Chapter 1 Introduction to Nonlinear Control

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Outline

1 Nonlinearity: 1.1 Nonlinear Function

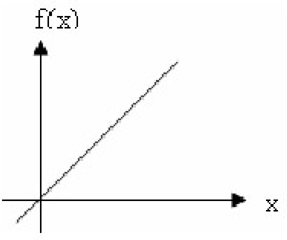


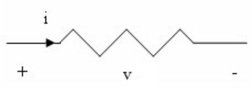
Fig. 1.1

$y = f(x)$ is linear,

iff $f(x_1 + x_2) = f(x_1) + f(x_2)$ and $f(\alpha x_1) = \alpha f(x_1)$, $\forall \alpha \in \Re$,

iff $y = kx$ with k some constant.

1.2 Nonlinear Elements in Dynamical Systems



$$i = h(v) = -v + \frac{1}{3}v^3$$

Fig. 1.2 : twin-tunnel diode

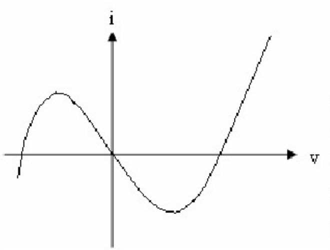


Fig. 1.3 : twin-tunnel diode

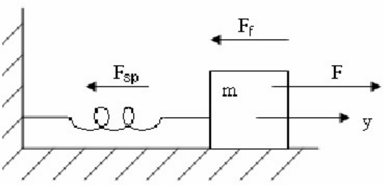


Fig. 1.4 : mass-spring-damper system

1.2 Nonlinear Elements in Dynamical Systems (cont.)

- 1.2.1 Nonlinear resistor
- 1.2.2 Nonlinear spring

$m\ddot{y} + F_f + F_{sp} = F$

F_{sp}: restoring force of the spring

$$F_{sp} = g(y) = \begin{cases} ky & k \text{ constant} \quad \text{linear spring} \\ ky + ka^2y^3 & \quad \text{hardening spring} \end{cases}$$

Beyond a certain displacement, a small displacement increment produces a large force increment.

- 1.2.3 Other nonlinearities

Saturation, Deadzone, on-off, backlash etc. (pp. 171-174)

1.3 Nonlinear Systems Examples
1.3.1 Duffing equation (mass-spring-damper system)

$$m\ddot{y} + F_f + F_{sp} = F$$

where,

$F_f = c\dot{y}$	is frictional force due to viscosity
$F_{sp} = k(1 + a^2y^2)y$	is hardening spring
$F = Acos\omega t$	is external force

$$\Rightarrow \quad m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

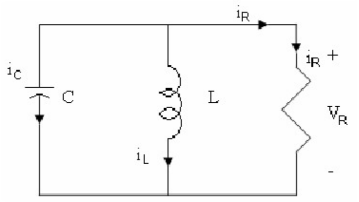


Fig. 1.5

Kirchhoff's Current Law

$$i_C + i_L + i_R = 0$$

$$\Rightarrow \; C\frac{dV_R}{dt} + \frac{1}{L}\int_{-\infty}^{t} V_R(\tau)d\tau + h(V_R) = 0$$

$$\Rightarrow \; C\frac{d^2V_R}{dt} + \frac{V_R}{L} + \frac{dh}{dV_R}\frac{dV_R}{dt} = 0$$

To be continued...

$$\Rightarrow \; C\frac{d^2V_R}{dt} + \frac{V_R}{L} + (-1 + V_R^2)\frac{dV_R}{dt} = 0$$

$$LC\frac{d^2V_R}{dt} + L(V_R^2 - 1)\dot{V}_R + V_R = 0$$

which is in the form of $m\ddot{y} + 2c(y^2 - 1)\dot{y} + ky = 0$ with $y = V_R$ where m, c, k are positive constant, and was used by Van der Pol to study oscillation in vacuum tube circuits.

- 1.3.3 **Other nonlinear systems**

Pendulum equation, robot arm manipulator, helicopter, spacecraft, neural networks, motors, etc.

1.4 Nonlinear Dynamical Systems in State Equations 1.4.1 State equation of Duffing equation

Given

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

Let $x_1 = y$ and $x_2 = \dot{y}$,

Then

$$\begin{aligned} \dot{x}_1 &= x_2 & (\neq Ax + Bu) \\ \dot{x}_2 &= \frac{1}{m}(-cx_2 - kx_1 - ka^2x_1^3 + A\cos\omega t) \\ y &= x_1 \end{aligned}$$

1.4.2 State equation of Van der Pol equation

Given

$$m\ddot{y} + 2c(y^2 - 1)\dot{y} + ky = 0$$

Let

$$x_1 = y$$
$$x_2 = \dot{y}$$

Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\{-2c(x_1^2 - 1)x_2 - kx_1\}$$
$$y = x_1$$

1.4.3 General representation of nonlinear dynamical systems

$$\dot{x}(t) = f(x(t), t) \quad , \quad x(0) = x_0 \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \qquad f(x,t) = \begin{bmatrix} f_1(x_1, \dots, x_n, t) \\ \dots \\ \dots \\ f_n(x_1, \dots, x_n, t) \end{bmatrix}$$

n is called the dimension of the system.

When $f(x,t) = f(x)$, i.e. $f(x,t)$ does not explicitly rely on t, we have

$$\dot{x}(t) = f(x(t)) \tag{2}$$

In this case, the system is called autonomous system, otherwise nonautonomous system.

1.4.3 General representation of nonlinear dynamical systems (cont.)

Van der Pol equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}\{-2c(x_1^2 - 1)x_2 - kx_1\} \end{bmatrix}$$

is an autonomous system while the Duffing equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}(-cx_2 - kx_1 - ka^2x_1^3 + A\cos\omega t) \\ y &= x_1 \end{aligned}$$

is nonautonomous.

The linear system

$$\dot{x} = Ax$$

is a special case of autonomous system when $f(x) = Ax$.

1.4.4 Nonlinear control systems

$$\dot{x} = f(x,u), \quad y = h(x,u) \tag{3}$$

where

$$x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \in \Re^n, \quad u = \begin{bmatrix} u_1 \\ \dots \\ u_m \end{bmatrix} \in \Re^m, \quad y = \begin{bmatrix} y_1 \\ \dots \\ y_p \end{bmatrix} \in \Re^p$$

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ \dots \\ f_n(x,u) \end{bmatrix} \in \Re^n, \quad h(x,u) = \begin{bmatrix} h_1(x,u) \\ \dots \\ h_p(x,u) \end{bmatrix} \in \Re^p$$

f and h are continuous functions of x and u.
Clearly (3) includes linear systems

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

as a special case with $f(x,u) = Ax + Bu$ and $h(x,u) = Cx + Du$.

1.4.5 Nonlinear control law

$$u = k(x,t) = \begin{bmatrix} k_1(x,t) \\ \ldots \\ k_m(x,t) \end{bmatrix} \in \Re^m \tag{4}$$

Applying (4) to (3) results in a closed-loop system of the form

$$\begin{aligned} \dot{x} &= f(x, k(x,t)) = f_c(x,t) \\ y &= h(x, k(x,t)) = h_c(x,t) \end{aligned} \tag{5}$$

which is a nonautonomous dynamic system. When $k(x,t)$ does not explicitly rely on t, i.e. $k(x,t) = k(x)$, (5) becomes

$$\begin{aligned} \dot{x} &= f(x, k(x)) = f_c(x) \\ y &= h(x, k(x)) = h_c(x) \end{aligned} \tag{6}$$

which is an autonomous system.

1.5 Solution of a Dynamical System

Consider

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x \in \Re^n, \quad f \in C^0 \tag{7}$$

A C^0 time function $x(t) \in \Re^n$ defined for $t \in [0, T), \ T > 0$, is said to be a solution of (7) satisfying the initial condition if

(i) $x(0) = x_0$ and

(ii) $\frac{dx(t)}{dt} = f(x(t)) \qquad 0 \leq t < T$.

x(t) is also called the system trajectory starting at x_0.

1.5 Solution of a Dynamical System (cont.)

In the special case where $f(x) = Ax$ with $A \in \Re^{n\times n}$, (7) always has a unique solution.

$$x(t) = e^{At}x_0, \quad 0 \le t < \infty, \quad \forall\ x_0$$

However, for nonlinear systems, (7) may not have a solution for some initial condition, or its solution for some initial condition may not be unique, or its solution may tend to infinity as t tends to some finite T.

Sometimes, to emphasize the reliance of a solution on the initial condition x_0, we may use the notation $x(t, x_0)$.

1.5 Solution of a Dynamical System (cont.)

Example 1 (finite escape time)

$$\dot{x} = -x + x^2, \quad x \in \Re^1, \quad x(0) = x_0$$

Solution: Integrating gives

$$\frac{dx}{-x + x^2} = dt \quad \Rightarrow \quad x(t) = \frac{x_0}{x_0 + (1 - x_0)e^t} \tag{8}$$

Verify: (i) $x(0) = x_0$, (ii) $\dot{x}(t) = \frac{-x_0(1-x_0)e^t}{(x_0+(1-x_0)e^t)^2}$.

On the other hand,

$$-x(t)+x^2(t) = \frac{-x_0}{x_0 + (1 - x_0)e^t} + \frac{x_0^2}{(x_0 + (1 - x_0)e^t)^2} = \frac{-x_0(1 - x_0)e^t}{(x_0 + (1 - x_0)e^t)^2}$$

Therefore

$$\dot{x} = -x + x^2$$

1.5 Solution of a Dynamical System (cont.)

But (8) may not be defined for all $t \geq 0$. In fact, it can be seen that when $x_0 > 1$, there exists a finite $t > 0$ such that

$$x_0 + (1 - x_0)e^t = 0 \quad \text{or} \quad t = \ln \frac{x_0}{x_0 - 1}$$

(8) is not defined at $t = \ln \frac{x_0}{x_0 - 1}$. On the other hand, when $x_0 \leq 1$, (8) is defined for all $t \geq 0$. In conclusion, (8) exists for $0 \leq t < T$ where

$$T = \begin{cases} \ln \frac{x_0}{x_0 - 1} & x_0 > 1 \\ \infty & x_0 \leq 1 \end{cases}$$

When T is finite, we say T is finite escape time.

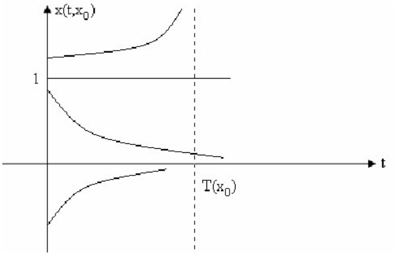


Fig. 1.6 Finite escape time

Example 2 (nonuniqueness of the solution)

$$\frac{dx}{dt} = 2\sqrt{x}, \qquad x(0) = 0$$

Solution:

$$x(t) = \begin{cases} 0 & 0 \le t < \infty \\ t^2 & 0 \le t < \infty \end{cases}$$

1.6 Equilibrium Points

An equilibrium of (7) is a vector $x^* \in \Re^n$ satisfying

$$f(x^*) = 0$$

Let $x(t, x^*) = x^*$, $\forall t \in [0, \infty)$. Then

(i) $x(0, x^*) = x^*$

(ii) $\frac{dx(t,x^*)}{dt} = f(x^*)$

Thus, the equilibrium of (7) is the solution of (7) over $[0, \infty)$ starting at x^*. That is, x^* has the property that once $x(t_0) = x^*$ for some t_0, then $x(t) = x^*$, $\forall\, t \geq t_0$.

1.6 Equilibrium Points (cont.)

For linear systems $\dot{x} = Ax$, x^* satisfies

$$Ax^* = 0$$

If A is invertible, $x^* = 0$ is the unique equilibrium. Otherwise, the solution of $Ax^* = 0$ is a linear subspace in $\Re^n$ each point of which is an equilibrium of $\dot{x} = Ax$. In other words, linear systems do not have multiple isolated equilibrium points.

On the other hand, a nonlinear system may have multiple isolated equilibrium points, e.g., the system $\dot{x} = -x + x^2$ has two isolated equilibrium points $x^* = \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right.$

1.6 Equilibrium Points (cont.)

Pendulum example

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

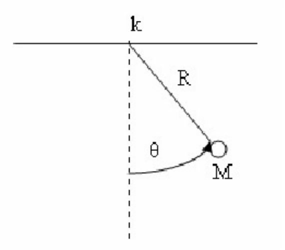


Fig. 1.7 Pendulum

where
k is the friction coefficient, M is mass
R is pendulum length, g is the gravity constant

1.6 Equilibrium Points (cont.)

Letting $x_1 = \theta$, $x_2 = \dot{\theta}$ gives

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR^2} x_2 - \frac{g}{R} \sin x_1 \end{aligned}$$

Letting $f(x) = 0$ gives

$$\begin{aligned} x_2 &= 0 \\ \sin x_1 &= 0 \end{aligned} \quad \Rightarrow \quad x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \cdots \right\}$$

Physically, these points correspond to the pendulum resting exactly at the vertical up and down positions.

2.1 Limit cycles (isolated periodic motion)

In addition to finite escape time, and multiple isolated equilibrium points, nonlinear systems have some other peculiar behaviors not seen in linear systems.

2.1 Limit cycles (isolated periodic motion)

Consider linear systems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

e.g. undamped mass spring system or LC circuit.

The solution is

$$x_1(t) = \sqrt{x_{10}^2 + x_{20}^2}\sin(\omega t + \theta)$$

$$x_2(t) = \sqrt{x_{10}^2 + x_{20}^2}\cos(\omega t + \theta)$$

where $\theta = tg^{-1}(\frac{x_{10}}{x_{20}})$.

2.1 Limit cycles (isolated periodic motion)

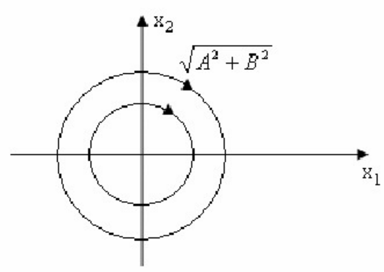


Fig. 1.8

The equation describes a periodical motion called harmonic oscillation. Since $x_1^2 + x_2^2 = x_{10}^2 + x_{20}^2$, the trajectories of the system are circles in the $x_1 - x_2$ plane whose radii depend on the initial condition.

2.1 Limit cycles (isolated periodic motion)

In general, for the linear system

$$\dot{x} = Ax, \quad x(0) = x_0$$

the solution is given by $x(t) = e^{At}x_0$, and its limiting behavior is

$$x(t) = e^{At}x_0 \begin{cases} \to 0 \text{ as } t \to \infty & A \text{ stable,} \\ \text{unbounded} & A \text{ unstable,} \\ \begin{array}{c}\text{periodic function depending on} \\ \text{both } A \text{ and initial condition}\end{array} & \text{marginally stable.} \end{cases}$$

2.1 Limit cycles (isolated periodic motion)

Next consider Van der Pol equation

$$\ddot{y} + 2c(y^2 - 1)\dot{y} + ky = 0$$

or

$$\dot{x}_1 = x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = -2c(x_1^2 - 1)x_2 - kx_1, \qquad x_2(0) = x_{20}$$

It can be shown that the trajectories of the system starting at any non-zero initial point converge to a closed-curve in the $x_1 - x_2$ plane.

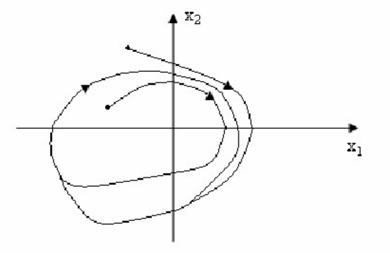


Fig. 1.9

2.1 Limit cycles (isolated periodic motion)

The closed-curve represents a periodic motion different from the harmonic oscillation as follows:
(a) isolated
(b) not depending on initial conditions
Such a periodic motion is called a limit cycle. It only happens to nonlinear systems.
Physical interpretation: due to the nonlinear damping $2c(y^2 - 1)$,
when y is large, the damper removes energy from the system
$\Rightarrow$ the motion is convergent, and
when y is small, the damper adds energy into the system
$\Rightarrow$ the motion is divergent.
Because the damping varies with y, the system motion can never grow unboundedly nor decay to zero.
Thus, it displays a sustained oscillation independent of initial conditions.

2.2 Bifurcations

Consider $m\ddot{y} + c\dot{y} + ka^2y^3 = A\cos\omega t$ or

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha x_1 - x_1^3$$

(can be viewed as an undamped Duffing equation)
α is a parameter.

The equilibrium points $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ satisfy

$$\begin{array}{c} x_2^* = 0 \\ -\alpha x_1^* - x_1^{*3} = 0 \\ \Leftrightarrow x_1^* = 0 \text{ and } x_1^{*2} + \alpha = 0 \end{array} \quad \Rightarrow x^* = \begin{bmatrix} x_1^* \\ 0 \end{bmatrix} \text{ where}$$

$$x_1^* = \begin{cases} 0, \ j\sqrt{\alpha}, \quad -j\sqrt{\alpha} & \alpha > 0 \\ 0 & \alpha = 0 \\ 0, \ \sqrt{-\alpha}, \quad -\sqrt{-\alpha} & \alpha < 0 \end{cases}, \text{ with } j = \sqrt{-1}.$$

2.2 Bifurcations (cont.)

That is, as α varies from positive to negative, three equilibrium points merge into one, and then split into three.

Such phenomenon as quantitative change of parameter leads to qualitative change of motion behavior is called bifurcation.

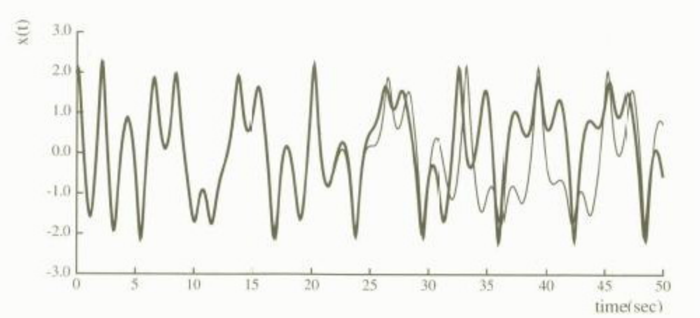
For this example, $\alpha = 0$ is called critical or bifurcation value.

2.3 Chaos

Consider $\ddot{y} + 0.05\dot{y} + y^3 = 7.5\cos t$ (Duffing equation)
Lightly damped, sinusoidally forced mechanical structure undergoing large elastic deflection.
Solution of this system for

$$x(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad x(0) = \begin{bmatrix} 3.01 \\ 4.01 \end{bmatrix}$$

is shown as follows (Fig 1.6 of the textbook).



2.3 Chaos (cont.)

Butterfly effect:

E. N. Lorenz, "Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" Dec. 29, 1972.

Small variation in initial conditions incurs radical changes in the solution - chaos.
Atmospheric dynamics display chaotic behavior, making long-term weather prediction impossible.

2.4 Other behaviors

Jump resonance, subharmonic generation, asynchronous quenching and frequency-amplitude dependence of free vibrations.

3 Nonlinear System Analysis and Design

◇ **Complexity**:
many peculiar behaviors.

◇ **Difficulty**:
closed-form solutions are usually unavailable, analytic, graphical, or approximate methods are adopted.

◇ **Major tools**:

- Phase plane methods: mainly for second order systems
- Describing functions: mainly for predicting limit cycles
- Lyapunov theory: indirect method or linearization, and direct method: construct a scalar function to study stability
- Other more recent methods: Input-output linearization, Sliding mode control, Backstepping design, Adaptive control, etc.