香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #3

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

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Problem 1

Does the system have any limit cycle?

$$\begin{aligned}\dot{x}_1 &= 2x_2^2 \sin x_2 \\ \dot{x}_2 &= 1 - \cos x_1 + 2x_2\end{aligned}$$

Solution:

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2 > 0 \tag{1}$$

By Bendixson's criterion, there are no periodic orbits. Therefore, there can be no limit cycle.

Problem 2

Consider the following nonlinear equation.

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^2 + 3x_2^3 \\ \dot{x}_2 &= x_3 - 2x_2x_1 - x_1x_3 \\ \dot{x}_3 &= 3x_1 + 2x_3x_2 - 2x_3 + u\end{aligned}$$

(a) Find the Jacobian linearization of the system at the origin.
(b) Using the Lyapunov's linearization method to determine the stability property of the closed-loop system under the state feedback control law $u = -Kx$ for $K = [-4 - 3 - 1]$.

Solution:

(a)

The Jacobian matrix of the nonlinear equation is (Close et al., 2001)

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \frac{\partial f_1(x)}{\partial x_3} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} \\ \frac{\partial f_3(x)}{\partial x_1} & \frac{\partial f_3(x)}{\partial x_2} & \frac{\partial f_3(x)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -2x_1 & 9x_2^2+1 & 0 \\ -2x_2 - x_3 & -2x_1 & -x_1+1 \\ 3 & 2x_3 & 2x_2-2 \end{bmatrix} \tag{2}$$

For the system at the origin,

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix} \tag{3}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

Therefore, the Jacobian linearization of the system at the origin is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{5}$$

(b)

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -4 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & 3 & -1 \end{bmatrix} \tag{6}$$

The eigenvalues of $A - BK$ are

$$\begin{aligned}\lambda_1 &= -1.5370 + 1.0064i \\ \lambda_2 &= -1.5370 - 1.0064i \\ \lambda_3 &= 2.0739\end{aligned} \tag{7}$$

Because the real part of one of the eigenvalues of $A - BK$–λ_3–is positive, the closed-loop system under the state feedback control law $u = -Kx$ for $K = \begin{bmatrix} -4 & -3 & -1 \end{bmatrix}$ is unstable.

Problem 3

The motion of the ball and beam system can be described by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= Bx_1(t)x_4^2(t) - BGsin(x_3(t))\\ \dot{x}_3(t) &= x_4(t)\\ \dot{x}_4(t) &= u(t)\\ y(t) &= x_1(t)\end{aligned}$$

where x_1 is the position of the ball, u is the torque applied to the beam, $G = 9.81\ m/s^2$ is the acceleration of gravity, and $B = 0.7134$ is a constant.
(a) Show that the Jacobian linearization of (1) at the origin is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -BG & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}u = Ax + bu$$

(b) Verify that the pair (A, b) is controllable, i.e., the matrix $[b\ Ab\ \cdots\ A^{n-1}b]$ is nonsingular.
(c) Using Arkerman's Formula to find K so that the eigenvalues of $A - bK$ are $\{-1, -2, -3, -24\}$.
(d) Simulate the closed-loop system composed of (1) and $u = -Kx$ with $x(0) = \alpha[1, 1, 1, 1]$ for $\alpha = 1, 20$ from $t = 0$ to $t = 20$. Is the equilibrium point $x = 0$ of the closed-loop system globally asymptotically stable?

Arkerman's Formula
Let $A \in R^{n\times n}$ and $b \in R^n$. Assume the pair (A, b) is controllable, i.e., the matrix $[b\ Ab\ \cdots\ A^{n-1}b]$ is nonsingular. Let $q(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$ which is called the desired polynomial.
Let $q(F) = F^n + \alpha_1 F^{n-1} + \cdots + \alpha_{n-1}F + \alpha_n I_n$
Then

$$K = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}_{1\times n}[b\ Ab\ \cdots\ A^{n-1}b]^{-1}q(F)$$

is such that

$$\det(sI - (A - BK)) = q(s)$$

Solution:

(a)

The Jacobian matrix of the nonlinear equation is

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix}\frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \frac{\partial f_1(x)}{\partial x_3} & \frac{\partial f_1(x)}{\partial x_4}\\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} & \frac{\partial f_2(x)}{\partial x_4}\\ \frac{\partial f_3(x)}{\partial x_1} & \frac{\partial f_3(x)}{\partial x_2} & \frac{\partial f_3(x)}{\partial x_3} & \frac{\partial f_3(x)}{\partial x_4}\\ \frac{\partial f_4(x)}{\partial x_1} & \frac{\partial f_4(x)}{\partial x_2} & \frac{\partial f_4(x)}{\partial x_3} & \frac{\partial f_4(x)}{\partial x_4}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ Bx_4^2(t) & 0 & -BG\cos(x_3(t)) & 2Bx_1(t)x_4(t)\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{8}$$

For the system at the origin,

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -BG & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{9}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

Therefore, the Jacobian linearization of the system at the origin is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{11}$$

(b)

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -BG \\ 0 & 0 & -BG & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = 4 \tag{13}$$

Therefore, the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$ is nonsingular, which indicates that the pair (A, b) is controllable.

(c)

According to Arkerman's Formula,

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -BG \\ 0 & 0 & -BG & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} q(A) \tag{14}$$

Because the eigenvalues of $A - bK$ are $\{-1, -2, -3, -24\}$ and $\det(sI - (A - BK)) = q(s)$, the desired polynomial is

$$q(s) = (s+1)(s+2)(s+3)(s+24) \tag{15}$$

Therefore,

$$\begin{aligned} q(A) &= (A+I)\cdot(A+2I)\cdot(A+3I)\cdot(A+24I) \\ &= \begin{bmatrix} 144 & 270 & 1084.8 & -210 \\ 0 & 144 & -1889.6 & -1084.8 \\ 0 & 0 & 144 & 270 \\ 0 & 0 & 0 & 144 \end{bmatrix} \end{aligned} \tag{16}$$

where I is the identity matrix. Then, we can get

$$\begin{aligned} K &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -6.9985 \\ 0 & 0 & -6.9985 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 144 & 270 & 1084.8 & -210 \\ 0 & 144 & -1889.6 & -1084.8 \\ 0 & 0 & 144 & 270 \\ 0 & 0 & 0 & 144 \end{bmatrix} \\ &= \begin{bmatrix} -20.5760 & -38.5799 & 155.0000 & 30.0000 \end{bmatrix} \end{aligned} \tag{17}$$

Instead of Arkerman's Formula, I also use another way to find out the answer, whose MATLAB codes are shown below:

```
clc; clear all;
B = 0.7134;
G = 9.81;
BG = B*G;
syms k1 k2 k3 k4
sp = sym('sp',[4 4 4]);
lambda = [-1 -2 -3 -24];
detsp = sym('detsp',[1 4]);
for i = 1:4
    sp(:,:,i) = [-lambda(i) 1 0 0;
        0 -lambda(i) -BG 0;
        0 0 -lambda(i) 1;
        -k1 -k2 -k3 -k4-lambda(i)];
    detsp(i) = det(sp(:,:,i));
end
[k1,k2,k3,k4] = solve(detsp==0);
```

(d)

I use the Simulink model as shown in Figure 1 to simulate the closed-loop system composed of (1) and $u = -Kx$ with $x(0) = \alpha \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ for $\alpha = 1$ and 20 from $t = 0$ to $t = 20$ s.

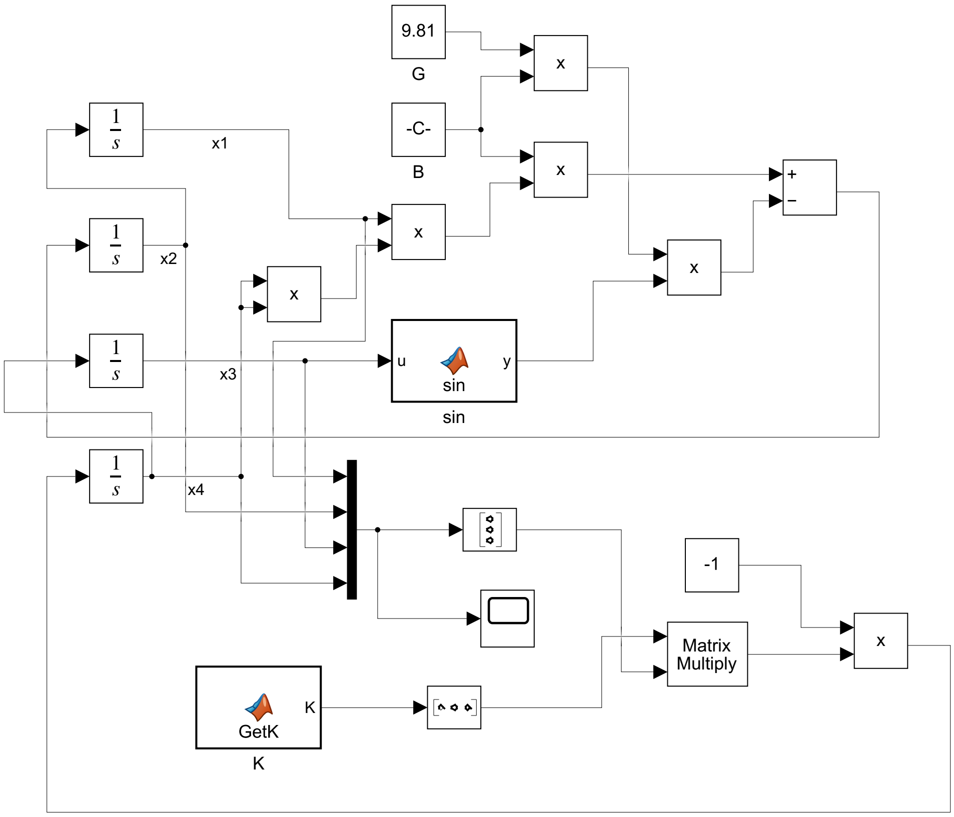

Figure 1: Simulink model to simulate the closed-loop system composed of (1) and $u = -Kx$ with $x(0) = \alpha \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ for $\alpha = 1$ and 20 from $t = 0$ to $t = 20$ s.

The results I get from the simulation is shown in Figure 2 and 3 for $\alpha = 1$ and 20, respectively.

From Figure 3, we can know that for the initial conditions of $x(0) = 20 \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, the system is unstable. Therefore, the equilibrium point $x = 0$ of the closed-loop system is not globally asymptotically stable.

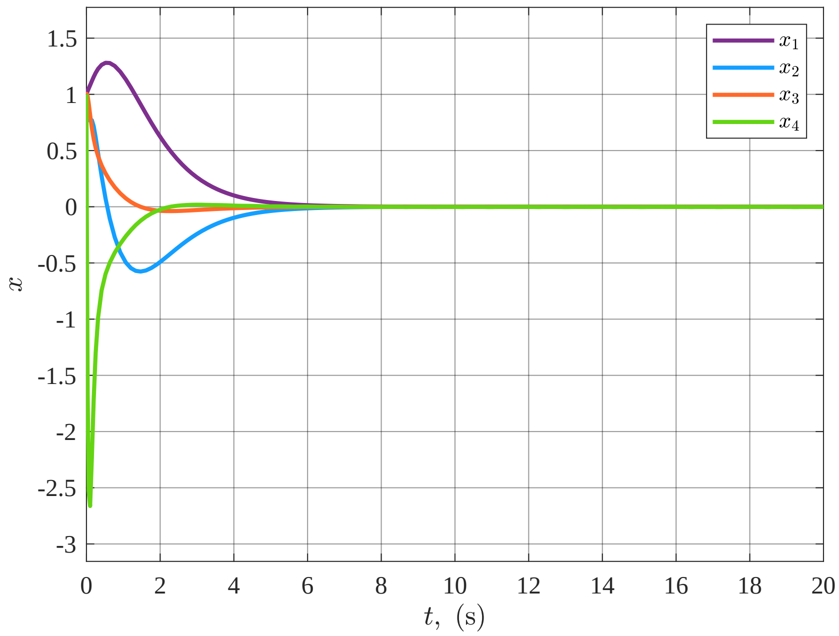

Figure 2: Simulation results for the system with $\alpha = 1$.

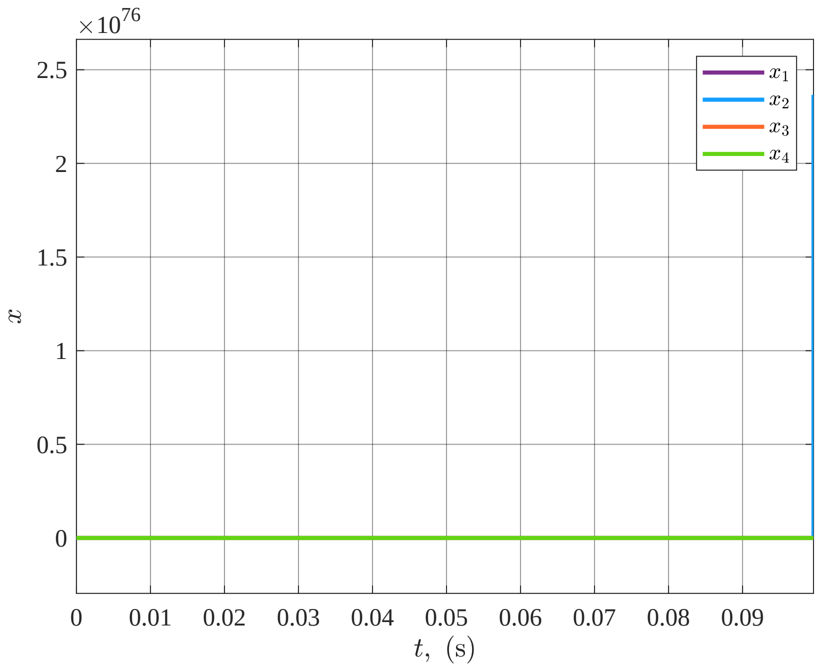

Figure 3: Simulation results for the system with $\alpha = 20$.

References

Close, C. M., Frederick, D. K., & Newell, J. C. (2001). *Modeling and analysis of dynamic systems*. John Wiley & Sons.