Chapter 0 Overview of Linear Control Systems

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Outline

1.1 N-th order ODE or transfer function

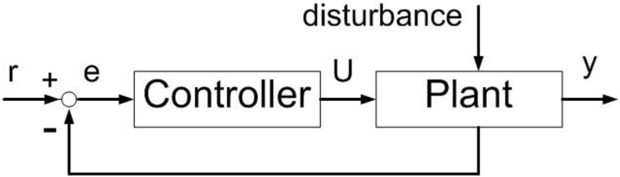
$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

or $\dfrac{Y(s)}{U(s)} = \dfrac{b_0 s^n + b_1 s^{(n-1)} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{(n-1)} + \cdots + a_{n-1}s + a_n}$

1.2 State space equations

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx + Du$$

2 Basic Control Problems



- Stabilization
- Asymptotic Tracking
- Disturbance Rejection/ Attenuation
- Robustness

3 Linear Control Methods

- **Classical**:

Laplace transformation, Bode diagram,

Root locus, Nyquist criterion, etc.

- **State Space**:

Pole placement, Observer design, Feedforward design,

Internal model design, Optimal control, etc.

4 Limitation of Linear Control

- Physical plants are inherently nonlinear.

  Linear modeling is only a simplification of real systems.

- To achieve a good performance in, e.g. high precision machine, high performance fighter airplane, nonlinear methodologies must be developed to handle nonlinearity of real systems.

5 Solution of the state equation satisfying initial condition

5.1 Initial value problem

Given

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad t \geq t_0,\ x \in R^n \tag{1}$$

where $A \in R^{n\times n}$ and $B \in R^{n\times m}$, $x_0 \in R^n$, and $u(t) \in R^m$.

Question: Find $x(t)$, $t \geq t_0$ s.t. $x(t_0) = x_0$ and $x(t)$ satisfies (1). Here x_0 is called the initial state of (1) and $u(t)$ is input.

5.2 Solution with $n = m = 1$

➢ Since $(e^{-At}x(t))' = (-Ae^{-At}x(t) + e^{-At}\dot{x}(t)) = e^{-At}(-Ax(t) + \dot{x}(t))$, (1) implies

$$(e^{-At}x(t))' \;=\; e^{-At}Bu(t) \tag{2}$$

Integrating on both sides of (2) from t_0 to t gives

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$\begin{aligned} \Rightarrow\; x(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ &= e^{A(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)d\tau\right] \end{aligned} \tag{3}$$

➢ This way of solving (1) is called separation of variables.

Verification:

(1) $x(t_0) = x_0$

(2)

$$\begin{aligned}\dot{x}(t) &= Ae^{A(t-t_0)}[x_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)d\tau] \\ &\quad + e^{A(t-t_0)}[0 + e^{-A(t-t_0)}Bu(t)] \\ &= Ax(t) + Bu(t)\end{aligned}$$

Note that use is made of the properties:

$$(e^{At})' = Ae^{At}, \quad e^0 = 1, \quad e^{A(t-\tau)} = e^{At}e^{-A\tau}$$

➢ Matrix exponential function e^{At}

Given $A \in R^{n \times n}$

$$e^{At} \triangleq I_n + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots = \sum_{j=0}^{\infty} \frac{A^jt^j}{j!} \tag{4}$$

where $A^0 = I_n$, and, for $k \geq 1$, $A^k = \underbrace{A \cdot A \cdot \cdots \cdot A}_{k \text{ terms}}$.

5.3 Solution for the general case

For example, given $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, then

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix},$$

..., and

$$A^k = \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}.$$

Therefore,

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} t + \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} \frac{t^2}{2!} + \\ & \quad \cdots + \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} \frac{t^k}{k!} + \cdots \\ &= \begin{pmatrix} \sum_{j=0}^{\infty} \frac{2^j t^j}{j!} & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{3^j t^j}{j!} \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \end{aligned}$$

5.3 Solution for the general case

➢ **Properties of e^{At}**

1. $e^{0_{n\times n}t} = I_n$
2. $\frac{de^{At}}{dt} = Ae^{At}$
3. $e^{A(t+\tau)} = e^{At}e^{A\tau}$
4. $\left(e^{At}\right)^{-1} = e^{-At}$

Verification of Property 2:

$$\begin{aligned}\frac{de^{At}}{dt} &= \frac{d}{dt}\left[I_n + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots\right] \\ &= 0 + A + \frac{A^2t}{1} + \cdots + \frac{A^kt^{k-1}}{(k-1)!} + \cdots = Ae^{At}\end{aligned}$$

5.4 Remarks

➢ The solution of (1) is still given by (3), that is

$$x(t) = e^{A(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)d\tau\right] \quad (5)$$

But e^{At} is a $n \times n$ matrix called matrix exponential function.

➢ Verification of (5):

(1) $x(t_0) = x_0$ clearly
(2)

$$\begin{aligned} \frac{dx(t)}{dt} &= Ae^{A(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)d\tau\right] \\ &\quad + e^{A(t-t_0)}\left[0 + e^{-A(t-t_0)}Bu(t)\right] \\ &= Ax(t) + Bu(t) \end{aligned}$$

5.4 General case

➢ Remarks:

1. The output

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{A(t-t_0)}\left[x_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)d\tau\right] + Du(t)
\end{aligned}
$$

2. The special case $t_0 = 0$:

$$
x(t) = e^{At}\left[x_0 + \int_{0}^{t} e^{-A\tau}Bu(\tau)d\tau\right]
$$

3. (5) is the unique solution of (1). It consists of two parts: zero input response + zero initial state response.