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Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #5

by

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Problem 1

Determine the stability of the following system at the origin. Indicate whether the stability is asymptotic, and whether it is global.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + \sin^4 x_1 - x_2^7\end{aligned} \tag{1}$$

Solution:

Here we use Lemma 2 of Invariant Set Theorem (Slotine et al., 1991). For the system (1), $c(x_1) = x_1^3 - \sin^4 x_1$ and $b(x_2) = x_2^7$. Therefore,

$$yb(y) = y^8 > 0, y \neq 0 \tag{2}$$

and

$$yc(y) = y\left(y^3 - \sin^4 y\right) \tag{3}$$

Let $f(x) = x^3 - \sin^4 x$.

$$f'(x) = -4\sin^3 x \cos x + 3x^2 \tag{4}$$

Because $\sin x \cos x \leq \frac{\sin^2 x + \cos^2 x}{2} = \frac{1}{2}$ (Cauchy–Schwarz inequality),

$$f'(x) = -4\sin^2 x \sin x \cos x + 3x^2 \geq -\sin^2 x + 3x^2 \geq -2x^2 = 3x^2 = x^2 \geq 0 \tag{5}$$

Hence, $f(x)$ is a monotonically increase function, And we note that $f(0) = 0$. Therefore, $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x < 0$. Therefore, we can conclude that

$$yc(y) = y\left(y^3 - \sin^4 y\right) > 0 \tag{6}$$

Hence, the system is asymptotically stable at the equilibrium point $x = 0$.

Moreover,

$$\lim_{|y|\to\infty} \int_0^y c(r)\, dr = \lim_{|y|\to\infty} \int_0^y \left(r^3 - \sin^4 r\right) dr > \lim_{|y|\to\infty} \int_0^y r^3 dr = \lim_{|y|\to\infty} \frac{1}{4} y^4 = \infty \tag{7}$$

Hence, the system is globally asymptotically stable at the equilibrium point $x = 0$.

Problem 2

Consider Lienard's equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 \sin^2 x_1 - b(x_1)\end{aligned} \tag{8}$$

where $b(y)$ is continuous function over $y \in \mathbf{R}$ and satisfies $yb(y) > 0$, $y \neq 0$.

(a) Show the following function

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} b(y)\,dy \tag{9}$$

is a Lyapunov function for the system.

(b) Show that the origin is locally asymptotically stable.

(c) Can you conclude global asymptotic stability of the origin based on this Lyapunov function? Why?

Solution:

(a) Because $yb(y) > 0$, $b(y) > 0$ for $y > 0$ and $b(y) < 0$ for $y < 0$. Hence, $\int_0^{x_1} b(y)\,dy > 0$ for $x_1 \neq 0$. Therefore,

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} b(y)\,dy > 0 \tag{10}$$

Taking the derivative of Equation (9) gets

$$\begin{aligned}\dot{V}(x_1, x_2) &= \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 \\ &= b(x_1)x_2 + x_2\left(-x_2 \sin^2 x_1 - b(x_1)\right) \\ &= x_2^2 \sin^2 x_1 \leq 0\end{aligned} \tag{11}$$

Therefore, $V(x)$ is positive definite, and $\dot{V}(x)$ is negative-semi definite, from which we can conclude that

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} b(y)\,dy \tag{12}$$

is a Lyapunov function for the system.

(b) Invariant Set Theorem is used in this question. For $x_1 \in (-\pi, \pi) - \{0\}$, if $\dot{V}(x_1, x_2) = 0$, $x_2 = 0$, so $\dot{x}_2 = 0$. Substituting $x_2 = 0$ and $\dot{x}_2 = 0$ into Equation (8), we can get $x_1 = 0$. Therefore, $\dot{V}_1(x) = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\{(x_1, x_2) \mid -\pi < x_1 < \pi, x_2 \in \mathbf{R}\}$. Therefore, the system is locally asymptotically stable.

(c) Global asymptotic stability of the origin based on this Lyapunov function can not be concluded because $\dot{V}(x_1, x_2) = 0$ when $\{(x_1, x_2) \mid x_1 = k\pi, k \in \mathbf{Z}, x_2 \in \mathbf{R}\}$ so we cannot find the invariant set.

Problem 3

Using the Krasovskii Theorem to show the global asymptotic stability of the equilibrium point at the origin of the following system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= x_1 - 3x_2 - x_2^5\end{aligned} \tag{13}$$

Solution:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 - 5x_2^4 \end{bmatrix} \tag{14}$$

$$F(x) = \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} = \begin{bmatrix} -6 & 2 \\ 2 & -6 - 10x_2^4 \end{bmatrix} < 0, \forall x \tag{15}$$

Thus, the equilibrium point is asymptotic stable.

Moreover,

$$V(x) = f^T(x) f(x) = (-3x_1 + x_2)^2 + \left(x_1 - 3x_2 - x_2^5\right)^2 \tag{16}$$

is radially unbounded. Thus, the equilibrium point is global asymptotic stable.

Problem 4

Consider the following system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c\left(x_1\right) - b\left(x_2\right)\end{aligned} \tag{17}$$

where the functions c and b are continuous satisfying the sign condition. Using the variable gradient method to derive a Lyapunov function for Equation (17) as follows:

$$V\left(x_1, x_2\right) = \int_0^{x_1} c\left(y\right) dy + \frac{1}{2}x_2^3 \tag{18}$$

Solution:

Let $\nabla V = \left[a_1 c\left(x_1\right), a_2 x_2\right]$. Because

$$\frac{\partial \nabla V_1}{x_2} = \frac{\partial a_1 c\left(x_1\right)}{x_2} = 0 = \frac{\partial a_2 x_2}{x_1} = \frac{\partial \nabla V_2}{x_1} \tag{19}$$

we can make $a_1 = a_2 = 1$. Then, $\dot{V}$ can be computed as

$$\dot{V} = \nabla V \dot{x} = c\left(x_1\right) x_2 + x_2\left(-c\left(x_1\right) - b\left(x_2\right)\right) = -x_2 b\left(x_2\right) \tag{20}$$

Bcause b is continuous satisfying the sign condition, $\dot{V} < 0$, which means $\dot{V}$ is negative definite. Therefore, the Lyapunov function can be expressed as

$$\begin{aligned}V\left(x\right) &= \int_0^{x_1} \nabla V_1\left(x_1, 0\right) dx_1 + \int_0^{x_2} \nabla V_2\left(x_1, x_2\right) dx_2 \\ &= \int_0^{x_1} c\left(x_1\right) dx_1 + \int_0^{x_2} x_2 dx_2 \\ &= \int_0^{x_1} c\left(y\right) dy + \frac{1}{2}x_2^2\end{aligned} \tag{21}$$

References

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.