Chapter 2 Phase Plane Analysis

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Outline

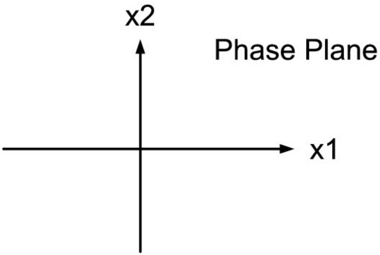
1 Introduction: 1.1 Phase Plane Method

Consider the following two dimensional autonomous system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \quad x_1(0) = x_{10} \\ \dot{x}_2 &= f_2(x_1, x_2), \quad x_2(0) = x_{20} \end{aligned} \tag{1}$$

Assume $x(t, x_0) = \begin{bmatrix} x_1(t, x_0) \\ x_2(t, x_0) \end{bmatrix}$, $t \in [0, \infty)$, is the system trajectory starting at x_0.

- The plane with $x_1(t, x_0)$ and $x_2(t, x_0)$ as coordinates is called phase plane.



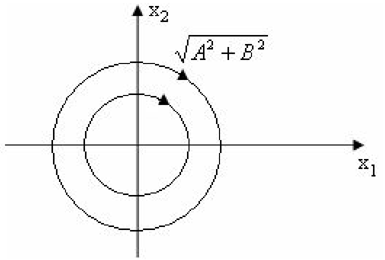
1.1 Phase Plane Method (cont.)

- The family of trajectories $x(t, x_0),\ t \in [0, \infty)$, plotted on the phase plane corresponding to various initial conditions x_0 is called phase portrait.

For example, consider the harmonic system:

$$
\begin{aligned}
\dot{x}_1 &= \omega x_2, \quad x_1(0) = A \\
\dot{x}_2 &= -\omega x_1, \quad x_2(0) = B \\
\Rightarrow \quad x_1(t, x_0) &= \sqrt{A^2 + B^2}\sin(\omega t + \theta) \\
x_2(t, x_0) &= \sqrt{A^2 + B^2}\cos(\omega t + \theta) \\
\Rightarrow \quad x_1^2 + x_2^2 &= A^2 + B^2
\end{aligned}
$$

1.1 Phase Plane Method (cont.)



- Usage: display graphically the nature of the system response.

For example, the above phase portraits show that the solution of the harmonic system starting from any initial condition is periodic, and neither converge to the origin nor diverge to infinity. Such a system is called marginally stable.

1.2 Slope of the Phase Trajectories

The slope of the phase trajectory passing through a point (x_1, x_2) is determined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

- It is unique at each (x_1, x_2) as long as $\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \neq 0$.
- It implies that the phase trajectories will not intersect.

1.3 Singular Points

$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

i.e. singular points are equilibrium points of two dimensional systems.

- At singular points, $\frac{dx_2}{dx_1} = \frac{0}{0}$. The slope is indeterminate. Many trajectories may intersect at such points.
- For linear systems, the stability behaviors are uniquely determined by the nature of singular points.
- However, for nonlinear systems, the behaviors may be more complex as illustrated in Fig. 2.2 of the textbook. The system may have multiple isolated singular points some of which are stable and some of which are not. The stable singular point may be globally asymptotically stable or just locally asymptotically stable.

1.3 Singular Points (cont.)

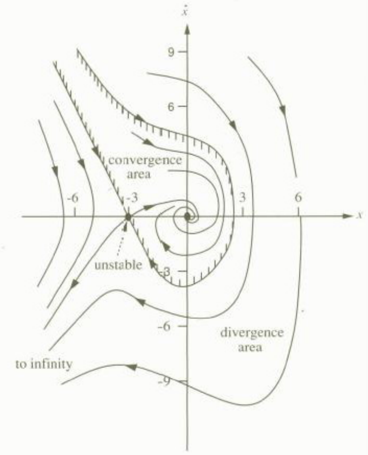


Figure 2.2 : The phase portrait of a nonlinear system

2 Constructing Phase Portraits

2.1 Analytic Method

Solving (1) to obtain $x_1(t, x_0)$ and $x_2(t, x_0)$.
Eliminating t to obtain an equation of the form $g(x_1, x_2, x_0) = 0$. For example,

$$
\begin{aligned}
x_1(t, x_0) &= \sqrt{A^2 + B^2}\sin(\omega t + \theta) \\
x_2(t, x_0) &= \sqrt{A^2 + B^2}\cos(\omega t + \theta) \\
\Rightarrow \quad x_1^2 + x_2^2 &= A^2 + B^2 \;\; = x_0^2
\end{aligned}
$$

2.2 Numerical Method

Using computer simulation. Problem 2 of Assignment 1.

3 Phase Plane Analysis of Linear Systems

- For classification of a singular point.
- Visuallizing the system trajectories.

3.1 Solution of Second Order Linear Systems

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

Assume A is nonsingular. Then the system has a unique equilibrium point at $x = 0$. Let $\lambda(A) = \{\lambda_1, \lambda_2\}$. Then, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ It is known that

$$x(t, x_0) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At} x_0 = \begin{cases} k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}, & \lambda_1 \neq \lambda_2 \\ (\alpha + \beta t) e^{\lambda_1 t}, & \lambda_1 = \lambda_2 \end{cases}$$

where $k_1,\ k_2$ or $\alpha,\ \beta \in R^2$ depend on A and x_0.

3.2 Types of Singular Points

Consider the following four cases based on the eigenvalue locations:

- Case 1: λ_1 and λ_2 are both real and have the same sign (stable or unstable node)
- Case 2: λ_1 and λ_2 are both real and have the opposite sign (saddle point)
- Case 3: λ_1 and λ_2 are complex conjugate with nonzero real part (stable or unstable focus)
- Case 4: λ_1 and λ_2 are complex conjugate with zero real part (center)

3.3 Phase Portraits Vs. Types of Singular Points

Case 1 (stable or unstable node) :

- If $\lambda_1 < 0$ and $\lambda_2 < 0$, $\quad x(t, x_0) \to 0$ exponentially as $t \to \infty$, as shown in Fig. 2.9(a) of the textbook. The singular point is called stable node.
- If $\lambda_1 > 0$ and $\lambda_2 > 0$, $\quad x(t, x_0) \to \infty$ exponentially as $t \to \infty$, as shown in Fig. 2.9(b). The singular point is called unstable node.

3.3 Phase Portraits Vs. Types of Singular Points (cont.)

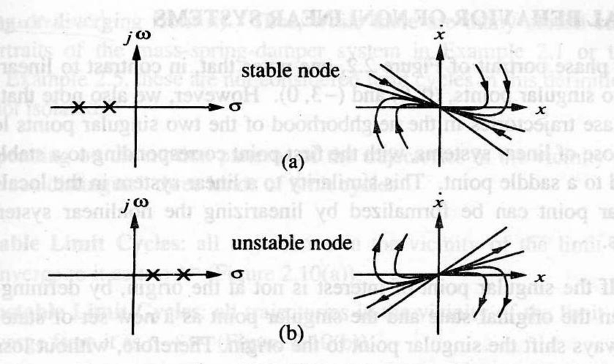


Fig. 2.9 (a)(b) of textbook

3.3 Phase Portraits Vs. Types of Singular Points (cont.)

Case 2 (saddle point):
Without loss of generality (WLG), assume $\lambda_1 < 0, \quad \lambda_2 > 0$. Then, $\lim_{t\to\infty} x(t, x_0) = \infty$ for $\forall\ x_0$ such that $k_2 \neq 0$ as shown in Fig. 2.9(c).
Two special cases,
a)

$$k_2 = 0, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_1 e^{\lambda_1 t} = \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} e^{\lambda_1 t} \quad \Rightarrow \quad \frac{x_1(t)}{x_2(t)} = \frac{k_{11}}{k_{12}}$$

(straight lines approaching the origin.)
b)

$$k_1 = 0, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_2 e^{\lambda_2 t} \quad \rightarrow \quad \infty, \quad t \rightarrow \infty$$

A saddle point is always unstable.

3.3 Phase Portraits Vs. Types of Singular Points (cont.)

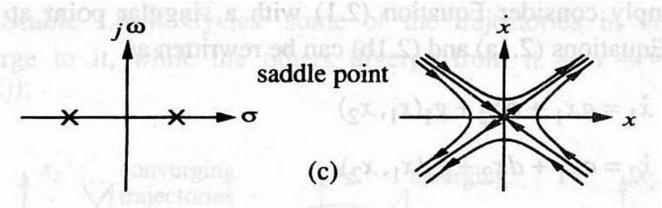


Fig. 2.9 (c) of textbook

3.3 Phase Portraits Vs. Types of Singular Points (cont.)

Case 3 (stable or unstable focus):
λ_1 and λ_2 are complex conjugate with nonzero real parts.
Let $\lambda_{1,2} = -\sigma \pm j\omega$. Then

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A(x_0)\sin(\omega t + \theta) \\ B(x_0)\cos(\omega t + \theta) \end{bmatrix} e^{-\sigma t}.$$

If $\sigma > 0$, then the singular point is called stable focus since $x(t, x_0) \to 0$ as $t \to \infty$ as shown in Fig. 2.9(d).
If $\sigma < 0$, then the singular point is called unstable focus since $x(t, x_0)$ goes unbounded as $t \to \infty$ as shown in Fig. 2.9(e).

3.3 Phase Portraits Vs.Types of Singular Points (cont.)

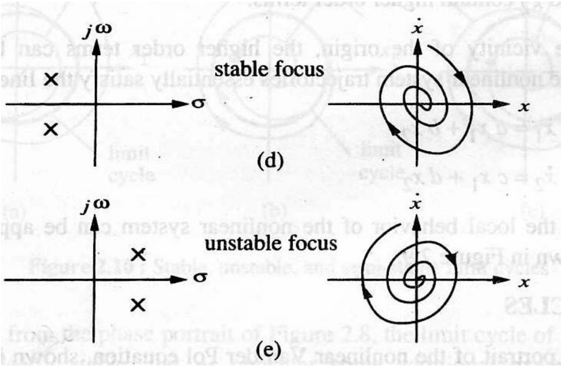


Fig. 2.9 (d)(e) of textbook

3.3 Phase Portraits Vs. Types of Singular Points (cont.)

Case 4 (center point)

$$\lambda_{1,2} = \pm j\omega$$

$$x(t, x_0) = \begin{bmatrix} x_1(t, x_0) \\ x_2(t, x_0) \end{bmatrix} = \begin{bmatrix} A\sin(\omega t + \theta) \\ B\cos(\omega t + \theta) \end{bmatrix}$$

$$\Rightarrow \qquad \frac{x_1^2}{A^2} + \frac{x_2^2}{B^2} = 1$$

$x(t, x_0)$ are ellipses and the singular point is the center of these ellipses as shown in Fig. 2.9(f).

3.3 Phase Portraits Vs. Types of Singular Points (cont.)

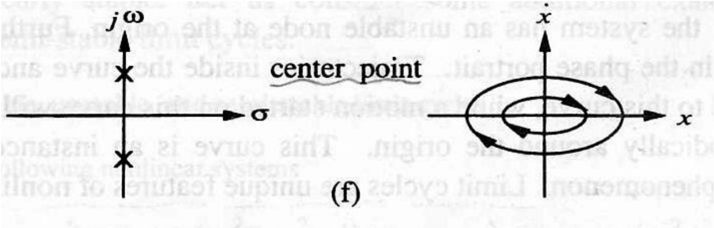


Fig. 2.9 (f) of textbook

3.4 Phase Portraits Vs. Types of Singular Points (cont.)

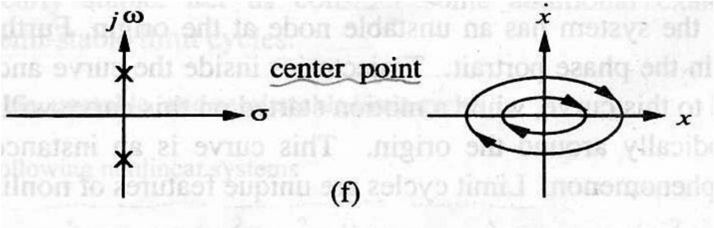


Fig. 2.9 (f) of textbook

3.5 Remark

Remark:
If A is singular, then at least one of the eigenvalues of A is at the origin. The solution of $Ax = 0$ is either a straight line when $rank(A) = 1$ or every point x is an equilibrium when $A = 0$. This case is less interesting as will be seen later.

4 Phase Plane Analysis of Nonlinear Systems
4.1 Jacobian Linearization

Consider the following two dimensional autonomous system:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Assume that

$$f_1(x_1^*, x_2^*) = 0$$
$$f_2(x_1^*, x_2^*) = 0$$

i.e. x^* is the equilibrium point. (E.P.)
Let

$$A_{x^*} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{x^*} = \left. \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, x_2) & \frac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1, x_2) & \frac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix} \right|_{x=x^*}$$

4.1 Jacobian Linearization (cont.)

Then Taylor expansion gives

$$f_1(x_1, x_2) = f_1(x_1^*, x_2^*) + a(x_1 - x_1^*) + b(x_2 - x_2^*) + o(x_1 - x_1^*, x_2 - x_2^*)$$
$$f_2(x_1, x_2) = f_2(x_1^*, x_2^*) + c(x_1 - x_1^*) + d(x_2 - x_2^*) + o(x_1 - x_1^*, x_2 - x_2^*)$$

where $o(x_1 - x_1^*, x_2 - x_2^*)$ means higher order terms in $(x_1 - x_1^*)$ and $(x_2 - x_2^*)$,
e.g. $(x_1 - x_1^*)^2, \quad (x_1 - x_1^*)(x_2 - x_2^*), \quad \ldots.$
Let $x_{\delta 1} = x_1 - x_1^*$, $x_{\delta 2} = x_2 - x_2^*$.
Then

$$\dot{x}_{\delta 1} = \dot{x}_1 = a x_{\delta 1} + b x_{\delta 2} + o(x_{\delta 1}, x_{\delta 2})$$
$$\dot{x}_{\delta 2} = \dot{x}_2 = c x_{\delta 1} + d x_{\delta 2} + o(x_{\delta 1}, x_{\delta 2})$$

4.1 Jacobian Linearization (cont.)

We call

$$\begin{aligned} \dot{x}_{\delta 1} &= a x_{\delta 1} + b x_{\delta 2} \\ \dot{x}_{\delta 2} &= c x_{\delta 1} + d x_{\delta 2} \end{aligned} \tag{2}$$

as the Jacobian linearization of (1) at E.P. $x = x^*$ and A_{x^*} is called Jacobian matrix of $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ at $x = x^*$.

To save notation, we will use

$$\begin{aligned} \dot{x}_1 &= a x_1 + b x_2 \\ \dot{x}_2 &= c x_1 + d x_2 \end{aligned} \quad \text{or} \quad \dot{x} = Ax \quad \text{where} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

to denote (2).

4.1 Jacobian Linearization (cont.)

Example:

Consider

$$\begin{aligned} \dot{x}_1 &= x_2 = f_1(x_1, x_2) \\ \dot{x}_2 &= -2x_1 - 0.5x_2 - x_1^2 = f_2(x_1, x_2) \end{aligned}$$

Clearly $f_1(0,0) = f_2(0,0) = 0$ and its Jacobian linearization at $x = 0$ is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 0.5x_2 \end{aligned}$$

4.2 Local Behavior of System (1)

It can be shown that the phase portraits of (1) in a neighborhood of an E.P. is quite similar to the phase portraits of (2) using the Lyapunov theory. For this reason, we call a singular point x^* of (1) as a node, saddle point, focus or center if $x_\delta = 0$ is the node, saddle point, focus, or center of (2).

4.2 Local Behavior of System (1) (cont.)

Example:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1x_2\end{aligned} \quad \Rightarrow \quad x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$A_{x^*} = \begin{bmatrix} -1 + x_2 & x_1 \\ 1 - 2x_2 & 1 - 2x_1 \end{bmatrix} \Rightarrow \quad A_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$\lambda(A_{(0,0)}) = \{1 \quad -1\} \Rightarrow$ saddle point of $\dot{x} = A_{(0,0)}x$.

$\lambda(A_{(1,1)}) = \{\frac{-1 \pm j\sqrt{3}}{2}\} \Rightarrow$ stable focus of $\dot{x} = A_{(1,1)}x$.

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a saddle point of (1), while

$x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a stable focus of (1).

4.3 Remark

Remark:
If the Jacobian matrix of A is singular, then, from the Lyapunov's linearization method to be studied in Chapter 3, the local behavior of the equilibrium of the nonlinear system in general cannot be inferred from the behavior of the system $\dot{x} = Ax$.

4.3 Three Types of Limit Cycles

4.3.1

- Limit Cycle: A limit cycle is defined as an isolated closed curve in the phase plane.
- Closed: the trajectory has to be closed, indicating the periodic nature of the motion, and
- Isolated: indicating the limiting nature of the cycle (with nearby trajectories converging or diverging from it)

Remark:

Let L denote a closed phase trajectory of a system. Let $\text{dist}(x, L) = \inf_{y \in L} \|x - y\|$ denote the distance of x to L. Then L is isolated if there exists some region $B = \{x \mid, \text{ dist}(x, L) < \varepsilon\}$ for some $\varepsilon > 0$, such that L is the only closed phase trajectory in the region B. An equilibrium x^* is a special case of a limit cycle. x^* is said to be isolated if, for some $\varepsilon > 0$, x^* is the only equilibrium inside the region $B = \{x \mid, \ \|x - x^*\| < \varepsilon\}$.

4.3.2 Three types of Limit cycles

- Stable Limit Cycles: $\lim_{t\to\infty} \text{dist}(x(t,x_0)-L)=0$ for all x_0 sufficiently close to L. That is, all trajectories in the vicinity of the limit cycle converge to it as $t\to\infty$
- Unstable Limit Cycles: all trajectories in the vicinity of the limit cycle diverge from it as $t\to\infty$
- Semi-Stable Limit Cycles: some of the trajectories in the vicinity of the limit cycle converge to it, while the others diverge from it as $t\to\infty$

4.3 Three Types of Limit Cycles (cont.)

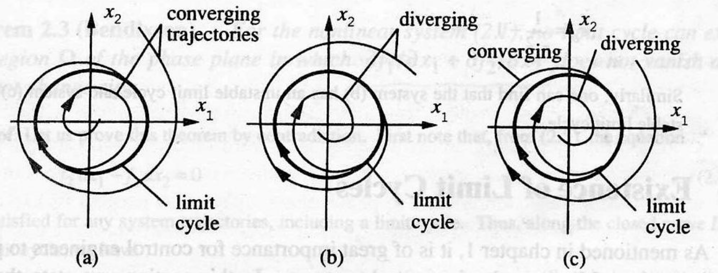


Figure 2.10 : Stable, unstable, and semi-stable limit cycles

4.3.3 Example

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1), \quad x_1(0) = x_{10}$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1), \; x_2(0) = x_{20}$$

Introducing polar coordinates

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}(\frac{x_2}{x_1})$$

gives

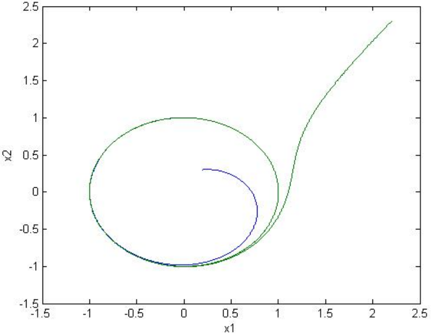
$$\begin{array}{ll} \frac{dr}{dt} = -r(r^2 - 1), & r(0) = r_0 \\ \frac{d\theta}{dt} = -1, & \theta(0) = \theta_0 \end{array} \quad \Rightarrow \quad \begin{array}{c} r(t) = \frac{1}{\sqrt{1 + c_0 e^{-2t}}} \\ \theta(t) = \theta_0 - t \end{array}$$

with $c_0 = \dfrac{1}{r_0^2} - 1$

If $c_0 = 0 \quad \Leftrightarrow \quad r_0 = 1, \quad$ then $r(t) = 1, \quad t \geq 0$

i.e. $x_1^2(t) + x_2^2(t) = 1$ is a limit cycle.

If $c_0 \neq 0, \quad$ then $\lim_{t\to\infty} r(t) = 1 \quad \Rightarrow$ stable limit cycle.



5 Existence of Limit Cycles

Limit cycles exist as the phenomenon in electronic oscillators, interaction of aerodynamics, aircraft wing flattering and structural vibration.

Theorem 2.1 (Poincare) [Index Theorem]

If a limit cycle exists in a second order system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

then

$$N = S + 1$$

where N is the number of nodes, centers and foci enclosed by a limit cycle, and S is the number of saddle points enclosed by a limit cycle.

Note: A limit cycle must enclose at least one equilibrium point.

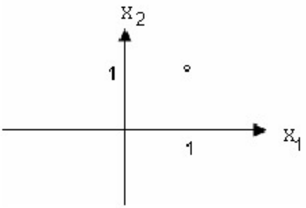
Example

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1x_2\end{aligned} \quad \Rightarrow \quad x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + x_2 & x_1 \\ 1 - 2x_2 & 1 - 2x_1 \end{bmatrix} \Rightarrow \quad A_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{saddle point}$$

and

$$A_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \text{stable focus since } \lambda(A_{(1,1)}) = \{\frac{-1 \pm j\sqrt{3}}{2}\}$$



5 Existence of Limit Cycles (cont.)

There is no limit cycle in the region $x_1^2 + x_2^2 < 2$ since $N(= 1) \neq S + 1(= 2)$.
The theorem is also called index theorem.

Theorem 2.2 (Poincare-Bendixson)

If a trajectory of the second-order system (1) remains in a finite region Ω, then one of the following is true.

- (a): The trajectory goes to an equilibrium point.
- (b): The trajectory tends to an asymptotic stable limit cycle.
- (c): The trajectory is itself a closed-curve.

Note: The theorem gives the asymptotic properties of the trajectories of second-order systems.
For linear systems, either (a) or (c) can happen.
For nonlinear systems, (b) may also happen.

5 Existence of Limit Cycles (cont.)

Theorem 2.3 (Bendixson)

For the nonlinear systems (1), no limit cycle can exist in a region Ω of the phase plane in which $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign.

Proof: From $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$, we have

$$f_2 dx_1 - f_1 dx_2 = 0 \tag{3}$$

for all $(x_1, x_2) \in \Omega$. Thus, if there exists a limit cycle in Ω which is a closed curve L, then

$$\int_L (f_1 dx_2 - f_2 dx_1) = 0. \tag{4}$$

By Stoke's Theorem, we have

$$\int_L (f_1 dx_2 - f_2 dx_1) = \int\int (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}) dx_1 dx_2 = 0 \tag{5}$$

where the integration on the right-hand side is carried out on the area enclosed by the limit cycle. Thus a contradiction occurs.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_2^2 \\ \dot{x}_2 &= -x_1 + x_1^2 x_2 \end{aligned} \quad \Rightarrow \quad \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 + x_2^2 > 0 \quad x \neq 0$$

No limit cycle can exist.
Note:

- (i): The origin is a center of the system. Theorem 2.1 cannot rule out the existence of a limit cycle encircling the origin since $N = S + 1$.
- (ii): All three theorems apply only to second-order systems.