Chapter 3 Fundamentals of Lyapunov Theory

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Outline

1 Introduction

1.1 Introduce the stability concepts for the system

$$\dot{x} = f(x), \quad x \in \Re^n \tag{1}$$

where it is assumed $f(0) = 0$.

- Stable
- Unstable
- Asymptotic stability
- Exponential stability
- Local and global asymptotic stability

1.2 Determine the stability of (1) without obtaining the solution of (1)

- Indirect method: linearization method
- Direct method: construct a Lyapunov function for (1)

1.3 Introduce some control approaches based on Lyapunov Theory

2 Concepts of stability 2.1 Notations

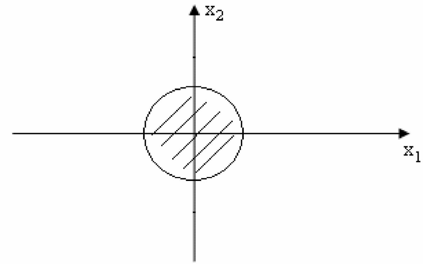


Fig. 3.1

Let $x = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix} \in R^n$. Then $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$

$B_R = \{x| \ \ \|x\| < R, \quad R > 0\}$ spherical region

$S_R = \{x| \ \ \|x\| = R, \quad R > 0\}$ sphere itself

2.2 Assumption

- $f(0) = 0$

Remark: If $f(x^*) = 0$ for some $x^* \neq 0$. Let $z = x - x^*$.
Then $\dot{z} = \dot{x} = f(x) = f(z + x^*) = \hat{f}(z)$.
Thus $z = 0$ is the equilibrium of $\dot{z} = \hat{f}(z)$.
Without loss of generality, we only need to consider the equilibrium at the origin.

2.3 Stability and instability

- 2.3.1 Definition 3.3 (P.48)

The equilibrium point $x = 0$ of (1) is said to be stable if, for any $R > 0$, there exists $r > 0$, such that if $\|x_0\| < r$, then $\|x(t, x_0)\| < R$ for all $t \geq 0$. Otherwise the equilibrium point is unstable.

- 2.3.2 Definition 3.4

The equilibrium point $x = 0$ of (1) is said to be asymptotically stable if

(a) It is stable.

(b) There exists some $r > 0$ such that

$$\|x_0\| < r \quad \Rightarrow \quad \lim_{t \to \infty} x(t, x_0) = 0.$$

Note: (b) is called convergence condition.

- 2.3.3 Remarks

(a) The concepts were introduced by Russian mathematician Lyapunov, often called (asymptotic) stability in the sense of Lyapunov (i.s.o.L.), in contrast with other stability concepts.

(b) Linear systems $\dot{x} = Ax$

$\dot{x} = Ax$ is (strictly) stable $\Leftrightarrow$ $x = 0$ is A.S. i.s.o.L.

$\dot{x} = Ax$ is marginally stable $\Leftrightarrow$ $x = 0$ is S. i.s.o.L.

$\dot{x} = Ax$ is unstable $\Leftrightarrow$ $x = 0$ is U.S. i.s.o.L.

- 2.3.3 Remarks (cont.)

(c) Geometric interpretation of stability (Fig. 3.3 of textbook)

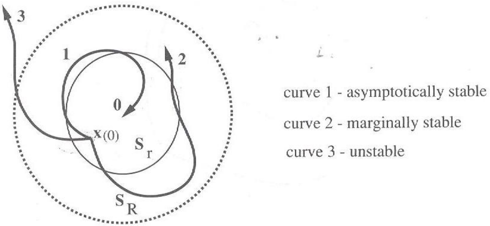


Figure 3.3 : Concepts of stability

$$\text{Given} B_R, \; \exists \; B_r \quad \text{such that} \quad x_0 \in B_r \; \Rightarrow x(t, x_0) \in B_R, \; \forall t > 0$$

- 2.3.3 Remarks (cont.)

(d) Asymptotic stability =Stability + Convergence.

Convergence does not imply stability (Fig. 3.5 of textbook)
Example (Vinogradov example)

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}, \quad \dot{x}_2 = \frac{x_2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}$$

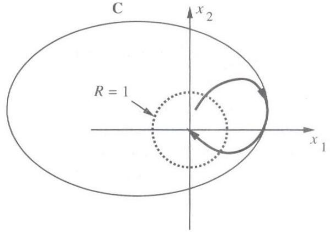


Figure 3.5 : State convergence does not imply stability

- 2.3.3 Remarks (cont.)

(e) Local A.S vs Global A.S.

$\lim_{t\to\infty} x(t, x_0) = 0$ is only required for all $x_0 \in B_r$ where r can be any arbitrarily small positive number.

The ball B_r is called a domain of attraction of the E.P.

If $B_r = R^n$, then the convergence is called global.

$$\text{S} + \text{G.C.} = \text{G.A.S.}$$

All (strictly) stable linear systems are G.A.S.

(a) Pendulum system

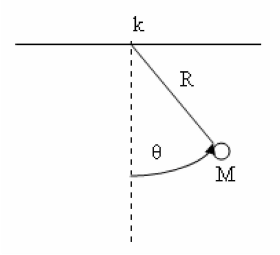


Fig. 3.2

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1$$
$$x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix} \right\}$$

2.4 Examples (cont.)

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{is} \quad \begin{cases} \text{(locally) A.S.} & k > 0 \\ S & k = 0 \end{cases}$$

$$x = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \quad \text{is} \quad \text{unstable}$$

(using either phase plane method or physical intuition)

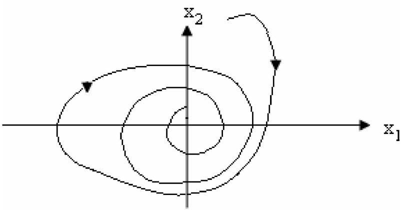


Fig. 3.3

An E.P. is G.A.S only if it is the unique E.P. of the system.

(b) Van der Pol equation

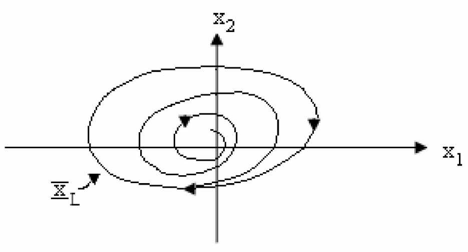


Fig. 3.4

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 0.2(x_1^2 - 1)x_2$$
$$\lim_{t\to\infty}(x(t, x_0) - x_L) = 0 \qquad \forall\ x_0 \neq 0.$$

Thus, $x = (0, 0)$ is an unstable E.P.

Definition 3.5 $x = 0$ is exponentially stable if there exists two strictly positive numbers α and λ such that

$$\|x(t, x_0)\| \le \alpha \|x_0\| e^{-\lambda t}, \quad \forall\, t > 0$$

for all $x_0 \in B_r$ where r is a positive number.

Example:

$$\begin{aligned} & \dot{x} = -(1 + \sin^2 x)x, \qquad x(0) = x_0, \qquad x \in \Re^1 \\ \Rightarrow \quad & x(t, x_0) = x_0 \exp\left[- \int_0^t 1 + \sin^2(x(\tau)) \right] d\tau \\ \Rightarrow \quad & \|x(t, x_0)\| \le \|x_0\| \exp(-t), \qquad \forall\, t > 0 \end{aligned}$$

2.5 Exponentially stable (cont.)

Remarks:

(a) E.S. $\Rightarrow$ $\lim_{t\to\infty} x(t, x_0) = 0$ at the rate faster than an exponential function. λ is called the rate of the exponential function, and $\tau = \frac{1}{\lambda}$ is called the time constant of the exponential function.

(b) Note that $e^{-1} = 0.3679$, $e^{-2} = 0.1353$, $e^{-3} = 0.0498$, $e^{-4} = 0.0183$. E.S. tells how fast the system trajectory approaches 0.

(c) Asymptotic convergence to the origin does not imply exponential convergence to the origin. For example, the function $x(t) = \frac{1}{1+t}$ converges to 0 as $t \to \infty$, but it does not converge to 0 exponentially.

(d) E.S $\Rightarrow$ A.S. $\Rightarrow$ S

2.6 Global A.S. or Global E.S.

Definition 3.6.

If A.S. or E.S. holds for any initial state x_0, the E.P. is said to be A.S. or E.S. in the large. It is also called G.A.S. or G.E.S.

For linear systems,

A.S. $\Leftrightarrow$ G.A.S. $\Leftrightarrow$ G.E.S.

instability $\Leftrightarrow$ exponential blowup.

3. Lyapunov's Indirect Method
3.1 Jacobian linearization

Consider (1), i.e., the system described by

$$\dot{x} = f(x)$$

where

$$x \in \Re^n, \quad f(x) = \begin{bmatrix} f_1(x) \\ \dots \\ f_n(x) \end{bmatrix}, \quad \text{and} \quad f(0) = 0.$$

We call $\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \dots & & \dots \\ \frac{\partial f_n(x)}{\partial x_1} & \dots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}$ the Jacobian matrix of $f(x)$.

Let $A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$. Then A is called the Jacobian matrix of $f(x)$ at $x = 0$.

3.1 Jacobian linearization (cont.)

The linear system

$$\dot{x} = Ax \tag{2}$$

is called the Jacobian linearization of (1), or the linear approximation of (1) at the E.P. $x = 0$.

Since $f(x) = Ax +$ higher-order terms in x, (2) is obtained from (1) by ignoring all higher-order terms.

Example

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a_1 x_2 - a_2 \sin x_1 = -a_1 x_2 - a_2(x_1 - \frac{1}{3!}x_1^3 + \frac{1}{5!}x_1^5 - ...)
\end{aligned}$$

Its Jacobian linearization at $x = 0$ is

$$\dot{x} = Ax \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$$

3.2 Lyapunov linearization method

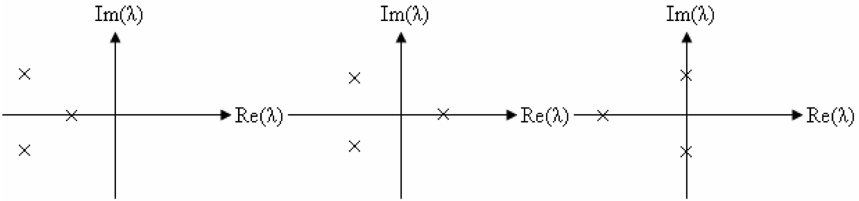
Theorem 3.1 Let $\lambda_1, ..., \lambda_n$ be eigenvalues of A.

◇ If $Re\{\lambda_i\} < 0$, $\forall\ i$, then E.P. of (1) is (locally) A.S.

◇ If there exists one i such that $Re\{\lambda_i\} > 0$, then the E.P. of (1) is unstable.

◇ If $Re\{\lambda_i\} \leq 0$, $\forall\ i$, and $Re\{\lambda_i\} = 0$ for at least one i, then the E.P. of (1) can be stable, unstable or A.S. Or nothing can be said about the stability property of (1).

Proof: using Lyapunov's direct method to be introduced later.



Examples (i)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a_1 x_2 - a_2 \sin x_1$$

where $a_1 = \frac{b}{MR^2} > 0, \quad a_2 = \frac{g}{R} > 0.$

$$x^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix} \right\}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -a_2 \cos x_1 & -a_1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} & \text{A.S.} \\ A_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ a_2 & -a_1 \end{bmatrix} & \text{unstable} \end{cases}$$

3.2 Lyapunov linearization method (cont.)

Examples (ii)

$$\dot{x} = ax + bx^5$$

Consider $\dot{x} = ax$.

$$a < 0 \quad \Rightarrow \quad x = 0 \quad \text{is A.S.}$$

$$a > 0 \quad \Rightarrow \quad x = 0 \quad \text{is U.S.}$$

$$a = 0 \quad \Rightarrow \quad \text{cannot tell by linearization method.}$$

4 Local Stabilization of Nonlinear Control Systems

4.1 Linearization of nonlinear control systems

Consider

$$\begin{array}{c} \dot{x} = f(x,u) \\ y = h(x) \end{array} \quad \text{where} \quad \begin{array}{c} x \in \Re^n \\ u \in \Re^m \\ y \in \Re^p \end{array} \tag{3}$$

$$f(0,0) = 0, \quad \text{and} \quad h(0) = 0$$

where

$$f(x,u) = \left[\begin{array}{c} f_1(x,u) \\ \ldots \\ f_n(x,u) \end{array} \right] \in \Re^n, \quad h(x) = \left[\begin{array}{c} h_1(x) \\ \ldots \\ h_p(x) \end{array} \right] \in \Re^p$$

Let $A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,\ u=0}$, $\quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,\ u=0}$, $\quad$ and $C = \left.\frac{\partial h(x)}{\partial x}\right|_{x=0}$

4.1 Linearization of nonlinear control systems (cont.)

where

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \cdots & \frac{\partial f_1(x,u)}{\partial x_n} \\ \cdots & & \cdots \\ \frac{\partial f_n(x,u)}{\partial x_1} & \cdots & \frac{\partial f_n(x,u)}{\partial x_n} \end{bmatrix},$$

$$\frac{\partial f(x,u)}{\partial u} = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial u_1} & \cdots & \frac{\partial f_1(x,u)}{\partial u_m} \\ \cdots & & \cdots \\ \frac{\partial f_n(x,u)}{\partial u_1} & \cdots & \frac{\partial f_n(x,u)}{\partial u_m} \end{bmatrix},$$

and

$$\frac{\partial h(x)}{\partial x} = \begin{bmatrix} \frac{\partial h_1(x)}{\partial x_1} & \cdots & \frac{\partial h_1(x)}{\partial x_n} \\ \cdots & & \cdots \\ \frac{\partial h_p(x)}{\partial x_1} & \cdots & \frac{\partial h_p(x)}{\partial x_n} \end{bmatrix}.$$

4.1 Linearization of nonlinear control systems (cont.)

Then $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$, and

$$\begin{aligned} \dot{x} &= f(x,u) = Ax + Bu + f_{h.o.t.}(x,u) \\ y &= Cx + h_{h.o.t.}(x) \end{aligned}$$

We call

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4}$$

the Jocabian linearization of (3) at $(x,u) = (0,0)$ or the linear perturbation model of (3).

4.1 Linearization of nonlinear control systems (cont.)

Example

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 + x_2^2 + \sin u \\ y &= x_1\end{aligned} \quad \Rightarrow \; A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},\; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\; C = [1, 0].$$

Therefore, $\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 + u \\ y &= x_1\end{aligned}$ is the Jacobian linearization at the origin.

4.2 Local stabilization of nonlinear systems

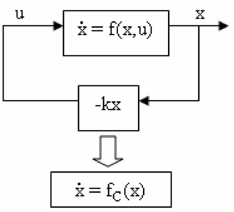
Consider $\dot{x} = f(x, u)$.
Under a linear state feedback control

$$u = -Kx \ \text{ where } K \in \Re^{m \times n},$$

the closed-loop system is

$$\dot{x} = f(x, -Kx) = f_c(x) \tag{5}$$

Clearly $f_c(0) = 0$.



4.2 Local stabilization of nonlinear systems (cont.)

Moreover,

$$\left.\frac{\partial f_c(x)}{\partial x}\right|_{x=0} = \left.\frac{\partial f(x,-Kx)}{\partial x}\right|_{x=0}$$
$$= \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} + \left.\frac{\partial f(x,u)}{\partial u}\frac{\partial u}{\partial x}\right|_{x=0,u=0}$$
$$= A - BK.$$

Thus the E.P. of $\dot{x} = f_c(x)$ at the origin is (locally) A.S. if all the eigenvalues of $A - BK$ have negative real parts.
Recall that if $\{A, B\}$ is controllable, i.e.

$$\text{rank } [B \;\; AB \;\; \cdots \;\; A^{n-1}B] = n,$$

then there exist matrices $K \in \Re^{m\times n}$ such that $A - BK$ is stable.
In particular, when $m = 1$, K can be calculated using Arkerman's formula so that $A - BK$ has desirable eigenvalues.

4.2 Local stabilization of nonlinear systems (cont.)

Remark

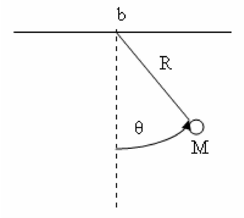
- (i) $\dot{x} = Ax + Bu$ is small perturbation model, valid only when x, u are small.
- (ii) The linear state feedback only guarantees local asymptotic stability, not global A.S.

5.1

Observation: If the total energy of a physical system is continuously dissipated, then the system must eventually settle down to an E.P.

Finite energy $\searrow$ no energy $\leftrightarrow$ no motion

Example (Pendulum system)



$$MR^2\ddot{\theta} + b\dot{\theta} + MgR\sin\theta = 0$$

5.1 (cont.)

or

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

where $x_1 = \theta$ and $x_2 = \dot{\theta}$, b is the friction coefficient at the hinge.

Total mechanical energy = kinetic energy + potential energy

$$\begin{aligned} V(x) &= \frac{1}{2}MR^2x_2^2 + \int_0^{x_1} MRg\sin x_1 dx_1 \\ &= \frac{1}{2}MR^2x_2^2 + MRg(1-\cos x_1). \end{aligned}$$

Due to energy consumption ($b > 0$), $V(x(t, x_0)) \searrow$ w.r.t. t, and

$$\lim_{t\to\infty} V(x(t, x_0)) = 0.$$

5.1 (cont.)

But since

$$V(x) = \quad 0 \quad \Rightarrow \; x = 0,$$

we have

$$\lim_{t\to\infty} V(x(t,x_0)) \quad = 0 \quad \Rightarrow \quad \lim_{t\to\infty} x(t,x_0) = 0.$$

Thus, we conclude that $x = 0$ is A.S.

Conclusion: The stability of a system may be related to the property of a scalar energy function:

- Zero energy corresponds to the equilibrium $(x = 0)$.
- The convergence of the total energy to zero $\Leftrightarrow$ A.S.
- Instability is related to the growth of the mechanical energy.

5.2 Positive definite functions and Lyapunov functions

(i) Definition 3.7

A scalar continuous function $V(x)$ is locally positive definite if $V(0) = 0$ and there exists a ball B_{R_0} such that $x \in B_{R_0}$ & $x \neq 0 \;\Rightarrow\; V(x) > 0$. If $B_{R_0} = \Re^n$, then $V(x)$ is globally positive definite.

(a) **Examples**

$$V(x) = x^T x$$

is globally positive definite.

$$V(x) = \frac{1}{2} M R^2 x_2^2 + M R g (1 - \cos x_1)$$

is locally positive definite since whenever $x_1^2 + x_2^2 < 1$, $x \neq 0$, $V(x) > 0$. But it is not globally positive definite since

$$V(2\pi, 0) = 0.$$

5.2 Positive definite functions and Lyapunov functions (cont.)

(b) **Geometric interpretation**

For $n = 2$ (Figs. 3.7 & 3.8 of the textbook).

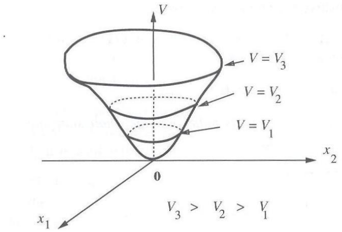


Figure 3.7 : Typical shape of a positive definite function $V(x_1, x_2)$

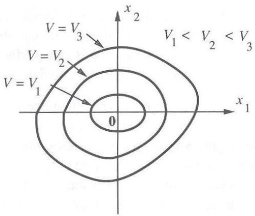


Figure 3.8 : Interpreting positive definite functions using contour curves

$V(x_1, x_2)$ typically corresponds to a surface looking like an upward cup. The contour curves $V(x_1, x_2) =$ constant represent a set of ovals surrounding the origin.

(ii) Related concepts

- $V(x)$ is negative definite if $-V(x)$ is positive definite.
- $V(x)$ is positive semi-definite if $V(0) = 0$ & $V(x) \geq 0$ for $x \neq 0$.
- $V(x)$ is negative semi-definite if $-V(x)$ is positive semi-definite.

Thus it can be seen that:

$V(x) = \frac{1}{2}MR^2x_2^2 + MRg(1 - \cos x_1)$ is positive semi-definite.

$V(x_1, x_2) = -x_1^2$ is negative semi-definite.

5.2 Positive definite functions and Lyapunov functions (cont.)

(iii) Lyapunov function

(a) **Definition 3.8**

If, in a ball B_R, the function $V(x)$ is P.D. and $\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \dots \frac{\partial V}{\partial x_n}\right]$ exists and is continuous, and if its time derivative along any state trajectory of system (1) is negative semi-definite, i.e.,

$$\dot{V}(x) \left(= \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f(x) \right) \leq 0,$$

then $V(x)$ is said to be a Lyapunov function for the system (1).

(b) $V(x) = \frac{1}{2}MR^2x_2^2 + MRg(1 - \cos x_1)$ is a Lyapunov function for the pendulum system since $V(x)$ is (locally) P.D. and

$$\dot{V}(x) = MRg \sin x_1 \dot{x}_1 + MR^2 x_2 \dot{x}_2$$

5.2 Positive definite functions and Lyapunov functions (cont.)

$$=(MRg\sin x_1)x_2 + MR^2x_2\left[-\frac{b}{MR^2}x_2 - \frac{g}{R}\sin x_1\right]$$
$$=-bx_2^2 \le 0.$$

(c) Geometric interpretation (Fig. 3.9 of the textbook)

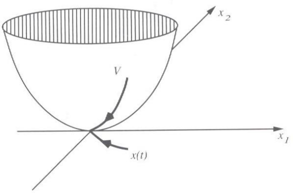


Figure 3.9 : Illustrating Definition 3.8 for $n = 2$

The point denoting the value of $V(x_1, x_2)$ always points down an inverted cup.

5.2 Positive definite functions and Lyapunov functions (cont.)

(d) Global: If ball $B_R = \Re^n$, Lyapunov function $V(x)$ becomes globally P.D. and $\dot{V}(x)$ becomes globally negative semi-definite.

(e) $V(x)$ is a generalization of a physical system's total energy function.

$$V(x) > 0, \quad x \neq 0, \quad V(x) = 0 \quad \Rightarrow \quad x = 0$$
$$\dot{V} \leq 0$$

5.3 Lyapunov Theorems

(i) **Theorem 3.2** (Local stability)

If, in a ball B_{R_0}, there exists a Lyapunov function $V(x)$ for (1), then the equilibrium point $x = 0$ is (locally) stable. If, in addition, $\dot{V}(x)$ is locally negative definite, then the stability is asymptotic.

Proof: Part I

Want to show: Given $R > 0$, $\exists r > 0$, such that
$x_0 \in B_r \quad \Rightarrow \quad x(t, x_0) \in B_R$, $\forall\ t > 0$. To this end, let

$$m = \min_{x \in S_R} V(x) \ \text{ where } \ S_R = \left\{ x | \ \|x\|^2 = \Re^2 \right\}$$

V is continuous & S_R compact $\quad \Rightarrow \quad m$ exists
V is p.d. $\quad \Rightarrow \quad m > 0.$
Furthermore, since $V(0) = 0$ & V is continuous, $\exists R > r > 0$ such that
$V(x) < m, \quad \forall\ x \in B_r.$

Consider $x(t, x_0)$ with $x_0 \in B_r$. Since $\dot{V}(x) \leq 0$, $V(x(t))$ is monotonously decreasing with respect to t, that is. $V(x(t, x_0)) \leq V(x(0, x_0)) = V(x_0) < m$, $\forall\ t > 0$. Thus, there exists no $T > 0$ such that $x(T, x_0) \in S_R$. Otherwise, $V(x(T, x_0)) \geq m$.

Part II

We have just shown that $x(t, x_0) \in B_R$, $\forall\ t > 0$, $\forall\ x_0 \in B_r$.
Since $V(x) \geq 0$ (lower bounded) and $\dot{V}(x(t)) \leq 0$ (monotonously decreasing w.r.t. t), there exist $L \geq 0$ such that
$V(x(t, x_0)) \geq L$ & $\lim_{t\to\infty} V(x(t, x_0)) = L$
We claim $L = 0$. Otherwise, if $L > 0$, then $\exists\ r_0 < R$ such that $V(x) < L$, $\forall\ x \in B_{r_0}$ since $V(0) = 0$, and $V(x)$ is continuous,
i.e. $x(t, x_0)$ never enters B_{r_0}.

Let

$$L_1 = \min_{r_0 \le \|x\| \le R} \left(-\dot{V}(x) \right). \text{ Then } \; L_1 > 0.$$

Therefore

$$-V\left(x(t, x_0)\right) + V_0(x_0) = \int_0^t -\dot{V}\left(x(t, x_0)\right) dt \ge \int_0^t L_1 dt = L_1 t$$

that is

$$V\left(x(t, x_0)\right) \le V(x_0) - L_1 t \tag{6}$$

But (6) implies $V\left(x(t, x_0)\right) < 0$ when $t > \frac{V(x_0)}{L_1}$, which is a contradiction to the fact that $V(x)$ is P.D.

5.2 Positive definite functions and Lyapunov functions (cont.)

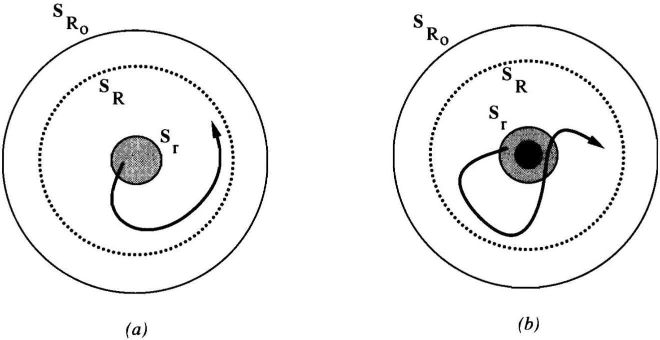


Figure 3.11 : Illustrating the proof of Theorem 3.2 for $n = 2$

Example

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - x_2$$
$$\dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1)$$

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ which is P.D.

$$\begin{aligned}\dot{V}(x_1, x_2) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1^2(x_1^2 + x_2^2 - 1) - x_2x_1 + x_2x_1 + x_2^2(x_1^2 + x_2^2 - 1) \\ &= (x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2) < 0 \qquad \forall\, x\end{aligned}$$

such that $0 < x_1^2 + x_2^2 < 1$. Therefore $\dot{V}(x)$ is N.D. and $x = 0$ is A.S.

(ii) **Theorem 3.3** (Global A.S.)
Assume that there exists a scalar function $V(x)$ such that

- $V(x)$ is globally P.D.
- $\dot{V}(x)$ is globally N.D.
- $V(x) \to \infty$ as $\|x\| \to \infty$ (radially unbounded)

Then the E.P. of (1) at $x = 0$ is globally A.S.

Remark: Why radially unbounded? To ensure the region defined $\{x \mid V(x) \leq l\}$ for any $l > 0$ is bounded so that $x(t, x_0)$ will not drift away from E.P. (Fig. 3.12 of the textbook). Let $l = \lim_{r\to\infty} \inf_{\|x\|\geq r} V(x)$. Then $V(x)$ is radially unbounded $\Leftrightarrow l = \infty$.

5.3 Lyapunov Theorems (cont.)

Outline of the Proof: The first two conditions guarantee stability, and also guarantee that, for any x_0, $V(x(t, x_0)) \leq V(x(0, x_0)) = V(x_0)$. This fact together with condition 3 further implies $x(t, x_0) \in B_R$ for some R for all $t \geq 0$ since $\{x \mid V(x) \leq V(x_0)\}$ for any x_0 is bounded. The rest of the proof is the same as the proof of part II of Theorem 3.2.

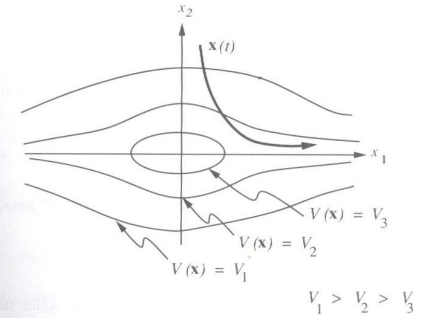


Figure 3.12 : Motivation of the radial unboundedness condition

It can be shown that

$$V(x) = \left[\frac{x_1^2}{1 + x_1^2}\right] + x_2^2 \qquad \text{is P.D.}$$

But it is not radially unbounded since $V(\infty, 0) = 1$. $V(x) = V_\alpha$, $V_\alpha > 1$ are open curves.

5.3 Lyapunov Theorems (cont.)

Lemma 1

Consider

$$\dot{x} + c(x) = 0, \ x \in \Re^1$$

where $c(x)$ is continuous, and satisfies the sign condition
$xc(x) > 0 \ \forall \ x \neq 0$.
Then $x = 0$ is G.A.S. E.P.

Proof

Let

$$V(x) = x^2, \quad \text{G.P.D.}$$

and radially unbounded,

$$\dot{V} = -2xc(x) < 0 \ \ \text{G.N.D.}$$

Example 1

$$\dot{x} = \sin^2 x - x \ i.e. \ c(x) = x - \sin^2 x \text{ which is continuous. Also}$$

$$xc(x) = x^2 - x\sin^2 x > 0 \ x \neq 0 \quad \text{since } \sin^2 x \leq |\sin x| < |x|.$$

Therefore the E.P. $x = 0$ is G.A.S.

Example 2

$$\dot{x} = -x^3$$

(cannot tell the stability property from linearization method)
i.e. $c(x) = x^3$.
Clearly $xc(x) = x^4 > 0 \ x \neq 0$. The E.P. $x = 0$ is G.A.S.

6 Invariant Set Theorem

6.1 Introduction:

Theorems 3.2 and 3.3 require the negative definiteness of $\dot{V}(x)$ to guarantee A.S.

In practice, it is relatively easy to find a Lyapunov function with $\dot{V}(x)$ negative semi-definite.

It is desirable to replace the negative definiteness of $\dot{V}(x)$ by some other condition.

6.2 Invariant Set

Definition 3.9

A set $G \subset \Re^n$ is an invariant set for (1) if every trajectory starting from a point in G remains in G for all future time. In other words, let $x(t, x_0)$ be the trajectory of (1) starting at x_0. Then G is an invariant set for (1) if and only if $\forall t > 0, x(t, x_0) \in G, \forall x_0 \in G$.

Examples

- (i) $\Re^n$ is an invariant set of (1).
- (ii) If x^* is an E.P. of (1), i.e., $f(x^*) = 0$, then, $G = \{x^*\}$ is an invariant set of (1).
- (iii) Limit cycle.

Exercise: Suppose $\dot{V}(x) \leq 0$ along the trajectory of (1). Let $\Omega_l = \{x \mid V(x) < l,\ l > 0\}$. Then Ω_l is an invariant set for (1).

Remark: Let R be a subset of $\Re^n$. A set G is said to be an invariant set of (1) in R if G is an invariant set for (1) and $G \subset R$. G is said to be the largest invariant set of (1) in R if G is the union of all invariant sets of (1) within R.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin \frac{\pi x_1}{2} - x_1 - |x_1^2 - 1| x_2^3. \end{aligned}$$

It can be seen that $G_0 = \{(0,0)\}$, $G_1 = \{(1,0)\}$, $G_2 = \{(-1,0)\}$, and the whole space $\Re^2$ are invariant sets of (1). Let $R = \{x \mid x_2 = 0\}$. Then the largest invariant set of (1) in R is $G_0 \cup G_1 \cup G_2$. Let $R = \{x \mid |x_1| < 1 \text{ and } x_2 = 0\}$. Then the largest invariant set of (1) in R is G_0.

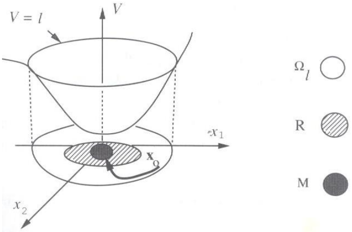
(i) **Theorem 3.4** (Local Invariant set theorem)
Consider $\dot{x} = f(x)$ where f is continuous and $V(x)$ is a scalar function with continuous first partial derivative s.t.

◇ for some $l > 0$, the region $\Omega_l \triangleq \{x \mid V(x) < l\}$ is bounded.

◇ $\dot{V}(x) \leq 0, \forall x \in \Omega_l$.

Let R be the set of all points within Ω_l where $\dot{V}(x) = 0$, and M be the largest invariant set in R. Then every solution $x(t)$ originating in Ω_l tends to M as $t \to \infty$.

(ii) **Geometric interpretation** (Figure 3.14 of the textbook)



(iii) **Remarks**

(a) Outline of the proof

Step 1. Using Barbalat's lemma in Section 4.3 to show $\lim_{t\to\infty} \dot{V}(x(t, x_0)) = 0, \forall x_0 \in \Omega_l$, i.e., all trajectories originating in Ω_l converge to R.

Step 2. Show that the trajectories cannot converge to just anywhere in the set R. They must converge to the largest invariant set M in R.

(b) When $\dot{V}$ is negative definite, $\dot{V} = 0 \Leftrightarrow x = 0$, i.e., $R = M = \{0\}$. Thus, Theorem 3.2 is a special case of Theorem 3.4.

(c) Let $l = \lim_{r\to\infty} \inf_{\|x\|\geq r} V(x)$. Since $V(x)$ is radially unbounded $\Leftrightarrow l = \infty$, Ω_l is bounded for all $l > 0$ if $V(x)$ is radially unbounded.

(iv) **Corollary**

Consider $\dot{x} = f(x)$ with f continuous, and let $V(x)$ be a scalar function with continuous partial derivative. Assume in some neighborhood Ω of the origin

- $V(x)$ is locally positive definite.
- $\dot{V}$ is negative semidefinite.
- The set $R = \{x \mid \dot{V}(x) = 0\} \bigcap \Omega$ contains no trajectories of $\dot{x} = f(x)$ other than the trivial trajectory $x = 0$.

Then, the E.P. $x = 0$ is A.S.. Further, the largest connected region of the form $\Omega_l = \{x \mid V(x) < l\}$ within Ω is a domain of attraction of the E.P.

Example(Pendulum)

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0 \tag{7}$$

Letting $x_1 = \theta, x_2 = \dot{\theta}$ gives

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - \sin x_1$$

Let $V(x) = (1 - \cos(x_1)) + \frac{x_2^2}{2}$ and $\Omega = \{-\pi < x_1 < \pi, -\infty < x_2 < \infty\}$. Then $V(x)$ is locally P.D. and

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= (\sin x_1)x_2 + x_2(-x_2 - \sin x_1) = -x_2^2 \leq 0.\end{aligned}$$

Thus, $\dot{V}(x)$ is negative semi-definite.

Now we have $R = \{x \mid \dot{V}(x) = 0\} \bigcap \Omega = \{x \mid x_2^2 = 0\} \bigcap \{-\pi < x_1 < \pi, -\infty < x_2 < \infty\} = \{(x_1, 0) | -\pi < x_1 < \pi\}$.

We claim $M = \{0\}$. Otherwise, $\exists x_1 \neq 0$ and $-\pi < x_1 < \pi$ s.t. $x_0 = [x_1 \;\; 0]^T \in M \subset R$.

Thus, $x(t, x_0) = [x_1(t, x_0) \;\; x_2(t, x_0)]^T \in M, \forall t > 0$ since M is invariant.

But $x_2(t, x_0) \equiv 0, \forall t > 0$ since $M \subset R$. Thus $0 = 0 - \sin(x_1(t, x_0)), \forall t > 0$ which is a contradiction.

Therefore $x_0 = 0$. Thus the origin is the only E.P. in R, and is asymptotically stable.

Remark: Theorem 3.4 can also estimate the domain of attraction. For the above example, the whole Ω is a domain of attraction.

6.4 Theorem 3.5 (Global Invariant Set Theorem)

Consider (1) with $f \in C$. Assume there exists $V \in C^{1}$[1] such that

(1) $\dot{V}(x) \leq 0$ over the whole space.

(2) $V(x) \to \infty$ as $\|x\| \to \infty$.

Let $R = \{x \mid \dot{V}(x) = 0\}$ and M be the largest invariant set in R. Then all solutions globally asymptotically converge to M as $t \to \infty$. Moreover, if $M = \{0\}$, then the E.P. is globally A.S.

[1] $f \in C^k$, $k \geq 0$, means the k^{th} derivatives of V exist and are continuous.

6.5 When does $M = \{0\}$?

If (1) satisfies the conditions of Theorems 3.4 and 3.5, then the trajectories of (1) are constrained by $(n+1)$ equations

$$\begin{aligned} \frac{dx(t,x_0)}{dt} &= f(x(t,x_0)),\ t \geq 0 \\ \lim_{t\to\infty} \dot{V}(x(t,x_0)) &= 0. \end{aligned} \tag{8}$$

Thus, if the only time function $x(t)$ that satisfies the following $(n+1)$ equations

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t)) \\ \dot{V}(x(t)) &= 0 \end{aligned}$$

is $x(t) = 0$, then necessarily $M = \{0\}$.

Pendulum Example

$$\begin{aligned} \dot{x}_1 &= x_2 && (9) \\ \dot{x}_2 &= -x_2(t) - \sin x_1(t) && (10) \\ x_2^2 &= 0 && (11) \end{aligned}$$

(11) $\Leftrightarrow x_2 = 0, \ \stackrel{(10)}{\Rightarrow} \sin x_1(t) = 0, \ \Rightarrow x_1(t) = 0.$

6.6 Lemma 2 (A.S. of the E.P. for a class of 2nd order systems)

Consider

$$\ddot{y} + b(\dot{y}) + c(y) = 0 \qquad y \in \Re^1 \tag{12}$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c(x_1) - b(x_2) \end{aligned}$$

where $c(y),\ b(y)$ are continuous satisfying the sign condition
$yc(y) > 0, \quad y \neq 0, \quad yb(y) > 0.$
Then, the E.P. of (12) is A.S. Moreover, if

$$\lim_{|y|\to\infty} \int_0^y c(r)dr = \infty$$

then the E.P. of (12) is G.A.S.

6.6 Lemma 2 (cont.) (A.S. of the E.P. for a class of 2nd order systems)

Proof:

Define

$$V(x_1, x_2) = \int_0^{x_1} c(y)dy + \frac{1}{2}x_2^2, \quad \text{G.P.D.}$$

Then

$$\begin{aligned}\dot{V}(x_1, x_2) &= \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 \\ &= c(x_1)x_2 + x_2(-c(x_1) - b(x_2)) = -x_2 b(x_2) \leq 0\end{aligned}$$

Thus, the E.P. is stable by Theorem 3.2. But we cannot conclude the A.S. by Theorem 3.2.

6.6 Lemma 2 (cont.) (A.S. of the E.P. for a class of 2nd order systems)

Next consider the invariant set theorem, we will show that the only trajectory that satisfies

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) && (13) \\ \dot{x}_2(t) &= -c(x_1(t)) - b(x_2(t)) && (14) \\ \dot{V}(x(t)) &= -x_2(t)b(x_2(t)) = 0 && (15) \end{aligned}$$

is $x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\forall t > 0$.

In fact,

$$\begin{aligned} (15) &\Rightarrow x_2(t) \equiv 0 && (16) \\ (16) &\Rightarrow \dot{x}_2(t) \equiv 0 \text{ and } b(x_2(t)) \equiv 0 \\ (14) &\Rightarrow c(x_1(t)) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0 \end{aligned}$$

Therefore, the E.P. at the origin is A.S.

6.6 Lemma 2 (cont.) (A.S. of the E.P. for a class of 2nd order systems)

Moreover, if $\lim_{|y|\to\infty} \int_0^y c(r)dr = \infty$
then $V(x)$ is radially unbounded. Then the E.P. is G.A.

Example

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^5 - x_2^3 + x_1^4 \sin^2 x_1 = -c(x_1) - b(x_2)\end{aligned}$$

where

$$c(x_1) = x_1^5 - x_1^4 \sin^2 x_1 \quad b(x_2) = x_2^3$$

Clearly

$$\begin{aligned}&yb(y) = y^4 > 0, \quad y \neq 0, \quad \text{and} \\ &yc(y) = y^6 - y^5 \sin^2 y > 0, \quad y \neq 0 \quad \text{since} \\ &\sin^2 y \leq |\sin y| < |y|, \quad y \neq 0\end{aligned}$$

6.6 Lemma 2 (cont.) (A.S. of the E.P. for a class of 2nd order systems)

Therefore, the E.P. is A.S.
Moreover

$$\lim_{|y|\to\infty}\int_0^y (r^5 - r^4\sin^2 r)dr = \infty$$

Thus, the E.P. is G.A.S.

7 Finding Lyapunov Function

7.1 Introduction

- Quadratic function
- Krasovskii's method
- The variable gradient method
- Physically motivated Lyapunov Function

7.2 Review of P.D. matrix

(a) $M \in \Re^{n \times n}$ is symmetric if $M^T = M$.

(b) M is P.D. if $x^T M x$ is P.D. The notation $M > 0$ denotes M is P.D.

M is positive semi-definite if $x^T M x$ is positive semi-definite. The notation $M \geq 0$ denotes M is positive semi-definite.

M is negative definite if $-M$ is P.D.

M is negative semi-definite if $-M$ is positive semi-definite.

(c) M is positive definite iff

$$\lambda_i(M) > 0, \qquad i = 1, ..., n$$

M is positive semi-definite if $\lambda_i(M) \geq 0 \qquad \forall\ i$.

If M is positive definite, then

$$\lambda_{\min}(M)\|x\|^2 \leq x^T M x \leq \lambda_{\max}(M)\|x\|^2$$

(d) Sylvester's theorem

Let

$$M = \begin{bmatrix} m_{11} & \dots & m_{1n} \\ \dots & & \dots \\ m_{n1} & \dots & m_{nn} \end{bmatrix}$$

and

$$M_i = \begin{bmatrix} m_{11} & \dots & m_{1i} \\ \dots & & \dots \\ m_{i1} & \dots & m_{ii} \end{bmatrix}. \tag{17}$$

Then $M > 0$ iff

$$\det(M_i) > 0, \quad i = 1, 2, \cdots, n$$

7.3 Lyapunov function for LTI systems

Consider

$$\dot{x} = Ax, \quad x \in \Re^n.$$

Let $V(x) = x^T P x$ where $P \in \Re^{n \times n}$ and $P = P^T$. Then

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P A x = x^T (A^T P + PA) x.$$

Clearly, if $P > 0$ and $A^T P + PA < 0$, then $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$ and $x = 0$ is asymptotically stable.

Question?

Given $Q = Q^T$, $Q > 0$, whether or not there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q? \quad \text{Lyapunov equation}$$

If yes, then $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$.

Theorem 3.6

The following are equivalent:

- (i) $x = 0$ is the asymptotically stable E.P. of $\dot{x} = Ax$.
- (ii) $Re\,(\lambda_i(A)) < 0$, $i = 1, 2, ..., n$.
- (iii) For any $Q \in \Re^{n\times n}$, $Q^T = Q$, $Q > 0$, there exists a unique $P \in \Re^{n\times n}$, $P = P^T$, $P > 0$ such that

$$A^T P + PA = -Q \tag{18}$$

Sketch of the Proof:

Let

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) dt. \quad (19)$$

Since A is strictly stable, there exist some $\alpha > 0$ and $\lambda > 0$ such that $|| \exp(At)|| \leq \alpha e^{-\lambda t}$, for all $t \geq 0$. Thus,

$$||P|| \leq \alpha^2 ||Q|| \int_0^\infty e^{-2\lambda t} dt = \frac{\alpha^2 ||Q||}{2\lambda}.$$

That is, P exists and is finite. Also P is P.D. since Q is. To show P satisfies (18), note that

$$\begin{aligned} -Q &= \int_0^\infty d[\exp(A^T t) Q \exp(At)] \\ &= \int_0^\infty [A^T \exp(A^T t) Q \exp(At) + \exp(A^T t) Q \exp(At) A] dt \\ &= A^T P + PA \end{aligned}$$

To show the uniqueness, let $A^T P_1 + P_1 A = -Q$. Then

$$
\begin{aligned}
P_1 &= -\int_0^\infty d[\exp(A^T t) P_1 \exp(At)] \\
&= -\int_0^\infty \exp(A^T t)(A^T P_1 + P_1 A)\exp(At) dt \\
&= \int_0^\infty \exp(A^T t) Q \exp(At) dt = P
\end{aligned}
$$

7.4 Solution of Lyapunov equation

Given $Q \in \Re^{n \times n}$, $Q = Q^T$, assume

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{12} & P_{22} & \dots & P_{2n} \\ \dots & & & \\ P_{1n} & \dots & \dots & P_{nn} \end{bmatrix}.$$

Since

$$P^T = P$$

equation

$$A^T P + PA = -Q$$

consists of $n + (n-1) + \dots + 1 = \dfrac{n(n+1)}{2}$ linearly independent equations in

$$\begin{array}{cccc} P_{11} & P_{12} & \dots & P_{1n} \\ & P_{22} & \dots & P_{2n} \\ & \dots & \dots & \\ & & P_{n-1\ n-1} & P_{n-1\ n} \\ & & & P_{nn} \end{array}$$

7.4 Solution of Lyapunov equation (cont.)

For instance, $A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0$.

Assume $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$.

Then $A^T P + PA = -Q$ gives

$$\begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

or

$$\begin{bmatrix} -16P_{12} & 4P_{11} - 12P_{12} - 8P_{22} \\ 4P_{11} - 12P_{12} - 8P_{22} & 8P_{12} - 24P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

or

$$\begin{array}{c} -16P_{12} = -1 \\ 8P_{12} - 24P_{22} = -1 \\ 4P_{11} - 12P_{12} - 8P_{22} = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} P_{12} = \frac{1}{16} \\ P_{22} = \frac{1}{16} \\ P_{11} = \frac{5}{16} \end{array} \quad \Rightarrow \quad P = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}.$$

7.5 Proof of Theorem 3.1 (Lyapunov's linearization theorem)

Let $A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$. Then

$$Re\{\lambda_i(A)\} < 0, \forall i \;\Rightarrow\; \text{A.S. of E.P. for } \dot{x} = f(x).$$

Proof: Since all eigenvalues of A have negative real part, by Theorem 3.6, for any $Q \in \Re^{n\times n}$, $Q = Q^T$, $Q > 0$, there exists a $P \in \Re^{n\times n}$, $P > 0$, $P = P^T$ such that

$$PA + A^T P = -Q.$$

Let $V(x) = x^T P x$. Then $V(x)$ is P.D. and

$$\dot{V}(x) = x^T P f(x) + f^T(x) P x.$$

By definition of A, there exists $g(x)$ such that

$$f(x) = Ax + g(x)$$

with $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$, since $f(x)$ is differentiable at $x = 0$.

7.5 Proof of Theorem 3.1 (cont.)

Thus,

$$\begin{aligned}\dot{V}(x) &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T Pg(x) \\ &= -x^T Qx + 2x^T Pg(x) \\ &\le -\lambda_{\min}(Q)\|x\|^2 + 2x^T Pg(x)\end{aligned}$$

since $x^T Qx \ge \lambda_{\min}(Q)\|x\|^2$. Moreover, since $g(x)$ is higher order in x, given any $\lambda > 0$, there exists $r_\lambda > 0$ such that $||g(x)|| < \frac{\lambda||x||}{2||P||}$, $\forall x \in B_{r_\lambda}$. Thus,

$$\left|x^T Pg(x)\right| \le ||x||||P||||g(x)|| < \frac{\lambda}{2}\|x\|^2 \quad \forall x \in B_{r_\lambda}$$

Therefore,

$$\dot{V}(x) \le -[\lambda_{\min}(Q) - \lambda]\|x\|_2^2 \quad \forall x \in B_{r_\lambda}$$

Letting $\lambda < \lambda_{\min}(Q)$ shows $\dot{V}(x)$ is N.D. for $x \in B_{r_\lambda}$. Thus the E.P. is A.S. by Theorem 3.2.

7.7 Theorem 3.7 (Krasovskii)

Consider (1), i.e., the following system

$$\dot{x} = f(x)$$

where $f \in C^1$. If the matrix $F(x) = \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x}$ is N.D. for $x \in B_r$ for some $r > 0$, then the E.P. of (1) at the origin is A.S. with the Lyapunov function $V(x) = f^T(x)f(x)$.

If $F(x)$ is globally N.D. and $V(x)$ is radially unbounded, then the E.P. is G.A.S.

Proof: Let us first note that, for any $y \in \Re^n$,

$$y^T F y = 2y^T \frac{\partial f}{\partial x} y.$$

Therefore, the assumption that $F(x)$ is N.D. $\Rightarrow \frac{\partial f}{\partial x}$ is invertible. Since $\frac{\partial f}{\partial x}$ is invertible and continuous, by inversion function theorem, the function f can be uniquely inverted in B_{r_0} for some $r_0 > 0$. This implies $f(x) \neq 0$ for $x \neq 0$ and $x \in B_{r_0}$. As a result, $V(x) = f^T(x)f(x)$ is P.D.

Also,

$$\dot{V} = f^T \dot{f} + \dot{f}^T f = f^T \frac{\partial f}{\partial x} f + f^T (\frac{\partial f}{\partial x})^T f = f^T F f$$

which is N.D.

An Example

Consider

$$\dot{x}_1 = -6x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

We have

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -6 & 2 \\ 2 & -6 - 6x_2^2 \end{bmatrix}$$
$$F(x) = \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} = \begin{bmatrix} -12 & 4 \\ 4 & -12 - 12x_2^2 \end{bmatrix} < 0, \quad \forall x$$

Thus the E.P. is A.S.
Moreover

$$V(x) = f^T(x)f(x) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

is radially unbounded. Thus, the E.P. is G.A.S.

7.8 The Variable Gradient Method

The variable gradient method is a formal approach to constructing Lyapunov functions. It involves assuming a certain form for the gradient of an unknown Lyapunov function, and then finding the Lyapunov function itself by integrating the assumed gradient.

Given a C^1 function $V : \Re^n \to \Re$, the gradient of $V(x)$ is defined as

$$\nabla V = \left[\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}\right].$$

If V is C^2, then ∇V satisfies the following **curl condition**:

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \quad (i, j = 1, 2, ..., n), \text{ where } \nabla V_i = \frac{\partial V}{\partial x_i}, \nabla V_j = \frac{\partial V}{\partial x_j}.$$

On the other hand, given an n dimensional vector valued C^1 function denoted by ∇V, if the component of ∇V satisfies the curl condition, then there exists a unique C^2 function V such that

$$\frac{\partial V}{\partial x_i} = \nabla V_i, \quad i = 1, 2, \cdots, n.$$

This function can be represented by the following integration:

$$V(x) = \int_0^x \nabla V d\mathbf{x}.$$

Since satisfaction of curl conditions implies that the above integration result is independent of the integration path, it is convenient to obtain V by integrating along a path which is parallel to each axis in turn:

$$\begin{aligned} V(x) = &\int_0^{x_1} \nabla V_1(x_1, 0, ..., 0) dx_1 + \int_0^{x_2} \nabla V_2(x_1, x_2, 0, ..., 0) dx_2 + \\ &\cdots + \int_0^{x_n} \nabla V_n(x_1, x_2, ..., x_n) dx_n \end{aligned} \tag{20}$$

7.8 The Variable Gradient Method (cont.)

The principle of the variable gradient method is to assume a specific form for the gradient ∇V, instead of assuming a specific form for the Lyapunov function itself.

Procedure for seeking a Lyapunov function:

- assume that the gradient function ∇V is of the form

$$\nabla V_i = \Sigma_{j=1}^{n} a_{ij} x_j \tag{21}$$

  where a_{ij}'s are coefficients to be determined.
- solve for coefficient a_{ij} so as to satisfy the curl equations
- restrict the coefficient in (21) so that $\dot{V}$ is negative semi-definite (at least locally)
- compute V from ∇V by integration (20)
- check whether V is positive definite

Example:

Use variable gradient method to find a Lyapunov function for the nonlinear system

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = -2x_2 + 2x_1x_2^2 \end{cases}$$

Solution:

Assume ∇V has the following form:

$$\nabla V_1 = a_{11}x_1 + a_{12}x_2$$
$$\nabla V_2 = a_{21}x_1 + a_{22}x_2$$

7.8 The Variable Gradient Method (cont.)

Solution (cont.):

By curl equation,

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1} \Rightarrow x_1 \frac{\partial a_{11}}{\partial x_2} + a_{12} + x_2 \frac{\partial a_{12}}{\partial x_2} = a_{21} + x_1 \frac{\partial a_{21}}{\partial x_1} + x_2 \frac{\partial a_{22}}{\partial x_1}$$

For simplicity, choosing $a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$ leads to

$$\nabla V_1 = x_1 \quad \nabla V_2 = x_2$$

Then $\dot{V}$ can be computed as

$$\dot{V} = \nabla V \dot{x} = -2x_1^2 - 2x_2^2(1 - x_1 x_2)$$

Thus $\dot{V}$ is locally N.D. in the region $(1 - x_1 x_2 > 0)$.

So $V(x) = \int_0^{x_1} x_1 dx_1 + \int_0^{x_2} x_2 dx_2 = \frac{x_1^2 + x_2^2}{2}$, which is P.D.

Therefore asymptotic stability is guaranteed.

7.8 The Variable Gradient Method (cont.)

Remark:

Coefficients a_{ij} can be chosen in other form. But a_{ij} are restricted by two conditions:

- the curl equations;
- $\dot{V}$ is negative semi-definite (at least locally).

Exercise: Consider the system in Lemma 2:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c(x_1) - b(x_2) \end{aligned} \tag{22}$$

Using the variable gradient method to derive a Lyapunov function for (22) as follows:

$$V(x_1, x_2) = \int_0^{x_1} c(y)dy + \frac{1}{2}x_2^2.$$

7.9 Physically Motivated Lyapunov Function

- Finding a Lyapunov function of a system often involves trial-and-error. However, since Lyapunov function is a generalization of energy function, engineering insight and physical knowledge may lead to an elegant Lyapunov analysis.
- **A robot arm manipulator example:** Shown in Figure 3.19 is a robot arm manipulator. It consists of a number of links connected by rotational or translational joints, with the last link equipped with some end-effector
- A fundamental task in robotic applications is for robot manipulators to transfer objects from one point to another, the so-called robot position control problem.

7.9 Physically Motivated Lyapunov Function

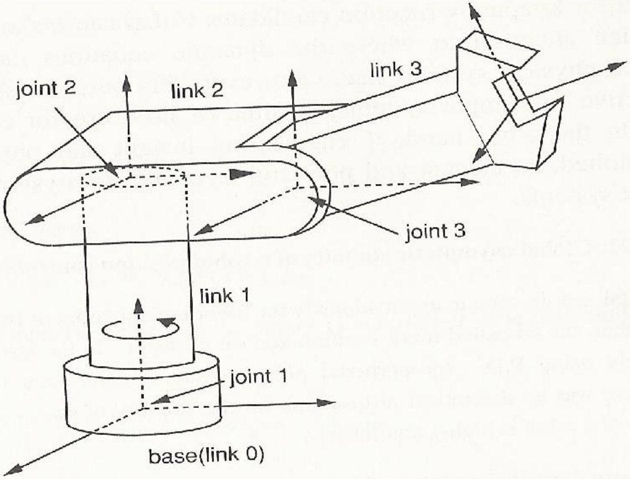


Figure 3.19 : A robot manipulator

7.9 Physically Motivated Lyapunov Function

- **The dynamics of an $n-$link robot arm**

$$H(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau \tag{23}$$

where q is an $n-$dimensional vector describing the joint positions of the robot, τ is the vector of input torques, g is the vector of gravitational torques, b represents the Coriolis and centripetal forces caused by the motion of the links, H the $n \times n$ inertia matrix of the robot arm and is positive definite for any q, and $b(q.\dot{q}) \in \Re^n$ satisfies $b(q,0) = 0$.

- **P.D. controller with a gravity compensation:**

$$\tau = -K_D\dot{q} - K_p q + g(q) \tag{24}$$

where K_D and K_p are constant positive definite $n \times n$ matrices.

- (23) contains hundreds of terms for the 5-link or 6-link robot arms. It is almost impossible to use trial-and-error to search for a Lyapunov function for the closed loop dynamics defined by (23) and (24).

7.9 Physically Motivated Lyapunov Function

- However, physical insights leads to the following Lyapunov function candidate

$$V = \frac{1}{2}[\dot{q}^T H \dot{q} + q^T K_p q] \tag{25}$$

  where the first term represents the kinetic energy of the manipulator, and the second term denotes the "artificial potential energy" associated with the virtual spring in the control law (24).
- The P.D. control term can be interpreted as mimicking a combination of dampers and springs.

7.9 Physically Motivated Lyapunov Function

- Physical insights also help derive the derivative of this function. In fact, the energy theorem in mechanics states that the rate of change of kinetic energy in a mechanical system is equal to the power provided by the external forces. Therefore,

$$\dot{V} = \dot{q}^T(\tau - g) + \dot{q}^T K_p q \tag{26}$$

  Substitution of the control law (24) in the above equation then leads to

$$\dot{V} = -\dot{q}^T K_D \dot{q} \tag{27}$$

- Thus, the system is stable. Since the arm cannot get "stuck" at any position such that $q \neq 0$, the system must also be asymptotically stable.

7.9 Physically Motivated Lyapunov Function

- In fact, by the invariant set theorem, $\lim_{t\to\infty}\dot{q}(t)=0$, which implies $\lim_{t\to\infty}\ddot{q}(t)=0$. Since the closed-loop system is

$$H(q)\ddot{q}+b(q,\dot{q})+K_D\dot{q}+K_pq=0, \tag{28}$$

we obtain

$$\lim_{t\to\infty}(b(q(t),0)+K_pq(t))=0. \tag{29}$$

Since $b(q,0)=0$ and K_p is positive definite, we have $\lim_{t\to\infty}q(t)=0$. That is, the system is actually globally asymptotically stable.

Remark:

Two lessons can be learned from this practical example.

- The first is that one should use as many as physical properties as possible in analyzing the behavior of a system;
- The second lesson is that physical concepts like energy may lead us to some uniquely powerful choices of Lyapunov functions.

7.10 Performance Analysis

Lyapunov functions can not only be used for stability analysis, but can also provide estimates of the transient performance of stable systems.

➢ **A simple convergence lemma**

Lemma 7.1

If a real function $W(t)$ *satisfies the inequality*

$$\dot{W}(t) + \alpha W(t) \leq 0 \tag{30}$$

where α *is a real number. Then*

$$W(t) \leq W(0)e^{-\alpha t} \tag{31}$$

Proof: Define a function $Z(t)$ by

$$Z(t) = \dot{W} + \alpha W \tag{32}$$

Equation (30) implies that $Z(t)$ is non-positive. The solution of the first-order equation (32) is

$$W(t) = W(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-r)} Z(t) dr$$

Because the second term in the right-hand-side of the above equation is non-positive, one has

$$W(t) \leq W(0)e^{-\alpha t}$$

7.10 Performance Analysis

Remark:

- The above lemma implies that, if W is a non-negative function, then the satisfaction of (30) guarantees the exponential convergence of W to zero. In particular, if $W = x^T P x$ with P a positive definite matrix, then $x(t)$ converges to the origin exponentially.
- In using Lyapunov's direct method for stability analysis, if one can manipulate $\dot{V}$ into the form of (30), the exponential convergence of V and the convergence rate can be inferred and, in turn, the exponential convergence rate of the state may then be determined if V is a quadratic function of the state.

7.10 Performance Analysis

➢ **Estimating convergence rates for linear systems**

Let us evaluate the convergence rate of a stable linear system based on the Lyapunov analysis. Let us denote the largest eigenvalue of the matrix P by $\lambda_{max}(P)$, the smallest eigenvalue of Q by $\lambda_{min}(Q)$, and their ratio $\lambda_{min}(Q)/\lambda_{max}(P)$ by γ. The positive definiteness of P and Q implies that these scalars are all strictly positive. Since matrix theory shows that

$$P \leq \lambda_{max}(P)I \qquad \lambda_{min}(Q)I \leq Q \tag{33}$$

We have

$$x^T Qx \geq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} x^T [\lambda_{max}(P)I]x \geq \gamma V \tag{34}$$

7.10 Performance Analysis

This implies that

$$\dot{V} \leq -\gamma V$$

This, according to Lemma 7.1, means that

$$x^T P x \leq V(0) e^{-\gamma t}$$

This, together with the fact $x^T P x \geq \lambda_{min}(P) \parallel x(t) \parallel^2$, implies that the state x converges to the origin with a rate of at least $\gamma/2$.

7.10 Performance Analysis

One might naturally wander how this convergence rate estimate varies with the choice of Q, and how it relates to the familiar notion of dominant pole in linear theory. An interesting result is that the convergence rate estimate is largest for $Q = I$. Indeed, let P_0 be the solution of the Lyapunov equation corresponding to $Q = I$:

$$A^T P_0 + P_0 A = -I$$

and let P be the solution corresponding to some other choice of Q

$$A^T P + PA = -Q_1$$

7.10 Performance Analysis

Without loss of generality, we can assume that $\lambda_{min}(Q_1) = 1$, since rescaling Q_1 will rescale P by the same factor, and therefore will not affect the value of the corresponding γ. Subtracting the above two equations yields

$$A^T(P - P_0) + (P - P_0)A = -(Q_1 - I)$$

Now since $\lambda_{min}(Q_1) = 1 = \lambda_{max}(I)$, the matrix $(Q_1 - I)$ is positive semi-definite, and hence the above equation implies that $(P - P_0)$ is positive semi-definite. Therefore

$$\lambda_{max}(P) \geq \lambda_{max}(P_0)$$

Since $\lambda_{min}(Q_1) = 1 = \lambda_{min}(I)$, $1/\lambda_{max}(P_0) \geq 1/\lambda_{max}(P)$. That is, the convergence rate estimate

$$\gamma = \lambda_{min}(Q)/\lambda_{max}(P)$$

corresponding to $Q = I$ is larger than (or equal to) that corresponding to $Q = Q_1$.

7.10 Performance Analysis

Remark:
If the stable matrix A is symmetric, then the meaning of this "optimal" value of γ, corresponding to the choice $Q = I$, can be interpreted easily. Indeed, all eigenvalues of A are then real, and furthermore A is diagonalizable, i.e., there exists a change of state coordinates such that in these coordinates A is diagonal. One immediately verifies that, in these coordinates, the matrix $P = -1/2A^{-1}$ verifies the Lyapunov equation for $Q = I$. Thus,

$$\gamma = 1/\lambda_{max}(P) = 2/\lambda_{max}(-A^{-1}) = 2\lambda_{min}(-A)$$

Therefore the cooresponding $\gamma/2$ is simply the absolute value of the dominant pole of the linear system. Furthermore, γ is obviously independent of the choice of state coordinates.