香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #6

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

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Problem 1

Show that the one-dimensional system $\dot{x} = -a(t)x$ where a(t) is continuous and nonnegative over $t \geq 0$ is exponentially stable if there exist a $T > 0$ such that, for any $t > 0$, $\int_t^{t+T} a(r)\,dr \geq \gamma$ for some $\gamma > 0$.

Hint: For any $t \geq t_0$, $e^{-\int_t^{t+T} a(r)dr} \leq e^{-\gamma} < 1$.

Solution:

For the system, $\dot{x} = -a(t)x$, the solutioin for $x(t)$ is

$$x(t) = x(t_0)\,e^{-\int_{t_0}^{t} a(x)dx} \tag{1}$$

If for any $t > 0$, $\int_t^{t+T} a(r)\,dr \geq \gamma$ for some $\gamma > 0$,

$$\int_{t_0}^{t} a(\tau)\,d\tau = \int_{t_0}^{t+T} a(\tau)\,d\tau + \int_{t_0+T}^{t_0+2T} a(\tau)\,d\tau + \cdots + \int_{t-T}^{t} a(\tau)\,d\tau \geq \frac{t-t_0}{T}\gamma \tag{2}$$

Therefore,

$$e^{-\int_{t_0}^{t} a(x)dx} \leq e^{-\frac{t-t_0}{T}\gamma} \tag{3}$$

Hence,

$$x(t) = x(t_0)\,e^{-\int_{t_0}^{t} a(x)dx} \leq x(t_0)\,e^{\frac{t_0}{T}\gamma}e^{-\frac{\gamma}{T}t} \tag{4}$$

We can conclude that $\dot{x} = -a(t)x$ is exponentially stable.

Problem 2

Condition (4.19) on the eigenvalues of $\boldsymbol{A}(t) + \boldsymbol{A}^T(t)$ is only, of course, a sufficient condition. For instance, show that the linear time-varying system associated with the matrix

$$\boldsymbol{A}(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix} \tag{5}$$

is globally asymptotically stable.

Solution:

$$\boldsymbol{A}(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + e^{\frac{t}{2}} x_2 \\ -x_2 \end{bmatrix} \tag{6}$$

we can obtain the solution for $x_2(t)$:

$$x_2(t) = x_2(t_0)\, e^{-(t-t_0)} \tag{7}$$

Because $\dot{x}_1 = -x_1 + e^{\frac{t}{2}} x_2$

$$\begin{aligned} x_1(t) &= x_1(t_0)\, e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-\tau)} e^{\frac{\tau}{2}} x_2(\tau)\, d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-\tau)} e^{\frac{\tau}{2}} x_2(t_0)\, e^{-(\tau-t_0)} d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + x_2(t_0)\, e^{-(t-t_0)} \int_{t_0}^{t} e^{\frac{\tau}{2}} d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + 2x_2(t_0)\, e^{-(t-t_0)} \left(e^{\frac{t}{2}} - e^{\frac{t_0}{2}}\right) \\ &= x_1(t_0)\, e^{-(t-t_0)} + 2x_2(t_0)\, e^{-\left(\frac{t}{2}-t_0\right)} - 2x_2(t_0)\, e^{-\left(t-\frac{3}{2}t_0\right)} \end{aligned} \tag{8}$$

We can conclude that $\dot{x} = -A(t)\,x$ is globally asymptotically stable since $\lim_{t\to\infty} x_i(t) = 0, i = 1, 2$.

Problem 3

Determine whether the following systems have a stable equilibrium. Indicate whether the stability is asymptotic, and whether it is global.

(a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9}$$

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\sin t \\ 0 & -(t+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{10}$$

(c)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

Solution:

(a)

$$A(t) = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10x_1 + e^{3t}x_2 \\ -2x_2 \end{bmatrix} \tag{12}$$

we can obtain the solution for $x_2(t)$:

$$x_2(t) = x_2(t_0)\, e^{-2(t-t_0)} \tag{13}$$

Because $\dot{x}_1 = -10x_1 + e^{3t}x_2$

$$\begin{aligned} x_1(t) &= x_1(t_0)\, e^{-10(t-t_0)} + \int_{t_0}^{t} e^{-10(t-\tau)} e^{3\tau} x_2(\tau)\, d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + \int_{t_0}^{t} e^{-10(t-\tau)} e^{3\tau} x_2(t_0)\, e^{-2(\tau-t_0)} d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + x_2(t_0)\, e^{-(10t-2t_0)} \int_{t_0}^{t} e^{11\tau} d\tau \\ &= x_1(t_0)\, e^{-(t-t_0)} + \frac{1}{11} x_2(t_0)\, e^{-(t-2t_0)} \left(e^{11t} - e^{11t_0}\right) \\ &= x_1(t_0)\, e^{-(t-t_0)} + \frac{1}{11} x_2(t_0)\, e^{t+3t_0} - \frac{1}{11} x_2(t_0)\, e^{-(10t-13t_0)} \end{aligned} \tag{14}$$

We can conclude that $\dot{x} = -A(t)\,x$ is unstable since $\lim_{t\to\infty} x_1(t) = \infty$.

(b)

$$A(t) + A^T(t) = \begin{bmatrix} -2 & 2\sin t \\ 2\sin t & -2(t+1) \end{bmatrix} \Longrightarrow -\left(A(t) + A^T(t)\right) = \begin{bmatrix} 2 & -2\sin t \\ -2\sin t & 2(t+1) \end{bmatrix} \tag{15}$$

The determinant of $-\left(A\left(t\right)+A^{T}\left(t\right)\right)$ is equal to

$$\det\left[-\left(A\left(t\right)+A^{T}\left(t\right)\right)\right]=4\left(t+1\right)-4\sin^{2}t\geq4\left(t+1\right)-4t>0 \tag{16}$$

Therefore, $-\left(A\left(t\right)+A^{T}\left(t\right)\right)$ is positive definite, which means $A\left(t\right)+A^{T}\left(t\right)$ is negative definite. Hence, $\lambda_{i}\left(A\left(t\right)+A^{T}\left(t\right)\right)<-\lambda$ for some $\lambda>0$. We can conclude that the system is globally asymptotically stable.

(c)

$$\boldsymbol{A}\left(t\right)=\begin{bmatrix}-1 & e^{2t}\\ 0 & -2\end{bmatrix}\Longrightarrow\begin{bmatrix}\dot{x}_{1}\\ \dot{x}_{2}\end{bmatrix}=\begin{bmatrix}-1x_{1}+e^{2t}x_{2}\\ -2x_{2}\end{bmatrix} \tag{17}$$

we can obtain the solution for $x_{2}\left(t\right)$:

$$x_{2}\left(t\right)=x_{2}\left(t_{0}\right)e^{-2\left(t-t_{0}\right)} \tag{18}$$

Because $\dot{x}_{1}=-10x_{1}+e^{3t}x_{2}$

$$\begin{aligned}x_{1}\left(t\right)&=x_{1}\left(t_{0}\right)e^{-\left(t-t_{0}\right)}+\int_{t_{0}}^{t}e^{-\left(t-\tau\right)}e^{3\tau}x_{2}\left(\tau\right)d\tau\\&=x_{1}\left(t_{0}\right)e^{-\left(t-t_{0}\right)}+\int_{t_{0}}^{t}e^{-\left(t-\tau\right)}e^{2\tau}x_{2}\left(t_{0}\right)e^{-2\left(\tau-t_{0}\right)}d\tau\\&=x_{1}\left(t_{0}\right)e^{-\left(t-t_{0}\right)}+x_{2}\left(t_{0}\right)e^{-\left(t-2t_{0}\right)}\int_{t_{0}}^{t}e^{\tau}d\tau\\&=x_{1}\left(t_{0}\right)e^{-\left(t-t_{0}\right)}+x_{2}\left(t_{0}\right)e^{-\left(t-2t_{0}\right)}\left(e^{t}-e^{t_{0}}\right)\\&=x_{1}\left(t_{0}\right)e^{-\left(t-t_{0}\right)}+x_{2}\left(t_{0}\right)e^{t_{0}}-x_{2}\left(t_{0}\right)e^{-\left(t-3t_{0}\right)}\end{aligned} \tag{19}$$

Since we can find constant $r\left(R,t_{0}\right)=\frac{R}{4}e^{-3t_{0}},\forall R>0$, such that $\left\|x\left(t_{0}\right)\right\|<r\Rightarrow\left\|x\left(t\right)\right\|<R,\ t\geq t_{0}$, so, the equilibrium point at origin is stable.

However, $\lim_{t\to\infty}x_{1}\left(t\right)=x_{2}\left(t_{0}\right)e^{t_{0}}$, so it is not asymptotically stable.

Problem 4

Show that the following system is globally exponentially stable with a detailed argument.

$$\begin{aligned}\dot{x}_1 &= -\left(5 + x_2^5 + x_3^8\right) x_1 \\ \dot{x}_2 &= -x_2 + 4x_3^2 \\ \dot{x}_3 &= -\left(2 + \sin t\right) x_3\end{aligned} \tag{20}$$

Solution:

Let $a(t) = 2 + \sin t$. Then

$$x_3(t) = x_3(t_0)\, e^{-\int_{t_0}^{t}(2+\sin\tau)d\tau} \Longrightarrow \|x_3\| e^{-(t-t_0)} \tag{21}$$

Therefore,

$$x_2(t) = e^{-(t-t_0)} x_2(t_0) + \int_{t_0}^{t} e^{-(t-\tau)} 4x_3^2(\tau)\, d\tau \tag{22}$$

Thus, it is ready to see that the system is globally exponentially stable upon using Proposition 1 on the x_1 subsystem (Slotine et al., 1991).

Problem 5

(i) For the autonomous system $\dot{x} = f(x), x \in R^n$, show that, if, in a certain neighborhood Ω of the origin, there exists a continuously differentiable scalar function $V(x)$ such that

- $V(0) = 0 \quad \forall t \geq 0$
- $V(x)$ can assume strictly positive values arbitrarily close to the origin.
- $\dot{V}(x)$ is positive definite (locally in Ω)

then the equilibrium point 0 is unstable.

(ii) Show that the E.P. of $\dot{x} = c(x)$ is unstable where $c(x)$ is continuous and satisfies $xc(x) > 0, x \neq 0$.

Hint: Let $R > 0$ be such that $\dot{V}$ is P.D. on $B_R = \left\{x \| \|x\|^2 \leq R^2\right\}$ and $B_R \subset \Omega$, and let

$$M = \max_{x \in B_R} V(x) \tag{23}$$

V is continuous & B_R compact $\Longrightarrow M$ exists. Also, $M > 0$ since $V(x)$ can assume strictly positive values arbitrarily close to the origin. For any $R > r > 0$, there exists $x(0)$ such that $0 < \|x(0)\| < r$, and $V(x(0)) = a > 0$. Since $\dot{V}(x)$ is positive definite (locally in Ω), $V(x(t)) > V(x(0)) > 0$ for all $t \geq 0$. Let $U = \{x | x \in B_R \text{ and } V(x)\} \geq a$. Then U is compact. Thus there exists $L > 0$ such that

$$L = \min_{x \in U} \left\{\dot{V}\right\} \tag{24}$$

If $\|x(t, x_0)\| < R$ for all $t \geq 0$, then

$$\begin{aligned} V(x(t, x_0)) - V_0(x_0) = \int_0^t \dot{V}(x(t, x_0))\, dt \geq \int_0^t L dt = Lt \\ \Longrightarrow V(x(t, x_0)) \geq V_0(x_0) + Lt > M \end{aligned} \tag{25}$$

when $t > \frac{M - V(x_0)}{L}$ which contradicts Equation (23). Thus, the E.P. is unstable.

Solution:

(i) Let $R > 0$ be such that $\dot{V}$ is P.D. on $B_R = \left\{x \| \|x\|^2 \leq R^2\right\}$ and $B_R \subset \Omega$, and let

$$M = \max_{x \in B_R} V(x) \tag{26}$$

V is continuous & B_R compact $\Longrightarrow M$ exists. Also, $M > 0$ since $V(x)$ can assume strictly positive values arbitrarily close to the origin. For any $R > r > 0$, there exists $x(0)$ such that $0 < \|x(0)\| < r$, and $V(x(0)) = a > 0$. Since $\dot{V}(x)$ is positive definite (locally in Ω), $V(x(t)) > V(x(0)) > 0$ for all $t \geq 0$. Let $U = \{x | x \in B_R \text{ and } V(x)\} \geq a$. Then U is compact. Thus there exists $L > 0$ such that

$$L = \min_{x \in U} \left\{\dot{V}\right\} \tag{27}$$

If $\|x(t, x_0)\| < R$ for all $t \geq 0$, then

$$V(x(t, x_0)) - V_0(x_0) = \int_0^t \dot{V}(x(t, x_0))\,dt \geq \int_0^t L dt = Lt$$
$$\Longrightarrow V(x(t, x_0)) \geq V_0(x_0) + Lt > M \tag{28}$$

when $t > \frac{M - V(x_0)}{L}$ which contradicts Equation (23). Thus, the E.P. is unstable.

(ii) If we take $V(x) = x^2$, we can see that $V(0) = 0 \quad \forall t \geq 0$. And $\dot{V}(x) = 2x\dot{x} = 2xc(x) > 0, \forall x \neq 0$, so $\dot{V}$ is globally positive definite.

Therefore, the equilibrium point of $\dot{x} = c(x)$ is unstable.

References

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.