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DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #4

by

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Problem 1

For the following systems, find the equilibrium points and determine their stability. Indicate whether the stability is asymptotic, and whether it is global.

(a)

$$\dot{x} = -x^3 + \sin^4 x \tag{1}$$

(b)

$$\dot{x} = (5 - x)^5 \tag{2}$$

Solution:

(a) The equilibrium points x^* satisfy $-x^{*3} + \sin^4 x^* = 0$. Obviously, $x^* = 0$ is one solution. Next, I will prove that $x^* = 0$ is the only solution. Let $f(x) = -x^3 + \sin^4 x$.

$$f'(x) = 4\sin^3 x \cos x - 3x^2 \tag{3}$$

Because $\sin x \cos x \le \frac{\sin^2 x + \cos^2 x}{2} = \frac{1}{2}$ (Cauchy–Schwarz inequality),

$$f'(x) = 4\sin^2 x \sin x \cos x - 3x^2 \le 2\sin^2 x - 3x^2 \le 2x^2 - 3x^2 = -x^2 \le 0 \tag{4}$$

Hence, $f(x)$ is a monotonically decreasing function, which indicates that $x^* = 0$ is the only solution. Therefore, the equilibrium point for this system is $x^* = 0$. Then, the following Lyapunov function is selected as the candidate:

$$V(x) = \frac{1}{2}x^2 \tag{5}$$

$V(x)$ is positive definite for $\forall x \in \mathbf{R} - \{0\}$. Taking the derivative of Equation (5) yields that

$$\dot{V}(x) = x\dot{x} = x\left(-x^3 + \sin^4 x\right) \tag{6}$$

From the analysis above for $f(x)$, it can be concluded that $f(x) > 0$ when $x < 0$ and $f(x) < 0$ when $x > 0$. Combing this conclusion with Equation (6) obtains $\dot{V}(x) < 0$ for $\forall x \in \mathbf{R} - \{0\}$, indicating that $\dot{V}(x)$ is negative definite. Therefore, the system is globally asymptotically stable.

(b) The equilibrium points x^* satisfy $(5 - x^*)^5 = 0$. Obviously, $x^* = 5$ is one solution and $(5 - x)^5$ is monotonically-decreasing. Therefore, the equilibrium point for this system is $x^* = 5$. Then, the following Lyapunov function is selected as the candidate:

$$V(x) = \frac{1}{2}(5 - x)^2 \tag{7}$$

$V(x)$ is positive definite for $\forall x \in \mathbf{R} - \{5\}$. Taking the derivative of Equation (7) yields that

$$\dot{V}(x) = -(5 - x)\dot{x} = -(5 - x)^6 < 0 \text{ for } \forall x \in \mathbf{R} - \{5\} \tag{8}$$

Therefore, $\dot{V}(x)$ is negative definite, which means that the system is globally asymptotically stable.

Problem 2

Consider the following pendulum equation:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a_1 \sin x_1 - a_2 x_2
\end{aligned} \tag{9}
$$

where $a_1 > 0$ and $a_2 > 0$.

(a) Show the equilibrium point $x = 0$ is stable using the Lyapunov function candidate $V(x) = a_1(1 - \cos x_1) + \frac{1}{2}x_2^2$. Can you conclude the asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$?

(b) Consider the Lyapunov function candidate

$$
V(x) = \frac{1}{2}\left(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2\right) + a_1(1 - \cos x_1) \tag{10}
$$

where $p_{22} = 1$ and $p_{11} = a_2 p_{12}$. Can you find appropriate value for p_{12} to conclude the asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$?

Solution:

(a)

$$
V(x) = a_1(1 - \cos x_1) + \frac{1}{2}x_2^2 > a_1(1 - \cos x_1) \geq a_1(1 - 1) = 0 \tag{11}
$$

$$
\dot{V}(x) = a_1 \sin x_1 \dot{x}_1 + x_2\dot{x}_2 = a_1 x_2 \sin x_1 + x_2(-a_1 \sin x_1 - a_2 x_2) = -a_2 x_2^2 \leq 0 \tag{12}
$$

Therefore, $V(x)$ is positive definite, and $\dot{V}(x)$ is negative semi-definite, which means I can not conclude the asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$.

(b)

$$
\begin{aligned}
V(x) &= \frac{1}{2}\left(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2\right) + a_1(1 - \cos x_1) \\
&= \frac{1}{2}\left(a_2p_{12}x_1^2 + 2p_{12}x_1x_2 + x_2^2\right) + a_1(1 - \cos x_1) \\
&> a_1(1 - \cos x_1) \geq a_1(1 - 1) = 0
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\dot{V}(x) &= p_{11}x_1\dot{x}_1 + p_{12}\dot{x}_1x_2 + p_{12}x_1\dot{x}_2 + p_{22}x_2\dot{x}_2 + a_1 \sin x_1 \dot{x}_1 \\
&= p_{11}x_1x_2 + p_{12}x_2^2 + p_{12}x_1(-a_1 \sin x_1 - a_2x_2) \\
&\quad + p_{22}x_2(-a_1 \sin x_1 - a_2x_2) + a_1x_2 \sin x_1 \\
&= a_2p_{12}x_1x_2 + p_{12}x_2^2 + p_{12}x_1(-a_1 \sin x_1 - a_2x_2) \\
&\quad + x_2(-a_1 \sin x_1 - a_2x_2) + a_1x_2 \sin x_1 \\
&= (p_{12} - a_2)x_2^2 - a_1p_{12}x_1 \sin x_1
\end{aligned} \tag{14}
$$

For $x_1 \in (-\pi, \pi)$ and $x_2 \in \mathbf{R}$, $0 < p_{12} < a_2$ is selected to make $\dot{V}(x)$ ND. Therefore, the appropriate value for p_{12}, i.e. $0 < p_{12} < a_2$, can be selected to conclude that the equilibrium point $x = 0$ is locally asymptotic stable.

However, I can not find appropriate value for p_{12} to conclude the globally asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$. Because for $\forall p_{12} \in \mathbf{R}^+$, $\exists k \in \mathbf{Z}$, which satisfies

$$k > -\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4} \tag{15}$$

so that when $x_1 = 2k\pi + \frac{3}{2}\pi$,

$$\begin{aligned}
\dot{V}(x) &= (p_{12} - a_2)\, x_2^2 - a_1 p_{12} x_1 \sin x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left(2k\pi + \frac{3}{2}\pi\right) \\
&> (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left[2\pi \left(-\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4}\right) + \frac{3}{2}\pi\right] = 0
\end{aligned} \tag{16}$$

and for $\forall p_{12} \in \mathbf{R}^-$, $\exists k \in \mathbf{Z}$, which satisfies

$$k < -\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4} \tag{17}$$

so that when $x_1 = 2k\pi + \frac{3}{2}\pi$,

$$\begin{aligned}
\dot{V}(x) &= (p_{12} - a_2)\, x_2^2 - a_1 p_{12} x_1 \sin x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} x_1 \\
&= (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left(2k\pi + \frac{3}{2}\pi\right) \\
&> (p_{12} - a_2)\, x_2^2 + a_1 p_{12} \left[2\pi \left(-\frac{(p_{12} - a_2)\, x_2^2}{2\pi a_1 p_{12}} - \frac{3}{4}\right) + \frac{3}{2}\pi\right] = 0
\end{aligned} \tag{18}$$

In addition, for $p_{12} = 0$, this situation has been discussed in (a). Therefore, the appropriate value for p_{12} to conclude the globally asymptotic stability of the equilibrium point $x = 0$ with this $V(x)$ can not be found.

Problem 3

Show that if symmetric p.d. matrices P and Q exists such that

$$A^T P + PA + 2\lambda P = -Q \tag{19}$$

then all the eigenvalues of A have a real part strictly less than $-\lambda$.

Solution:

Consider the linear homogeneous continuous-time system

$$\dot{x}(t) = (A + \lambda I)\, x(t) \tag{20}$$

Let us associate with this system and the equilibrium point $x^* = 0$ the quadratic function

$$V(x) = x^T P x \tag{21}$$

where P is symmetric and positive definite. This V is continuous and has continuous first partial derivatives. Furthermore, since P is positive definite, the origin is the unique minimum point of V. Thus in terms of general characteristics, such a positive definite quadratic form is a suitable candidate for a Lyapunov function. It remains, of course, to determine how $\dot{V}(x)$ is influenced by the dynamics of the system.

We have

$$\begin{aligned} \dot{V}(x) &= \frac{d}{dt} x^T P x \\ &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (A + \lambda I) P x + x^T P (A + \lambda I) x \\ &= x^T \left(A^T P + PA + 2\lambda P\right) x \\ &= -x^T Q x \end{aligned} \tag{22}$$

Because matrix Q is symmetric p.d., $\dot{V}(x) < 0$ for $\forall x \in \mathbf{R} - \{0\}$, indicating that $\dot{V}(x)$ is ND. Therefore, the system is globally asymptotically stable. To ensure the system is global asymptotically stable, the real parts of the eigenvalues of $(A + \lambda I)$ need to be always negative, which means all the eigenvalues of A have a real part strictly less than $-\lambda$ (Luenberger, 1979).

Problem 4

For the linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -6x_1 - 5x_2 \end{aligned} \tag{23}$$

(a) what can you say about its stability and asymptotic stability from the candidate Lyapunov functions

$$\begin{aligned} V_1(x) &= 6x_1^2 + x_2^2 \\ V_2(x) &= x_1^2 + x_2^2 - x_1x_2 \end{aligned} \tag{24}$$

(b) For $Q = I$, solve the Lyapunov equation for a symmetric p.d. matrix P.

$$A^T P + PA = -Q \tag{25}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \tag{26}$$

Solution:

(a) For the candidate V_1,

$$V_1(x) = 6x_1^2 + x_2^2 > 0 \text{ for } \forall x \in \mathbf{R}^2 - \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \tag{27}$$

Therefore, $V_1(x)$ is positive definite. Taking the derivative of Equation (27) yields that

$$\dot{V}_1(x) = 12x_1\dot{x}_1 + 2x_2\dot{x}_2 = 12x_1x_2 + 2x_2(-6x_1 - 5x_2) = -10x_2^2 \le 0 \tag{28}$$

Therefore, $\dot{V}(x)$ is negative semi-definite, which means the system is stable at the equilibrium point $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

P.S. If Invariant Set Theorem is used in this question, asymptotic stability can be concluded. Because let $R = \mathbf{R}^2$, if $\dot{V}_1(x) = 0$, $x_2 = 0$, so $\dot{x}_2 = 0$. Substituting $x_2 = 0$ and $\dot{x}_2 = 0$ into Equation (23), we can get $x_1 = 0$. Therefore, $\dot{V}_1(x) = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Therefore, the system is asymptotically stable.

For the candidate V_2,

$$V_2(x) = x_1^2 + x_2^2 - x_1x_2 = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0 \text{ for } \forall x \in \mathbf{R}^2 - \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \tag{29}$$

Therefore, $V_2(x)$ is positive definite. Taking the derivative of Equation (29) yields that

$$\begin{aligned}\dot{V}_2(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 - \dot{x}_1x_2 - x_1\dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-6x_1 - 5x_2) - x_2^2 - x_1(-6x_1 - 5x_2) \\ &= 6x_1^2 - 11x_2^2 - 5x_1x_2 \\ &= (6x_1 - 11x_2)(x_1 + x_2)\end{aligned} \tag{30}$$

The sign of $\dot{V}_2(x)$ can not be told from Equation (30). Therefore, this candidate Lyapunov functions can not conclude the stability of the equilibrium point $x = 0$.

(b) Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, where $p_{12} = p_{21}$.

$$\begin{aligned}A^TP + PA = -Q &\Leftrightarrow \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} -6p_{21} & -6p_{22} \\ p_{11} - 5p_{21} & p_{12} - 5p_{22} \end{bmatrix} + \begin{bmatrix} -6p_{12} & p_{11} - 5p_{12} \\ -6p_{22} & p_{21} - 5p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} -6p_{21} - 6p_{12} & -6p_{22} + p_{11} - 5p_{12} \\ p_{11} - 5p_{21} - 6p_{22} & p_{12} - 5p_{22} + p_{21} - 5p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned} \tag{31}$$

Therefore, we can know that

$$\begin{cases} -6p_{21} - 6p_{12} = -1 \\ -6p_{22} + p_{11} - 5p_{12} = 0 \\ p_{11} - 5p_{21} - 6p_{22} = 0 \\ p_{12} - 5p_{22} + p_{21} - 5p_{22} = -1 \end{cases} \tag{32}$$

Solving Equation (32) yields that

$$P = \begin{bmatrix} \frac{67}{60} & \frac{1}{12} \\ \frac{1}{12} & \frac{7}{60} \end{bmatrix} \tag{33}$$

References

Luenberger, D. G. (1979). *Introduction to dynamic systems: theory, models, and applications*, volume 1. Wiley New York.