香港中文大學
The Chinese University of Hong Kong

THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5070 Nonlinear Control Systems

Assignment #7

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Show that if a function $x : [0, \infty) \to R^n$ is uniformly continuous, and there exists a positive definite quadratic function $V(x)$ such that

$$\int_0^\infty V(x(t))\, dt < 0 \tag{1}$$

then $x(t)$ tends to zero as $t \to \infty$.

Solution:

Because $V(x)$ is a positive definite quadratic function, we can express $V(x(t))$ as follows

$$V(x(t)) = x^T P x \tag{2}$$

where P is a positive definite matrix.

Define $f(t) = \int_0^t x(\tau)\, d\tau$. We claim that $f(t)$ has a finite limit as $t \to \infty$. Otherwise, if

$$\lim_{t\to\infty} f(t) = \int_0^\infty x(\tau)\, d\tau = \infty \tag{3}$$

we will have

$$\int_0^\infty \|x(\tau)\|^2 d\tau = \infty \tag{4}$$

In addition,

$$\int_0^\infty \|x(\tau)\|^2 d\tau \le \lambda_{max}(P) \int_0^\infty x^T P x dt = \lambda_{max}(P) \int_0^\infty V(x(t))\, dt \tag{5}$$

Hence,

$$\int_0^\infty V(x(t))\, dt = \infty \tag{6}$$

which is contradicted to Equation (1). Therefore, $f(t)$ has a finite limit as $t \to \infty$.

Besides, because $x : [0, \infty) \to R^n$ is uniformly continuous, that is $\dot{f}(t)$ is uniformly continuous, by Barbalat's Lemma 4.2, we can conclude that $\dot{f}(t) \to 0$ as $t \to 0$, that is $x(t)$ tends to zero as $t \to \infty$.

Problem 2

Consider the following one dimensional single-input nonlinear control system

$$\dot{x} = \theta g(x, t) + u \tag{7}$$

where θ is some constant parameter, and $g(x, t)$ is some bounded smooth function defined for all t and x.

(a) Assuming θ is known, show that, under the following state feedback nonlinear controller

$$u = -\theta g(x, t) - kx \tag{8}$$

where $k > 0$, the equilibrium point of the closed-loop system is globally asymptotically stable.

(b) If θ is unknown, the feedback controller $u = -\theta g(x, t) - kx$ is not implementable. One can use adaptive control to control the system. Show that, under the following adaptive controller

$$\begin{aligned} u &= -\hat{\theta} g(x, t) - kx \\ \dot{\hat{\theta}} &= g(x, t)\, x \end{aligned} \tag{9}$$

the closed-loop system takes the following form

$$\begin{aligned} \dot{x} &= \phi g(x, t) - kx \\ \dot{\hat{\theta}} &= g(x, t)\, x \end{aligned} \tag{10}$$

where $\phi = \theta - \hat{\theta}$ (you can interpret $\hat{\theta}$ as an estimation of θ).

(c) Using a Lyapunov-like function $V = x^2 + \phi^2$ to show that both x and $\hat{\theta}$ are bounded and $\lim_{t\to\infty} x(t) = 0$.

(d) For $g(x, t) = cos(x)\, sin(t)$ and $k = 2$, do the simulation for the closed-loop system using MATLAB with $x(0) = 0$ and $\hat{\theta}(0) = 1$. Plot $x(t)$; $\hat{\theta}(t)$; $\phi(t)$; $u(t)$ for $0 < t < 40$ seconds.

Solution:

(a) Substituting the controller in Equation (8) into the system in Equation (7) yields that

$$\dot{x} = \theta g(x, t) + u = \theta g(x, t) - \theta g(x, t) - kx = -kx \tag{11}$$

Because $k > 0$, according to linear system stability theory, $-k$ is Hurwitz. Therefore, the equilibrium point of the closed-loop system is globally asymptotically stable.

(b) Substituting the controller in Equation (9) into the system in Equation (7) yields that

$$\begin{aligned} \dot{x} &= \theta g\left(x,t\right)+u \\ u &= -\hat{\theta} g\left(x,t\right)-kx \\ \dot{\hat{\theta}} &= g\left(x,t\right)x \end{aligned} \implies \begin{aligned} \dot{x} &= \phi g\left(x,t\right)-kx \\ \dot{\hat{\theta}} &= g\left(x,t\right)x \end{aligned} \tag{12}$$

where $\phi = \theta - \hat{\theta}$ (you can interpret $\hat{\theta}$ as an estimation of θ).

(c) $V = x^2 + \phi^2$ is lower bounded obviously. And its derivative

$$\begin{aligned} \dot{V} &= 2x\dot{x} + 2\phi\dot{\phi} \\ &= 2x\left(\phi g\left(x,t\right)-kx\right)+2\phi\left(\dot{\theta}-\dot{\hat{\theta}}\right) \\ &= 2x\left(\phi g\left(x,t\right)-kx\right)+2\phi\left(0-g\left(x,t\right)x\right) \\ &= -2kx^2 \le 0 \end{aligned} \tag{13}$$

This implies that $V\left(x\left(t\right)\right) \le V\left(x\left(0\right)\right), \forall t > 0$, which indicate that x and ϕ should all be bounded. Taking the derivative to $\dot{V}$ yields that

$$\ddot{V} = -4kx\dot{x} = -4kx\left(\phi g\left(x,t\right)-kx\right) \tag{14}$$

Here, x, ϕ, and $g\left(x,t\right)$ are all bounded. Therefore, $\ddot{V}$ is bounded. According to Barbalat's Lemma 4.1, $\dot{V}$ is uniformly continuous. Again, using Barbalat's Lemma 4.3, $\dot{V}\left(x,t\right) \to 0$ as $t \to \infty$, which means $\lim_{t\to\infty} x\left(t\right) = 0$.

(d) We set $\theta = \pi$ and use following Simulink to get the results:

And we use the following code to plot the results:

```
clear all; clc;
figg1 = openfig('x.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "x"+".pdf";
saveas(gcf,filename);
close(figg1);
figg2 = openfig('theta.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\hat{\theta}$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
```

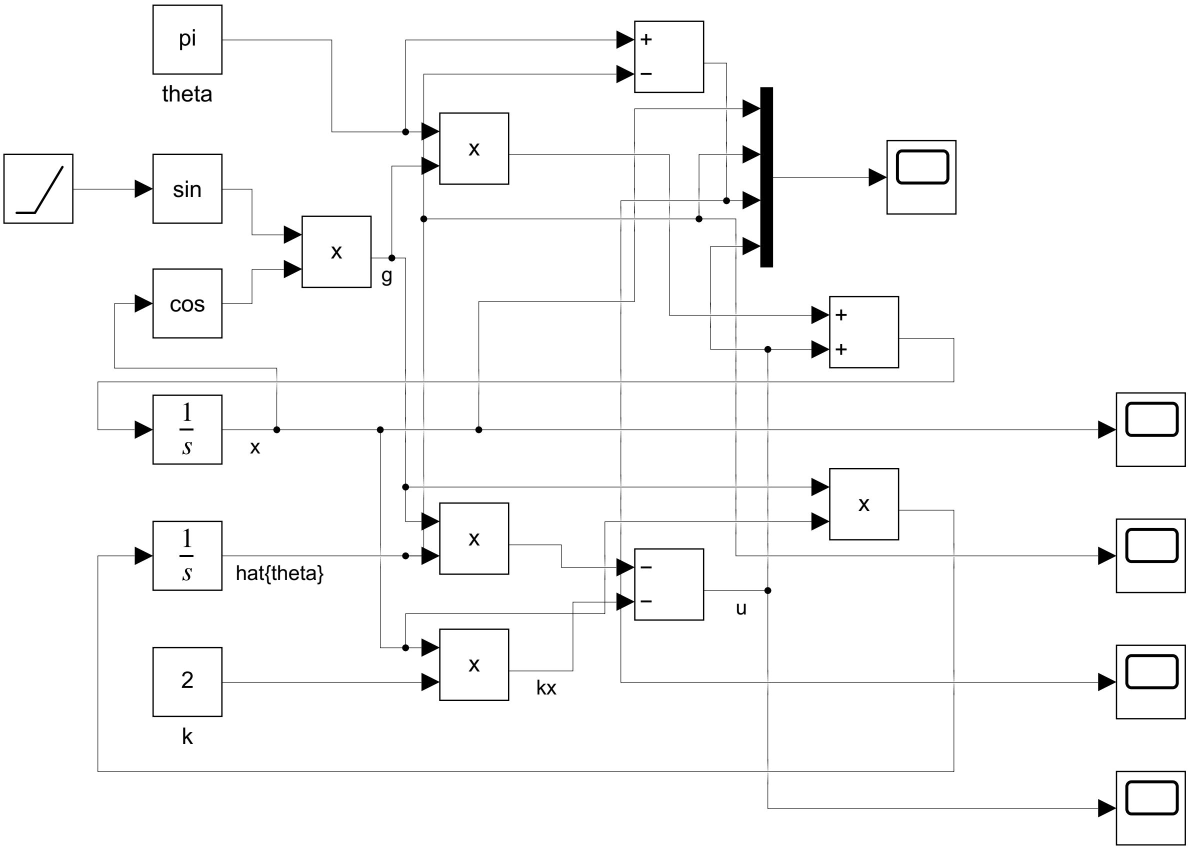

Figure 1: Block diagram for the system.

```
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "theta"+".pdf";
saveas(gcf,filename);
close(figg2);
figg3 = openfig('phi.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\phi$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "phi"+".pdf";
saveas(gcf,filename);
close(figg3);
figg4 = openfig('u.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$u$', 'interpreter','latex');
title('');
a = get(gca,'XTickLabel');
```

```
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "u"+".pdf";
saveas(gcf,filename);
close(figg4);
```

The results are shown as follows:

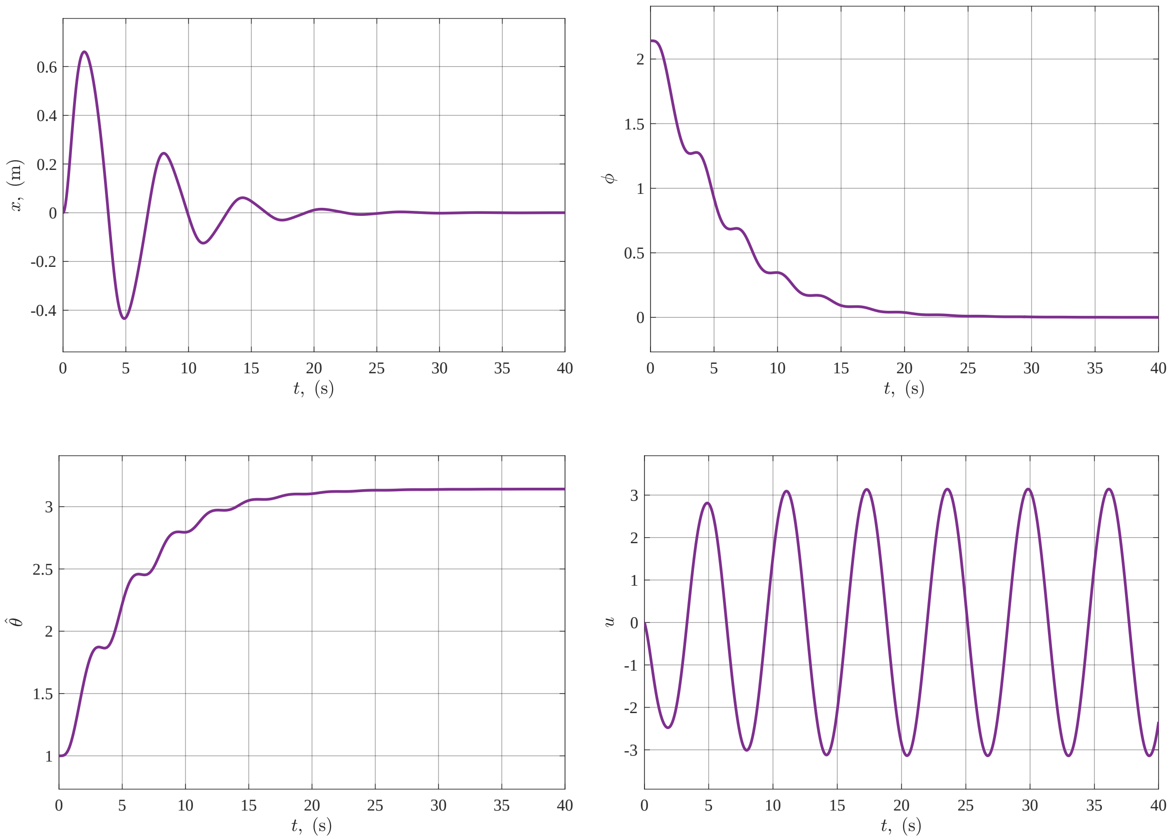

Figure 2: Simulation results.

Problem 3

Consider the system $\dot{x} = f(x)$ where $x \in R^n$, and f is continuously differentiable with $f(0) = 0$. Let J be the Jacobian matrix of f at the origin. It is known from the Lyapunov's linearization method that the equilibrium at $x = 0$ of this system is unstable if at least one of the eigenvalues of J has positive real part. Prove a special case of this result by assuming that $n = 2$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Hint: Let $V(x) = x_1^2 - x_2^2$.

Solution:

Because $J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, the system can be described as

$$\begin{cases} \dot{x}_1 = x_1 + g_1(x_1, x_2) \\ \dot{x}_2 = -2x_2 + g_2(x_1, x_2) \end{cases} \tag{15}$$

where $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are higher order terms.

Define $V(x) = x_1^2 - x_2^2$ satisfies $V(0) = 0$ and we can assume positive values arbitrarily near the origin. Take the time derivative to $V(x)$:

$$\begin{aligned} \dot{V}(x) &= 2x_1\dot{x}_1 - 2x_2\dot{x}_2 \\ &= 2x_1(x_1 + g_1(x_1, x_2)) - 2x_2(-2x_2 + g_2(x_1, x_2)) \\ &= 2x_1^2 + 4x_2^2 + 2x_1 g_1(x_1, x_2) - 2x_2 g_2(x_1, x_2) \end{aligned} \tag{16}$$

Because $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are higher order terms, $\forall \alpha_{1,2} > 0$, $\exists r > 0$ such that $\|g_i(x_1, x_2)\| < \alpha_i \|x_i\|, i = 1, 2, \forall x \in B_r$. Therefore,

$$\begin{aligned} \dot{V}(x) &= 2x_1^2 + 4x_2^2 + 2x_1 g_1(x_1, x_2) - 2x_2 g_2(x_1, x_2) \\ &\geq 2\|x_1\|^2 + 4\|x_2\|^2 - 2\alpha_1\|x_1\|^2 - 2\alpha_2\|x_2\|^2 \\ &= (2 - 2\alpha_1)\|x_1\|^2 + (4 - 2\alpha_2)\|x_2\|^2 \end{aligned} \tag{17}$$

as long as we select $0 < \alpha_1 < 1$ and $0 < \alpha_2 < \frac{1}{2}$, $\dot{V}(x)$ can defined as positive definite in B_r.

Problem 4

Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_1 + u \end{aligned} \tag{18}$$

using backstepping to design a state feedback controller to globally stabilize the origin.

Solution:

Letting $u = u_a - x_1$ gives

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= u_a \end{aligned} \tag{19}$$

In this form, $\eta = x_1$ and $\zeta = x_2$.

$$f(\eta) = 0 \tag{20}$$

and

$$g(\eta) = \eta \tag{21}$$

Therefore,

$$\phi(x_1) = \frac{-\alpha\eta - f(\eta)}{g(\eta)} = -\alpha \tag{22}$$

Here, $\alpha > 0$. The closed-loop system has a control Lyapunov function

$$V(x_1) = \frac{1}{2}x_1^2 \tag{23}$$

By Lemma 1, we can have a control Lyapunov function

$$\begin{aligned} V_\alpha(x_1, x_2) &= V(x_1) + \frac{1}{2}(x_2 - \phi(x_1))^2 \\ &= \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \alpha)^2 \end{aligned} \tag{24}$$

with respect to

$$\begin{aligned} u_a &= \phi_\alpha(x_1, x_2) \\ &= \frac{\partial\phi(\eta)}{\partial\eta}[f(\eta) + g(\eta)\zeta] - \frac{\partial V(\eta)}{\partial\eta}g(\eta) - k(\zeta - \phi(\eta)) = -x_1^2 - k(x_2 + \alpha) \end{aligned} \tag{25}$$

where $k > 0$. Therefore,

$$u = u_a - x_1 = -x_1^2 - x_1 - k(x_2 + \alpha) \tag{26}$$

Problem 5

(a) Using the Lyapunov function candidate $V(x) = x^2$ to determine the stability of the origin of the following system

$$\dot{x} = -x^3 + x^2 \sin^2 x \tag{27}$$

(b) Using backstepping to design a state feedback controller to globally stabilize the origin of the following system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 \sin^2 x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_3 + \exp(x_2) u \end{aligned} \tag{28}$$

Solution:

(a) $V(x) = x^2$ is positive definite. Taking the time derivative to $V(x)$ yields

$$\dot{V}(x) = 2x\dot{x} = 2x\left(-x^3 + x^2 \sin^2 x\right) = -2\left(x^4 - x^3 \sin^2 x\right) < -2\left(x^4 - x^3 x\right) = 0 \tag{29}$$

Hence, $\dot{V}(x)$ is negative definite. Therefore, the system is globally asymptotically stable.

(b) Letting $u = \exp(-x_2)(u_a - x_3)$ gives

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 \sin^2 x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_a \end{aligned} \tag{30}$$

In this form, $\eta = x_1$ and $\zeta = x_2$.

$$f(\eta) = -x_1^3 \tag{31}$$

and

$$g(\eta) = x_1 \sin^2 x_1 \tag{32}$$

From part (a), we can have a control Lyapunov function

$$V(x_1) = \frac{1}{2} x_1^2 \tag{33}$$

with respect to

$$\phi(x_1) = x_1 \tag{34}$$

By Lemma 1, we can have a control Lyapunov function

$$\begin{aligned} V_\alpha(x_1, x_2) &= V(x_1) + \frac{1}{2}(x_2 - \phi(x_1))^2 \\ &= \frac{1}{2} x_1^2 + \frac{1}{2}(x_2 - x_1)^2 \end{aligned} \tag{35}$$

with respect to

$$\begin{aligned} \phi_\alpha\left(x_1, x_2\right) & \\ &= \frac{\partial \phi\left(\eta\right)}{\partial \eta}\left[f\left(\eta\right)+g\left(\eta\right)\zeta\right]-\frac{\partial V\left(\eta\right)}{\partial \eta}g\left(\eta\right)-k\left(\zeta-\phi\left(\eta\right)\right) \\ &= -x_1^3+x_1x_2\sin^2 x_1-x_1^2\sin^2 x_1-k_1\left(x_2-x_1\right) \end{aligned} \tag{36}$$

where $k_1>0$. Applying the extension of Lemma 1 to the whole system yields

$$\begin{aligned} V\left(x_1, x_2, x_3\right) &= V_\alpha\left(x_1, x_2\right)+\frac{1}{2}\left(x_3-\phi\left(x_1, x_2\right)\right)^2 \\ &= \frac{1}{2}x_1^2+\frac{1}{2}\left(x_2-x_1\right)^2 \\ &\quad +\frac{1}{2}\left(x_3+x_1^3-x_1x_2\sin^2 x_1+x_1^2\sin^2 x_1+k_1\left(x_2-x_1\right)\right)^2 \end{aligned} \tag{37}$$

with respect to

$$\begin{aligned} u_a &= \phi_\alpha\left(x_1, x_2, x_3\right) \\ &= \frac{\partial \phi_\alpha\left(x_1, x_2\right)}{\partial\left(x_1, x_2\right)}\left[f\left(x_1, x_2\right)+g\left(x_1, x_2\right)x_3\right]-\frac{\partial V_\alpha\left(x_1, x_2\right)}{\partial\left(x_1, x_2\right)}g\left(x_1, x_2\right)-k\left(x_3-\phi_\alpha\left(x_1, x_2\right)\right) \\ &= \left(-3x_1^2+x_2\sin^2 x_1+2x_1x_2\sin x_1\cos x_1-2x_1\sin^2 x_1-2x_1^2\sin x_1\cos x_1+k_1\right)\left(-x_1^3+x_1x_2\sin^2 x_1\right) \\ &\quad +\left(x_1\sin^2 x_1+k_1\right)x_3-\left(x_2-x_1\right)-k_2\left(x_3+x_1^3-x_1x_2\sin^2 x_1+x_1^2\sin^2 x_1+k_1\left(x_2-x_1\right)\right) \end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned} u &= \exp\left(-x_2\right)\left(u_a-x_3\right) \\ &= \exp\left(-x_2\right)\left(-3x_1^2+x_2\sin^2 x_1+2x_1x_2\sin x_1\cos x_1-2x_1\sin^2 x_1-2x_1^2\sin x_1\cos x_1+k_1\right) \\ &\quad \cdot\left(-x_1^3+x_1x_2\sin^2 x_1\right)+\exp\left(-x_2\right)\left(x_1\sin^2 x_1+k_1\right)x_3-\exp\left(-x_2\right)\left(x_2-x_1\right) \\ &\quad -k_2\exp\left(-x_2\right)\left(x_3+x_1^3-x_1x_2\sin^2 x_1+x_1^2\sin^2 x_1+k_1\left(x_2-x_1\right)\right) \end{aligned} \tag{39}$$