Slotine et al., 1991).

(b)

When the state starts on the unit circle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from the outside. When $r < 1$, then $\dot{r} < 0$. This implies that the state tends to diverge from it. Therefore, the limit cycle is semi-stable.

Problem 3

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_1 + bx_2 - x_1^2 x_2 - x_1^3 \end{aligned}$$

Show that there can be no limit cycle if $b < 0$.

Solution:

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = b - x_1^2 < 0 \tag{23}$$

By Bendixson's criterion, there are no periodic orbits. Therefore, there can be no limit cycle if $b < 0$.

References

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.