香港中文大學
The Chinese University of Hong Kong

The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5070 Nonlinear Control Systems

Assignment #8

by

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2022-23 Term 1

Problem 1

Consider the controlled van del Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon\left(1 - x_1^2\right)x_2 + u, \epsilon > 0 \\ y &= x_1 \end{aligned} \tag{1}$$

(a) Calculate the relative degree of the system.

(b) Find a state feedback control law so that the equilibrium point at the origin of the closed-loop is globally asymptotically stable.

Solution:

(a)

$$\dot{y} = \dot{x}_1 = x_2 \tag{2}$$

$$\ddot{y} = \dot{x}_2 = -x_1 + \epsilon\left(1 - x_1^2\right)x_2 + u = \alpha(x) + \beta(x)u \tag{3}$$

Therefore, the relative degree of the system is 2.

(b) The state feedback control law so that the equilibrium point at the origin of the closed-loop is globally asymptotically stable is shown as follows:

$$\begin{aligned} u &= \frac{y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)} - \alpha(x)}{\beta(x)} \\ &= -\alpha_1 \dot{e} - \alpha_2 e - \alpha(x) \\ &= -\alpha_1 \dot{y} - \alpha_2 y - \alpha(x) \\ &= -\alpha_1 x_2 - \alpha_2 x_1 + x_1 - \epsilon\left(1 - x_1^2\right)x_2 \end{aligned} \tag{4}$$

Because $\rho = n$, the closed-loop system can always be made an asymptotically stable linear system.

Choosing $\alpha_1 = 1$ and $\alpha_2 = 2$, the control law becomes

$$u = -x_2 - x_1 \tag{5}$$

Problem 2

The motion equation of a single-link robot manipulator is given by

$$J\ddot{\theta} + MgL\sin\theta = u \tag{6}$$

(a) Give the state space equation of (6) with $x_1 = \theta$, $x_2 = \dot{\theta}$, and $y = x_1$

(b) Assume $J = 5$, $gL = 1$, and $M = 10$. Let $y_d(t)$ be a sufficiently smooth time function over $t \in [0, \infty)$. Let $e(t) = y(t) - y_d(t)$. Design a state feedback control law so that $e(t)$ satisfies $\ddot{e}(t) + 2\dot{e}(t) + e(t) = 0$.

(c) Check your design in simple simulation for $y_d(t)$ to be a unit step input, and a sinusoidal function $\sin t$, respectively.

Solution:

(a) The state space equation of (6) with $x_1 = \theta$, $x_2 = \dot{\theta}$, and $y = x_1$ is shown as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{MgL\sin x_1}{J} + \tfrac{1}{J}u \\ y &= x_1 \end{aligned} \tag{7}$$

(b) Because $J = 5$, $gL = 1$, and $M = 10$, the state space equation of (6) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\sin x_1 + 0.2u \\ y &= x_1 \end{aligned} \tag{8}$$

Then, we will find the relative degree ρ:

$$\dot{y} = \dot{x}_1 = x_2 \tag{9}$$

$$\ddot{y} = \dot{x}_2 = -2\sin x_1 + 0.2u = \alpha(x) + \beta(x)u \tag{10}$$

Therefore, the relative degree of the system is 2. state feedback control law so that the system is globally asymptotically stable

$$\begin{aligned} u &= \frac{y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)} - \alpha(x)}{\beta(x)} \\ &= \frac{\ddot{y}_d(t) - \alpha_1\dot{e}(t) - \alpha_2 e(t) - \alpha(x)}{\beta(x)} \\ &= 5\left(\ddot{y}_d(t) - \alpha_1\dot{e}(t) - \alpha_2 e(t) + 2\sin x_1\right) \end{aligned} \tag{11}$$

Substituting Equation (11) into Equation (10) obtains

$$\ddot{y}(t) = \ddot{y}_d(t) - \alpha_1\dot{e}(t) - \alpha_2 e(t) \tag{12}$$

That is

$$\ddot{e}(t) + \alpha_1 \dot{e}(t) + \alpha_2 e(t) = 0 \tag{13}$$

Therefore, $\alpha_1 = 2$ and $\alpha_2 = 1$ can satisfy the requirements of $\ddot{e}(t) + 2\dot{e}(t) + e(t) = 0$. Hence, the state feedback control law is

$$u = 5\left(\ddot{y}_d(t) - 2(x_2 - \dot{y}_d) - (x_1 - y_d) + 2\sin x_1\right) \tag{14}$$

(c) We use the following Simulink to get the results:

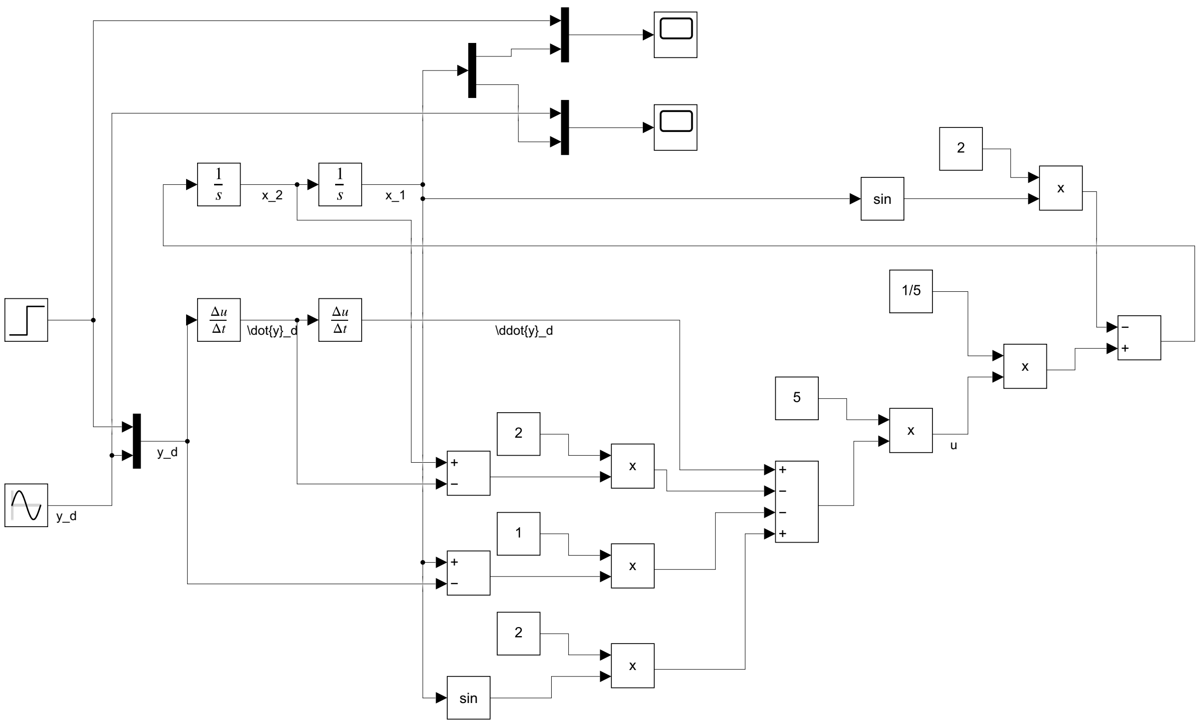


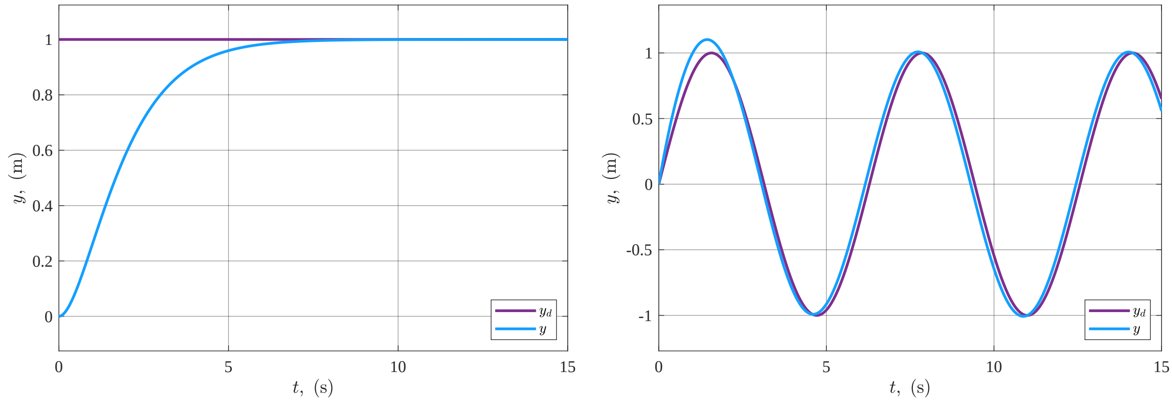
Figure 1: Block diagram for the system.

And we use the following code to plot the results:

```
clear all; clc;
figg1 = openfig('Q2Step.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$y, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
legend('$y_d$', '$y$', 'interpreter','latex','Location','southeast');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2Step"+".pdf";
saveas(gcf,filename);
close(figg1);
figg2 = openfig('Q2Sine.fig','reuse');
```

```
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$y, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
legend('$y_d$', '$y$', 'interpreter','latex','Location','southeast');
title('');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2Sine"+".pdf";
saveas(gcf,filename);
close(figg2);
```

The results are shown as follows:



(a) Step input.

(b) Sinusoidal input.

Figure 2: Simulation results.

Problem 3

Another way to achieve asymptotic tracking: Consider

$$y^{(n)} = \alpha(x) + \beta(x) u \tag{15}$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \alpha(x) + \beta(x) u \\ y &= x_1 \end{aligned} \tag{16}$$

where $x = \begin{bmatrix} y & \dot{y} & \cdots & y^{(n-1)} \end{bmatrix}$, $\alpha(x)$ and $\beta(x)$ are known and $\beta(x) \neq 0$ for all x. Given $y_d(t)$, let $e(t) = y(t) - y_d(t)$ and define

$$s = e^{(n-1)} + \alpha_1 e^{(n-2)} + \ldots + \alpha_{n-1} e \tag{17}$$

where α_1, ..., α_{n-1} are such that

$$\lambda^{n-1} + \alpha_1 \lambda^{n-2} + \ldots + \alpha_{n-2}\lambda + \alpha_{n-1} \tag{18}$$

is a stable polynomial.

(a) Design a control law such that

$$\dot{s} + ks = 0 \tag{19}$$

where $k > 0$.

(b) Show that the control law achieves $\lim_{t\to\infty} e(t) = 0$.

(c) Show that, when $y_d = 0$, the closed-loop system is globally asymptotically stable.

Solution:

(a) To achieve asymptotic tracking for Equation (16), note that using an input transformation

$$\alpha(x) + \beta(x) u = u_a \tag{20}$$

or

$$u = \frac{u_a - \alpha(x)}{\beta(x)} \tag{21}$$

gives

$$y^{(n)} = u_a \tag{22}$$

which is in the chain integrator form. In order to achieve Equation (19) with $\rho = n$, i.e.,

$$\begin{aligned}&\left(y^{(n)} - y_d^{(n)}\right) + (\alpha_1 + k)\left(y^{(n-1)} - y_d^{(n-1)}\right) + (\alpha_2 + k\alpha_1)\left(y^{(n-2)} - y_d^{(n-2)}\right)\\&+ \ldots + (\alpha_{n-1} + k\alpha_{n-2})\left(y^{(1)} - y_d^{(1)}\right) + k\alpha_{n-1}(y - y_d) = 0\end{aligned} \tag{23}$$

Substituting Equation (22) into (23) gives

$$\begin{aligned}&\left(u_a - y_d^{(n)}\right) + (\alpha_1 + k)\left(y^{(n-1)} - y_d^{(n-1)}\right) + (\alpha_2 + k\alpha_1)\left(y^{(n-2)} - y_d^{(n-2)}\right)\\&+ \ldots + (\alpha_{n-1} + k\alpha_{n-2})\left(y^{(1)} - y_d^{(1)}\right) + k\alpha_{n-1}(y - y_d) = 0\\\Longrightarrow & u_a = y_d^{(n)} - \sum_{i=1}^{n}(\alpha_i + k\alpha_{i-1})\, e^{n-i}\end{aligned} \tag{24}$$

Here $\alpha_0 = 1$ and $\alpha_n = 0$. Thus,

$$u = \frac{u_a - \alpha(x)}{\beta(x)} = \frac{y_d^{(n)} - \sum_{i=1}^{n}(\alpha_i + k\alpha_{i-1})\, e^{n-i} - \alpha(x)}{\beta(x)} \tag{25}$$

(b) Because

$$s = e^{(n-1)} + \alpha_1 e^{(n-2)} + \ldots + \alpha_{n-1} e \tag{26}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are such that

$$\lambda^{n-1} + \alpha_1\lambda^{n-2} + \ldots + \alpha_{n-2}\lambda + \alpha_{n-1} \tag{27}$$

is a stable polynomial, for

$$\dot{s} = e^{(n)} + \alpha_1 e^{(n-1)} + \ldots + \alpha_{n-1} e^{(1)} \tag{28}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are such that

$$\lambda^{n} + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-2}\lambda^2 + \alpha_{n-1}\lambda \tag{29}$$

is also a stable polynomial by Routh-Hurwitz stability criterion. Therefore,

$$\begin{aligned}\dot{s} + ks = e^{(n)} &+ (\alpha_1 + k)\, e^{(n-1)} + (\alpha_2 + k\alpha_1)\, e^{(n-2)}\\&+ \ldots + (\alpha_{n-1} + k\alpha_{n-2})\, e^{(1)} + k\alpha_{n-1} e = 0\end{aligned} \tag{30}$$

where $\alpha_i + k\alpha_{i-1}$, $i = 1, \ldots, n$ are such that

$$\lambda^{n} + (\alpha_1 + k)\,\lambda^{n-1} + \ldots + (\alpha_{n-1} + k\alpha_{n-2})\,\lambda + k\alpha_{n-1} \tag{31}$$

is also a stable polynomial. As a result, e satisfies $\lim_{t\to\infty} e(t) = 0$.

(c) When $y_d = 0$, the closed-loop system is

$$\dot{x} = Ax + Bk\left(x, 0, \ldots, 0\right) = \left(A - B\left[k\alpha_{n-1}, \left(\alpha_{n-1} + k\alpha_{n-2}\right), \ldots, \left(\alpha_1 + k\right)\right]\right)x$$
$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -k\alpha_{n-1} & -\left(\alpha_{n-1} + k\alpha_{n-2}\right) & -\left(\alpha_{n-2} + k\alpha_{n-3}\right) & \ldots & -\left(\alpha_1 + k\right) \end{bmatrix} x \tag{32}$$

Clearly, $\left(A - B\left[k\alpha_{n-1}, \left(\alpha_{n-1} + k\alpha_{n-2}\right), \ldots, \left(\alpha_1 + k\right)\right]\right)$ is a companion matrix with its characteristic polynomial being

$$\lambda^n + \left(\alpha_1 + k\right)\lambda^{n-1} + \ldots + \left(\alpha_{n-1} + k\alpha_{n-2}\right)\lambda + k\alpha_{n-1} \tag{33}$$

Thus, when $y_d = 0$, the closed-loop system is globally asymptotically stable.