Chapter 4 Advanced Stability Theory

Professor Jie Huang

Dept. Mechanical and Automation Engineering
The Chinese University of Hong Kong

Outline

1 Introduction

1.1 Topics

(i) Stability concepts for non-autonomous systems

$$\dot{x} = f(x, t), \quad t \geq t_0 \tag{1}$$

- Stability vs uniform stability
- Instability theorems
- The converse theorems (existence of Lyapunov functions)

(ii) Barbalat's Lemma (useful for adaptive control)

(iii) Absolute stability (classical results)

(iv) Total stability

1.2 Non-autonomous systems

1.2.1 Equilibrium point

x^* is the E.P. of (1) if

$$f(x^*, t) \equiv 0, \quad \forall t \geq t_0$$

Example 1

$$\dot{x} = -\frac{a(t)x}{1+x^2}$$

$x^* = 0$ is the E.P.
For linear system $\dot{x} = A(t)x$, $x^* = 0$ is always an E.P., and is the unique E.P. if $A(t)$ is nonsingular for some $t \geq t_0$.

1.2 Non-autonomous systems (cont.)

1.2.2 Shift-invariant

For autonomous systems

$$\dot{x} = f(x), \quad t \geq 0 \tag{2}$$

Let $x(t)$ be the solution of (2) satisfying $x(0) = x_0$ and let $y(t) = x(t - t_0)$. Then $y(t_0) = x(0) = x_0$ and

$$\frac{dy(t)}{dt} = \frac{dx(t-t_0)}{dt} = f(x(t-t_0)) = f(y(t))$$

That is, $x(t - t_0)$ is the solution of (2) satisfying $x(t_0 - t_0) = x_0$. This property is called shift-invariant. As a result, we can always assume the initial time to be zero.

1.2 Non-autonomous systems (cont.)

But for non-autonomous systems, if $x(t)$ satisfies

$$\frac{dx(t)}{dt} = f(x(t), t).$$

Let $y(t) = x(t - t_0)$. Then

$$\frac{dy(t)}{dt} = \frac{dx(t - t_0)}{dt} = f(x(t - t_0), t - t_0) = f(y(t), t - t_0) \neq f(y(t), t).$$

Therefore, $x(t - t_0)$ may not be the solution of (1).
Thus, the initial time matters, and the shift-invariant property does not hold.

1.2.3 Nominal motion and motion stability

Let $x^*(t)$ be the solution of (2) satisfying $x^*(0) = x_0$, i.e.

$$\dot{x}^*(t) = f(x^*(t)), \quad x^*(0) = x_0.$$

$x^*(t)$ is called the nominal motion.

Let $x(t)$ be the solution of (2) satisfying $x(0) = x_0 + \delta x_0$, i.e., $x(t)$ is such that

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 + \delta x_0.$$

δx_0 is called the perturbation of the initial condition and $x(t)$ is called the perturbed motion.

Let $e(t) = x(t) - x^*(t)$. Then

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{x}^*(t) = f(x(t)) - f(x^*(t)) \\ &= f(x^*(t) + e(t)) - f(x^*(t)) \\ &= g(e, t), \qquad t \geq 0 \\ e(0) &= \delta x_0 \end{aligned}$$

1.2 Non-autonomous systems (cont.)

Clearly, $g(0,t) = 0$, $\forall t \geq 0$. Therefore, the origin is an E.P. of the non-autonomous system

$$\dot{e} = g(e,t).$$

We have seen that

- (i) The stability property of a nominal motion of a system can be converted into the stability property of an E.P. of the error system.
- (ii) A non-autonomous system can arise from studying the motion stability of an autonomous system.

(i) **Definition 4.1**
The E.P. $x^* = 0$ of (1) is stable at t_0 if $\forall R > 0$, $\exists$ constant $r(R, t_0) > 0$ such that

$$\|x(t_0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq t_0$$

Otherwise, the E.P. $x^* = 0$ is unstable.

(ii) **Definition 4.2**
The E.P. $x^* = 0$ is A.S. at time t_0 if

- It is stable at t_0.
- $\exists r(t_0) > 0$ such that $\|x(t_0)\| < r(t_0) \ \Rightarrow \ \lim_{t\to\infty} x(t) = 0$

Note:

- initial time t_0 matters
- r is allowed to rely on t_0

(iii) **Definition 4.3**

The E.P. $x^* = 0$ is E.S. if there exist $\alpha > 0$ and $\lambda > 0$ such that, for all sufficiently small $x(t_0)$,

$$\|x(t)\| \leq \alpha \|x_0\| e^{-\lambda(t-t_0)} \qquad t \geq t_0$$

(iv) **Definition 4.4**

The E.P. $x^* = 0$ is G.A.S. if $x^* = 0$ is stable at t_0 and, $\forall\ x(t_0)$,

$$\lim_{t\to\infty} x(t) = 0$$

2.2 Example 4.2

$$
\begin{aligned}
\dot{x}(t) = -a(t)x(t),\ x \in \Re^1 &\Rightarrow \frac{dx}{dt} = -a(t)x(t) \\
&\Rightarrow \frac{dx}{x} = -a(t)dt \\
&\Rightarrow \ln \frac{x(t)}{x(t_0)} = \int_{t_0}^{t} -a(\tau)d\tau \\
&\Rightarrow x(t) = x(t_0)\exp\left[-\int_{t_0}^{t} a(\tau)d\tau\right].
\end{aligned}
$$

Thus, the E.P. of the system

- (i) is stable if $a(t) \geq 0,\ \forall\ t \geq t_0$
- (ii) A.S. if $a(t) \geq 0,\ \forall\ t \geq t_0$ and $\int_0^\infty a(\tau)d\tau = \infty$
- (iii) E.S. if $a(t) \geq 0,\ \forall\ t \geq t_0$ and $\exists\ T > 0$ such that $\forall\ t \geq 0$

$$\int_t^{t+T} a(\tau)d\tau \geq r, \qquad \text{with r being a positive number}$$

2.2 Example 4.2 (cont.)

(i) and (ii) are clear, and (iii) is left for assignment.

Examples

◇ $\dot{x} = \frac{-x}{1+t^2}$ is stable at any t_0 since

$$a(t) = \frac{1}{(1+t)^2} \geq 0, \quad \forall t \geq t_0$$

but not A.S. since

$$\int_0^t a(\tau)d\tau = \int_0^t \frac{1}{(1+\tau)^2}d\tau = \frac{-1}{1+t} + 1 \to 1 \text{ as } t \to \infty$$

◇ $\dot{x} = \frac{-x}{1+t}$ is A.S. since

$$\int_0^t \frac{d\tau}{1+\tau} = \ln(1+t) \to \infty \text{ as } t \to \infty \tag{3}$$

2.2 Example 4.2 (cont.)

◇ $\dot{x} = -tx$ is E.S. since

$$\int_t^{t+T} \tau d\tau = \frac{\tau^2}{2}\Big|_t^{t+T} = \frac{(t+T)^2 - t^2}{2} = \frac{T(2t+T)}{2} \geq \frac{T^2}{2}$$

◇ $\dot{x} = -\dfrac{x}{1+\sin^2 x}$ is E.S. since

$$x(t) = x(t_0) \exp\left[-\int_{t_0}^{t} \frac{1}{1+\sin x^2(\tau)} d\tau\right]$$

$$\|x(t)\| \leq \|x(t_0)\| \exp\left[-\int_{t_0}^{t} \frac{1}{2} d\tau\right] \leq \|x(t_0)\| \exp\left[-\frac{(t-t_0)}{2}\right]$$

2.3 Uniformity in stability concepts

2.3.1 Introduction

- It is desirable for a system to have a certain uniformity in its behavior regardless of when the operation starts.
- Non-autonomous systems with uniform properties have some desirable ability to withstand disturbance.

2.3.2 Definitions

Definitions 4.5

The E.P. $x^* = 0$ is locally uniformly stable (U.S.) if the scalar r in Definition 4.1 can be chosen independent of t_0, i.e. $r = r(R)$

Definition 4.6

The E.P. $x^* = 0$ is locally U.A.S. if

◇ It is U.S.

◇ There exists a ball of attraction B_{R_0} whose radius is independent of t_0 such that any system trajectory with initial states in B_{R_0} converges to $x^* = 0$ uniformly in t_0.

2.3 Uniformity in stability concepts (cont.)

The uniform convergence in terms of t_0 means that, for all R_1 and R_2 satisfying $0 < R_2 < R_1 < R_0$, $\exists\ T(R_1, R_2) > 0$ such that $\forall\ t_0 \geq 0$,

$$\|x(t_0)\| < R_1 \quad \Rightarrow \quad \|x(t)\| \leq R_2, \quad \forall t \geq t_0 + T(R_1, R_2)$$

i.e. the state trajectory, starting from within B_{R_1}, will converge to a smaller ball B_{R_2} after a time period T which is independent of t_0.

By definition,

U.A.S. $\Rightarrow$ A.S., but the converse is generally not true.

2.3.3 Example 4.3

$$\dot{x} = -\frac{x}{1+t} \;\Rightarrow\; x(t) = \frac{1+t_0}{1+t}x(t_0)$$

The E.P. $x^* = 0$ is clearly A.S. but not uniformly A.S. In fact, for $\forall 0 < R_2 < R_1$, and, for any $|x(t_0)| < R_1$,

$$t > \frac{(1+t_0)R_1}{R_2} - 1 \geq \frac{(1+t_0)|x(t_0)|}{R_2} - 1 \Rightarrow \; |x(t)| \leq \frac{(1+t_0)|x(t_0)|}{1+t} < R_2$$

Let $t = t_0 + T(R_1, R_2, t_0) = \frac{(1+t_0)R_1}{R_2} - 1$. Then $T(R_1, R_2, t_0) = \frac{(1+t_0)(R_1 - R_2)}{R_2}$. Then

$$|x(t)| < R_2, \quad \forall t \geq t_0 + T(R_1, R_2, t_0)$$

But T cannot be made independent of t_0. This is because a larger t_0 requires a longer time to get close to the origin.

2.3.4 Remark

- (i) E.S. $\Rightarrow$ U.A.S.
- (ii) For linear systems, E.S. $\Leftrightarrow$ U.A.S.
- (iii) Global U.A.S. (replacing B_{R_0} by $\Re^n$)
- (iv) For autonomous systems,
  S $\Leftrightarrow$ U.S.
  U.A.S. $\Leftrightarrow$ A.S.

3.1 P.D. functions and decrescent functions

- Direct method
- Linearization method

3.1.1 Definition 4.7

A scalar continuous time-varying function $V(x,t)$ is locally positive definite if $V(0,t)=0$, $\forall t \geq 0$ and there exists a time-invariant P.D. function $V_0(x)$ such that

$$\forall t \geq 0, \quad V(x,t) \geq V_0(x) \tag{4}$$

i.e. a time-varying function is locally P.D. if it dominates a time-invariant locally P.D. function. Similar definitions can be obtained for negative definite, positive semi-definite, negative semi-definite or global positive definite functions.

3.1 P.D. functions and decrescent functions (cont.)

3.1.2 Definition 4.8

A scalar function $V(x,t)$ is decrescent if $V(0,t)=0$, $\forall t \geq 0$, and if $\exists$ a time-invariant P.D. function $V_1(x)$ such that

$$\forall t \geq 0, \qquad V(x,t) \leq V_1(x)$$

i.e. a scalar function $V(x,t)$ is decrescent if it is dominated by a time-invariant locally P.D. function.

A time-invariant P.D. function is always decrescent.

3.1.3 Example

$$\begin{aligned} V_0(x) = x_1^2 + x_2^2 \leq V(x,t) &= (1+\sin^2 t)(x_1^2+x_2^2) \\ &\leq 2(x_1^2+x_2^2) = V_1(x) \end{aligned}$$

3.2 Lyapunov Theorem

3.2.1

Let $V(x,t)$ be a time-varying function, its derivative along the trajectory of system (1) is

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \tag{5}$$

3.2.2

$V(x,t)$ is called a Lyapunov fucntion for (1) if $V(x,t) \in C^1$ and $\exists B_r$ for some $r > 0$ such that, in B_r, V is positive definite and $\dot{V}$, its derivative along the system trajectories, is negative semi-definite.

3.2.3 Theorem 4.1

◇ Stability: If, in a ball B_{R_0} around the E.P. $x^* = 0$, there exists a scalar function $V(x,t) \in C^1$ such that

- 1. V is P.D.
- 2. $\dot{V}$ is N.S.D.

then the E.P. $x^* = 0$ is stable in the sense of Lyapunov.

◇ Uniform stability and uniform asymptotic stability: If, furthermore,

- 3. V is decrescent

then $x^* = 0$ is U.S. If condition 2 is strengthened by requiring that $\dot{V}$ be negative definite, then $x^* = 0$ is uniformly A.S.

◇ Global U.A.S.: If the ball B_{R_0} is replaced by $\Re^n$, and condition 1, the strengthened condition 2, condition 3, and the condition

- 4. $V(x,t)$ is radially unbounded.

are all satisfied, then $x^* = 0$ is G.U.A.S.

3.2.4 Example 4.5 (G.A.S)

$$\begin{aligned} \dot{x}_1 &= -x_1(t) - e^{-2t}x_2(t) \\ \dot{x}_2 &= x_1 - x_2(t) \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -e^{-2t} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Choose

$$\begin{aligned} V(x,t) &= x_1^2 + (1+e^{-2t})x_2^2 \\ &= [x_1 \;\; x_2] \begin{bmatrix} 1 & 0 \\ 0 & 1+e^{-2t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T P(t) x \end{aligned}$$

which is P.D. and decrescent.
Furthermore

$$\begin{aligned} \dot{V}(x,t) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= -2e^{-2t}x_2^2 + 2x_1(-x_1 - e^{-2t}x_2) + 2(1+e^{-2t})x_2(-x_1 - x_2) \end{aligned}$$

$$= -2[x_1^2 - x_1x_2 + x_2^2(1 + 2e^{-2t})]$$
$$\leq -2(x_1^2 - x_1x_2 + x_2^2) = -(x_1 - x_2)^2 - x_1^2 - x_2^2$$

Therefore $\dot{V}$ is N.D. $\Rightarrow$ The E.P. is G.A.S.

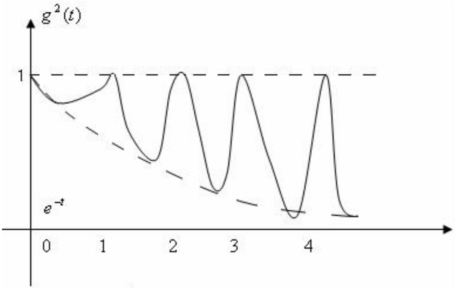
3.2.5 Example 4.6 (importance of the decrescence condition)

Consider

$$\dot{x} = \frac{\dot{g}(t)}{g(t)}x, \quad x \in \Re^1 \tag{6}$$

where $g(t)$ is a continuously differentiable function which coincides with the function $e^{-\frac{t}{2}}$ except around some peaks where it reaches the value 1. The curve of $g^2(t)$ is shown in Figure 4.2.

3.2 Lyapunov Theorem (cont.)



The function $g^2(t)$

Assume $g(t) = 1 \quad t = 0, 1, ...$ and the width of the peak corresponding to abcissa n is assumed to be smaller than $\left(\frac{1}{2}\right)^n$. Thus,

$$\int_0^{\infty} g^2(\tau) d\tau < \int_0^{\infty} e^{-\tau} d\tau + \sum_{n=1}^{\infty} \frac{1}{2^n} = 2$$

3.2 Lyapunov Theorem (cont.)

Let

$$V(x,t) = \frac{x^2}{g^2(t)}\left[3 - \int_0^t g^2(\tau)d\tau\right] > x^2$$

which is P.D. but not decrescent.
Then $\dot{V} = -x^2$ which is N.D. Yet the solution of (6) is

$$x(t) = \frac{g(t)}{g(t_0)}x(t_0)$$

Therefore $x^* = 0$ is not A.S.

3.2.6 Example

$$\ddot{x} + c(t)\dot{x} + k_0 x = 0$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_0 x_1 - c(t) x_2 \end{aligned} \quad \Rightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_0 & -c(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A(t)x$$

where $c(t) \geq 0$ is a time-varying damping coefficient, and $k_0 > 0$ is a spring constant. This system is not necessarily A.S. In fact, let $c(t) = (2 + e^t)$ and $k_0 = 1$. Then with $x_1(0) = 2$, $x_2(0) = -1$, we have

$$x_1(t) = 1 + e^{-t}, \quad x_2(t) = -e^{-t},$$

which is not A.S.

Interpretation: the damping increases so fast that the system gets "stuck" at $x = 1$.

3.2 Lyapunov Theorem (cont.)

Remark: Various stability concepts with respect to the equilibrium point can also be stated in terms of class $\mathcal{K}$, and class $\mathcal{KL}$ functions.

Class $\mathcal{K}$ function:
A continuous function $\alpha : [0, a) \to \mathcal{R}^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$, and is said to be class $\mathcal{K}_\infty$ if, in addition, $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Class $\mathcal{KL}$ function:
A continuous function $\beta : [0, a) \times [0, \infty) \to \mathcal{R}^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the function $\beta(\cdot, s)$ is a class $\mathcal{K}$ function defined on $[0, a)$, and, for each fixed r, the function $\beta(r, \cdot) : [0, \infty) \to [0, \infty)$ is decreasing, and $\beta(r, s) \to 0$ as $s \to \infty$.

Examples:
$\alpha(r) = r, r^2$ belong to class $\mathcal{K}_\infty$. $\beta(r, s) = \alpha(r)e^{-\lambda s}$, where $\alpha(\cdot)$ is a class $\mathcal{K}$ function and $\lambda > 0$, belongs to class $\mathcal{KL}$

3.2 Remark

Lemma: Let $V : B_R \times [0, \infty) \to \Re$ for some $R > 0$ be a continuous positive definite function. Let $B_r \subset B_R$ for some $R > r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(||x||) \leq V(x, t) \leq \alpha_2(||x||)$$

for all $x \in B_r$. If $B_R = \Re^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in \Re^n$. Moreover, if $V(x, t)$ is radially unbounded, then α_1 and α_2 can be chosen to be class $\mathcal{K}_\infty$ functions.

See the following reference:

H. K. Khalil, Nonlinear Systems, third edition, Prentice Hall, 2002.

3.2 Lyapunov Theorem (cont.)

Remark. *The equilibrium point of the system (1) is*

1. uniformly stable (US), if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c, independent of t_0, such that
$$||x(t)|| \leq \alpha(||x(t_0)||), \forall\, t \geq t_0, \forall\, ||x(t_0)|| < c; \tag{7}$$
2. uniformly asymptotically stable (UAS), if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive constant c, independent of t_0, such that
$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \forall\, t \geq t_0, \forall\, ||x(t_0)|| < c; \tag{8}$$
3. exponentially stable (ES) if (8) is satisfied with $\beta(r,s) = re^{-\lambda s}$, for some $r > 0$, $\lambda > 0$.
4. globally uniformly stable (GUS) if (7) is satisfied with $\alpha \in \mathcal{K}_\infty$ and $c = \infty$;
5. globally uniformly asymptotically stable (GUAS) if (8) is satisfied with $c = \infty$; and
6. globally exponentially stable (GES) if (8) is satisfied with $\beta(r,s) = re^{-\lambda s}$ with $c = \infty$, for some $r > 0$, $\lambda > 0$.

Remark. Suppose there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c, independent of t_0, such that

$$||x(t)|| \leq \alpha(||x(t_0)||), \forall\, t \geq t_0, \forall\, ||x(t_0)|| < c.$$

Then, the equilibrium point at the origin is uniformly stable.

Proof: For any $0 < R < \alpha(b)$ with $0 < b < a$, let $0 < r < \min\{c, \alpha^{-1}(R)\}$, which is independent of t_0. Then

$$||x(t)|| \leq \alpha(||x(t_0)||) \leq \alpha(r) < R, \forall\, t \geq t_0, \forall\, ||x(t_0)|| < r.$$

Exercise:
Suppose there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a positive constant c, independent of t_0, such that

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \forall\, t \geq t_0, \forall\, ||x(t_0)|| < c.$$

Show that the equilibrium point at the origin is uniformly asymptotically stable.

3.2 Lyapunov Theorem (cont.)

Theorem. Let $V : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}^+$ be a C^1 function such that, for some class $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, defined on $[0, d)$,

(i) $$\alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|)$$

(ii) $\dot{V}(x,t) \stackrel{def}{=} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq 0$ for all $\|x\| < d$, and all $t \geq t_0$.

Then the origin of system (1) is uniformly stable. If, (ii) is replaced by

(iii) $\dot{V}(x,t) \leq -\alpha_3(\|x\|)$, for all $\|x\| < d$, and all $t \geq t_0$, where $\alpha_3(\cdot)$ is some class $\mathcal{K}$ function defined on $[0, d)$,

then the origin is uniformly asymptotically stable.
If $d = \infty$, and $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathcal{K}_\infty$ functions, then the origin is uniformly globally asymptotically stable.
If $\alpha_i(r) = k_i r^\lambda$ on $[0, d)$, $k_i > 0$, $\lambda > 0$, $i = 1, 2, 3$, then the origin is exponentially stable. If $d = \infty$, then the origin is globally exponentially stable.

4 Linear Time-varying Systems

4.1 Consider systems

$$\dot{x} = A(t)x, \qquad t \geq 0 \tag{9}$$

where $A(t) \in \Re^{n\times n}$ and is continuous in t for all t.

4.2 Corollary 1

The system (9) is uniformly asymptotically stable if $\exists \lambda > 0$ such that $\lambda_i(A(t) + A^T(t)) \leq -\lambda$, $\forall t \geq 0$, $i = 1, ..., n$.

Proof: Let $V = x^T x$. Then

$$\dot{V} = x^T\dot{x} + \dot{x}^T x = x^T(A(t) + A^T(t))x \leq -\lambda x^T x$$

By Theorem 4.1, (9) is uniformly asymptotically stable.
Remark: Note that $\dot{V} \leq -\lambda V \Rightarrow \ x^T x = V(t) \leq V(0)e^{-\lambda t}$ (A simple exercise). Thus (9) is actually exponentially stable.
In general, for (9), U.A.S. $\Leftrightarrow$ E.S.

4.3 Corollary 2

Assume $\exists P(t) \in \Re^{n \times n}$ which is C^1 and symmetric, and there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1 I \leq P(t) \leq c_2 I \qquad \forall\ t \geq 0$$

Further assume for some $Q(t) \in \Re^{n \times n}$, continuous and symmetric such that

$$Q(t) \geq c_3 I > 0$$
$$\dot{P} + P(t)A(t) + A^T(t)P(t) = -Q(t)$$

Then (9) is U.A.S.

Proof: Let $V(t,x) = x^T P(t) x$. Then

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2$$

Thus, $V(t,x)$ is P.D. and decrescent. Moreover, $\forall t \geq 0$,

$$\begin{aligned} \dot{V} &= x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x \\ &= x^T (\dot{P} + P(t)A(t) + A^T(t)P(t)) x = -x^T Q(t) x \leq -c_3 \|x\|^2 \end{aligned}$$

4.3 Corollary 2 (cont.)

Remark: In general, for (9), U.A.S. $\Leftrightarrow$ E.S.

Example 4.7

Given $A(t) = \begin{bmatrix} -1 & -e^{-2t} \\ 1 & -1 \end{bmatrix}$

let $P(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1+e^{-2t} \end{bmatrix}$

Then

$$\dot{P} + PA + A^T P$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & -2e^{-2t} \end{bmatrix} + \begin{bmatrix} -1-e^{-2t} & e^{-2t} \\ 1+e^{-2t} & -1-e^{-2t} \end{bmatrix} + \begin{bmatrix} -1 & 1+e^{-2t} \\ -e^{-2t} & -1-e^{-2t} \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 1 \\ 1 & -2-4e^{-2t} \end{bmatrix} = -Q \leq \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Therefore, the system is U.A.S.

4.4 Perturbed linear systems

Proposition 1

$$\dot{x} = A_1 x + g(t, x) \tag{10}$$

where $A_1 \in \Re^{n\times n}$ is constant and Hurwitz, and $g(t,x)$ is continuous satisfying $||g(t,x)|| \le \gamma(t)||x||,\ t \ge 0$ where $\gamma : \Re \to \Re$ is continuous and $\lim_{t\to\infty} \gamma(t) = 0$. Then (10) is G.E.S.

Hint of Proof: Let $V(x) = x^T P x$ where P is positive definite and symmetric such that $PA_1 + A_1^T P = -I_n$.

Corollary Consider

$$\dot{x} = (A_1 + A_2(t))x \tag{11}$$

where $A_1 \in \Re^{n\times n}$ is constant and Hurwitz, and $A_2(t)$ is such that

$$A_2(t) \to \ 0 \quad \text{as } t \to \infty$$

or

$$\int_0^\infty \|A_2(t)\| dt < \infty$$

Then (10) is E.S.

Example 4.8

$$\dot{x}_1 = (-5 + x_2^5 + x_3^8)x_1, \ \dot{x}_2 = -x_2 + 4x_3^2, \ \dot{x}_3 = -(2 + \sin t)x_3$$

Let $a(t) = 2 + \sin t$. Then

$$x_3(t) = x_3(t_0) \exp\left[-\int_{t_0}^{t} (2 + \sin t) dr\right] \Rightarrow \|x_3\| \leq \|x_3(t_0)\| \exp^{-(t-t_0)}$$

Therefore

$$x_2(t) = e^{-(t-t_0)} x_2(t_0) + \int_{t_0}^{t} e^{-(t-\tau)} 4x_3^2(\tau) d\tau$$

Thus, it is ready to see that the system is G.E.S upon using Proposition 1 on the x_1 subsystem.

Proposition 2

Consider (9), and assume $Re(\lambda_i[(A(t))]) \leq -\alpha$ for some $\alpha > 0$, $\forall\ i$. Further assume $A(t)$ is bounded, and

$$\int_0^\infty A^T(t)A(t)dt < \infty$$

Then the system is globally E.S.

5.1

Given

$$\dot{x} = f(x,t) \text{ with } f(0,t) = 0, \quad \forall t \geq 0$$

where $x \in \Re^n$, $f: \ B_r \times [0,\infty) \to \Re^n$ is C^1.

Further suppose the Jacobian matrix $\left[\dfrac{\partial f}{\partial x}\right]$ is bounded and Lipschitz on B_r uniformly in t, i.e., $\forall 1 \leq i \leq n$, some constant $L_1 > 0$,

$$\left\| \frac{\partial f_i}{\partial x}(x_a,t) - \frac{\partial f_i}{\partial x}(x_b,t) \right\|_2 \leq L_1 \left\| x_a - x_b \right\|_2,$$

$$\forall \ x_a, x_b \in B_r, \ \forall t \geq 0.$$

Let

$$A(t) = \left(\frac{\partial f}{\partial x} \right)\bigg|_{x=0}$$

Then

$$\dot{x} = A(t)x + f_{h.o.t.}(x,t)$$

where

$$\|f_{h.o.t.}(x,t)\| \leq L\|x\|^2, \ \ \forall t \geq 0, \ \forall x \in B_r \text{ with } L = \sqrt{n}L_1.$$

The linear-time-varying system

$$\dot{x} = A(t)x$$

is said to be the linearization of the nonlinear system around the E.P. $x^* = 0$.

5.2 Theorem 4.2

If the linearization system is uniformly A.S., then the E.P. $x^* = 0$ of the original system is also U.A.S.

5.3 Theorem 4.3

If $A(t)$ is constant, and one or more of the eigenvalues of $A(t)$ has positive real part, then the E.P. $x^* = 0$ of the original system is unstable.

Remark: By the mean value theorem,

$$f_i(x,t) = f_i(0,t) + \frac{\partial f_i}{\partial x}(\xi x, t)x$$

for some $0 \le \xi \le 1$. Since $f(0,t) = 0$ for all $t \ge 0$, we have

$$f_i(x,t) = \frac{\partial f_i}{\partial x}(\xi x, t)x = \frac{\partial f_i}{\partial x}(0,t)x + \left(\frac{\partial f_i}{\partial x}(\xi x, t) - \frac{\partial f_i}{\partial x}(0,t)\right)x$$

Hence,

$$f(x,t) = A(t)x + f_{h.o.t.}(x,t)$$

where

$$f_{h.o.t.}(x,t) = \left(\frac{\partial f}{\partial x}(\xi x, t) - \frac{\partial f}{\partial x}(0,t)\right)x$$

5 The Linearization Method (cont.)

The function $f_{h.o.t.}(x,t)$ satisfies, $\forall t \geq 0, \ \forall x \in B_r$,

$$\|f_{h.o.t.}(x,t)\|^2 \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}(\xi x, t) - \frac{\partial f_i}{\partial x}(0,t) \right\|^2 \right) \|x\|^2 \leq L^2 \|x\|^4$$

where $L = \sqrt{n} L_1$.

Proof of Theorem 4.2: Let $V(x,t) = x^T P(t) x$ where $P(t)$ is the positive definite and symmetric solution of the equation $\dot{P} + P(t)A(t) + A^T(t)P(t) = -Q(t)$ for some continuous, positive definite, and symmetric matrix $Q(t)$. The rest of the proof is the same as that of Theorem 3.1.

5.4 Instability Theorems

There are three instability theorems which can be used to ascertain instability of the equilibrium point of a dynamic system. One of the instability theorems is stated as follows.

5.4 Theorem 4.4 (First instability theorem)

If, in a certain neighborhood Ω of the origin, there exist a continuously differentiable, decrescent scalar function $V(x,t)$ such that

- $V(0,t) = 0 \ \ \forall t \geq t_0$
- $V(x,t_0)$ can assume strictly positive values arbitrarily close to the origin
- $\dot{V}(x,t)$ is positive definite (locally in Ω)

then the equilibrium point 0 at time t_0 is unstable.

Remark (i) V does not have to be P.D., e.g., $V(x) = x_1^2 - x_2^2$ is not P.D., but it can assume positive values arbitrarily near the origin. ($V(x) = x_1^2$ along the line $x_2 = 0$)

(ii) For autonomous systems, V can be a function of x only.

(iii) The proof is left as problem 4 of Assignment 6.

5.4 Instability Theorem

Example 4.9 Consider the system

$$\dot{x}_1 = 2x_2 + x_1(x_1^2 + 2x_2^4)$$
$$\dot{x}_2 = -2x_1 + x_2(x_1^2 + x_2^4)$$

The eigenvalues of the Jacobian linearization of the system are $+2j$ and $-2j$, indicating the inability of Lyapunov's linearization method for this system. However, if we take

$$V = \frac{1}{2}(x_1^2 + x_2^2)$$

its derivative is

$$\dot{V} = (x_1^2 + x_2^2)(x_1^2 + x_2^4)$$

Thus, by Theorem 4.4, the E.P. of the system at the origin is unstable.
Remark: This theorem can also be used to prove the second part of Theorem 3.1.

6 Existence of Lyapunov Functions

Introduction

Existence of Lyapunov function $\Rightarrow$ stability

Existence of Lyapunov function with $\dot{V}$ N.D. $\Rightarrow$ A.S.

What happen if a system is known to be stable or A.S.? Converse Lyapunov Theorems give answers.

These theorems are useful in stability analysis of interconnected systems or adaptive systems.

6 Existence of Lyapunov Functions (cont.)

Theorem 4.7 (Stability)

If the origin of (1) is stable, there exists a positive definite function $V(x,t)$ with a non-positive derivative.

Theorem 4.8 (U.A.S.)

If the E.P. at the origin is U.A.S., there exists a P.D. and decrescent function $V(x,t)$ with a negative definite derivative.

The theorem is useful in establishing robustness of U.A.S. systems to persistent disturbance.

6 Existence of Lyapunov Functions (cont.)

Theorem 4.9 (E.S.)

If the vector function $f(x,t)$ in (1) has continuous and bounded first partial derivatives w.r.t. x and t, for all $x \in B_r$ for some $r > 0$ and for all $t \geq 0$, then the E.P. at the origin is E.S. iff $\exists$ a function $V(x,t)$ and $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$ such that $\forall x \in B_r$, $\forall t \geq 0$

$$\alpha_1 \|x\|^2 \leq V(x,t) \leq \alpha_2 \|x\|^2$$
$$\dot{V} \leq -\alpha_3 \|x\|^2$$
$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4 \|x\|$$

The theorem is useful for estimating the convergence rate and stability analysis of interconnected systems.

7.1.1

Play a role for non-autonomous systems similar to what invariant set theorem does for autonomous system (invariant set theorem does not apply to non-autonomous system).

Barbalat's Lemma is useful in stability analysis for non-autonomous systems when $\dot{V}(x,t)$ is negative semi-definite.

7.1.2 Some facts

$\diamond$ $\dot{f} \to 0$ does not imply f converges, e.g., let $f(t) = \sin(\log t)$

$$\dot{f}(t) = \frac{\cos(\log t)}{t} \to 0 \text{ as } t \to \infty$$

$\diamond$ f converges does not imply $\dot{f} \to 0$
e.g., $f(t) = e^{-t}\sin(e^{2t}) \to 0$ as $t \to \infty$,
but $\dot{f}(t) = -e^{-t}\sin(e^{2t}) + e^{2t}e^{-t}\cos(e^{2t})$ is unbounded.
$\diamond$ If f is lower bounded and decreasing ($\dot{f} \leq 0$), then it converges to a limit.

7.2 Barbalat's Lemma

(i) A function $g(t)$ is continuous on $[0, \infty)$ if

$$\forall t_1 > 0,\ \forall\ R > 0,\ \exists \eta(R, t_1) > 0 \text{ such that}$$
$$\forall t \geq 0,\ |t - t_1| < \eta \Rightarrow |g(t) - g(t_1)| < R$$

(ii) $g(t)$ is uniformly continuous on $[0, \infty)$ if

$$\forall R > 0,\ \exists \eta(R) > 0 \text{ such that}$$
$$\forall t_1 \geq 0,\ \forall t \geq 0,\ |t - t_1| < \eta \Rightarrow |g(t) - g(t_1)| < R$$

i.e., $\exists \eta$ which does not depend on the specific t_1, and in particular, η does not shrink as $t_1 \to \infty$.

7.2 Barbalat's Lemma (cont.)

7.2.1

If $\dot{g}(t)$ is bounded, i.e., $|\dot{g}(t)| < M$ for some $M > 0$, $t \in [0, \infty)$, then $g(t)$ is uniformly continuous in $[0, \infty)$ since

$$g(t) - g(t_1) = \dot{g}(t_2)(t - t_1), \ \forall t, \forall t_1 \text{ where } t \leq t_2 \leq t_1$$

7.2.2 Lemma 4.2

Assume $f(t)$ is continuously differentiable for $t \geq t_0$ for some t_0, and $f(t)$ has a finite limit as $t \to \infty$, and $\dot{f}$ is uniformly continuous. Then $\dot{f}(t) \to 0$ as $t \to \infty$.

Proof: Assume $\dot{f}(t)$ does not approach zero as $t \to \infty$. Then, for any $\varepsilon_0 > 0$, there exist $T > 0$, $t > T$, such that $|\dot{f}(t)| \geq \varepsilon_0$. Thus, there exists an infinite sequence $t_1, t_2, ...$ satisfying $t_i \to \infty$ as $i \to \infty$ such that $|\dot{f}(t_i)| \geq \varepsilon_0$. Since $\dot{f}(t)$ is uniformly continuous, $\exists \eta > 0$ such that

$$|\dot{f}(t') - \dot{f}(t'')| < \frac{\varepsilon_0}{2} \text{ for } \forall t', t'' \text{ satisfying } |t' - t''| < \eta$$

Thus, $\forall t_i$,

$$|\dot{f}(t)| = |\dot{f}(t) - \dot{f}(t_i) + \dot{f}(t_i)| \geq |\dot{f}(t_i)| - |\dot{f}(t) - \dot{f}(t_i)| > \frac{\varepsilon_0}{2}$$

whenever $|t - t_i| < \eta$. Hence, for any t_i

$$\left| \int_{t_i-\eta}^{t_i+\eta} \dot{f}(t)dt \right| = \int_{t_i-\eta}^{t_i+\eta} |\dot{f}(t)|dt \geq \frac{\varepsilon_0}{2} 2\eta = \varepsilon_0 \eta$$

where the left equality is due to the fact that $\dot{f}$ keeps a constant sign over integration interval since $\dot{f}$ is continuous and $|\dot{f}(t)| > \frac{\varepsilon_0}{2} > 0$.
Thus, the integral $\int_{t_0}^{t} \dot{f}(\tau)d\tau = f(t) - f(t_0)$ cannot have a limit as $t \to \infty$ **(Why? Exercise)**, a contradiction.

Corollary: If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and is such that $\ddot{f}$ exists and is bounded, then $\dot{f}(t) \to 0$ as $t \to \infty$

Example 4.12

Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where A is strictly stable and u is bounded. Show that $y(t)$ is uniformly continuous.

Proof: the state x is bounded since A is strictly stable and u is bounded because

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Thus, $\dot{x}$ is also bounded since $\dot{x} = Ax + Bu$.
Therefore $\dot{y} = C\dot{x}$ is also bounded. Thus, $y(t)$ is uniformly continuous.

7.2.3 Lemma 4.3

If a scalar function $V(x,t)$ satisfies the following conditions

- $\diamond$ $V(x,t)$ is lower bounded
- $\diamond$ $\dot{V}(x,t)$ is negative semi-definite
- $\diamond$ $\dot{V}(x,t)$ is uniformly continuous in time

then $\dot{V}(x,t) \to 0$ as $t \to \infty$.

Remark:

$$\lim_{t\to\infty} V(t) = V_\infty \le V(x(0),0)$$

7.2.4 Example 4.13

Consider a second order system

$$\dot{e} = -e + \theta\omega(t)$$
$$\dot{\theta} = -e\omega(t)$$

where e and θ are the two states, and $\omega(t)$ is a bounded continuous function. Let us analyze the asymptotic properties of the system.

Consider the lower bounded function

$$V = e^2 + \theta^2$$

Its derivative is

$$\dot{V} = 2e\dot{e} + 2\theta\dot{\theta} = 2e(-e + \theta\omega) + 2\theta(-e\omega) = -2e^2 \leq 0$$

This implies that $V(t) \leq V(0)$, and therefore, that e and θ are bounded.

But we cannot use the invariant set theorem to conclude $\dot{V} = -2e^2 \to 0$ as $t \to \infty$ because the system is not autonomous. Nevertheless, we can use Lemma 4.3.

Note that the derivative of $\dot{V}$ is

$$\ddot{V} = -4e\dot{e} = -4e(-e + \theta\omega)$$

This shows that $\ddot{V}$ is bounded since e, θ and ω are bounded. Hence, $\dot{V}$ is uniformly continuous. By Lemma 4.3, $e(t) \to 0$ as $t \to \infty$.

Remark:

An analysis based on Barbalat's Lemma is called a Lyapunov like analysis. It is usually used for stability analysis of a closed-loop system resulting from adaptive control (Chapter 8). V does not have to be P.D. and can be a lower bounded function. $\dot{V}$ must be uniformly continuous.

8 Total Stability

Total stability is concerned with systems of the form

$$\dot{x} = f(x,t) + g(x,t) \tag{12}$$

where $f(0,t) = 0$ but $g(0,t)$ may not be zero. Thus the origin is not necessarily an equilibrium point for (12). The term $g(x,t)$ can be considered a perturbation term for the following unperturbed system

$$\dot{x} = f(x,t) \tag{13}$$

which has an equilibrium point at the origin.
It is desirable to derive a boundedness condition for the perturbed equation (12) from the stability properties of the associated unperturbed system (13). The concept of total stability characterizes the ability of a system to withstand small persistent disturbances, and is defined as follows:

8 Total Stability (cont.)

Definition 4.13

The equilibrium point $x = 0$ for the unperturbed system (13) is said to be totally stable if for every $\varepsilon \geq 0$, two numbers δ_1 and δ_2 exist such that $\|x(t_0)\| < \delta_1$ and $\|g(x,t)\| < \delta_2$ imply that every solution $x(t)$ of the perturbed system (12) satisfies the condition $\|x(t)\| < \varepsilon$.

Remark

- The above definition means that an equilibrium point is totally stable if the state of the perturbed system can be kept arbitrarily close to zero by restricting the initial state and the perturbation to be sufficiently small.
- Let $u(t) = g(x,t)$. Then total stability can be viewed as a local version (with small input) of BIBO (bounded input bounded output) stability.
- If the unperturbed system is linear and time-invariant, then total stability is guaranteed by the asymptotic stability of the unperturbed system $\left(x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\right)$.

Theorem 4.14

If the equilibrium point of (13) is uniformly asymptotically stable, then it is totally stable.

Remarks

- Uniformly asymptotically stable systems can withstand small disturbances.
- Total stability of a system may be established by Theorem 4.8.
- Note that asymptotic stability is not sufficient to guarantee the total stability of a nonlinear system as can be verified by counter-examples.
- Exponentially stable systems are always totally stable because exponential stability implies uniform asymptotic stability.

8 Total Stability (cont.)

Example 4.22

Consider the system $\ddot{x} + 2\dot{x}^3 + 3x = d(t)$ which can be put in the standard form

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = -b(x_2) - c(x_1) + d(t)$$

where $b(x_2) = 2x_2^3$, $c(x_1) = 3x_1$, and $d(t)$ is the perturbation. Using Lemma 2 of Chapter 3 shows that the equilibrium point of the unperturbed system is (uniformly) globally asymptotically stable. Thus, Theorem 4.14 shows that the system can withstand small disturbances $d(t)$.

Remark

- Total stability guarantees boundedness to only small-disturbance, and requires only local uniform asymptotic stability of the E.P.
- The global uniform asymptotic stability cannot guarantee the boundedness of state in the presence of large (though still bounded) perturbation. (See the following counter-example)

Example 4.23

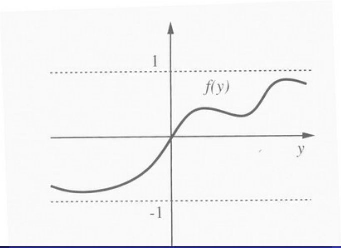
The nonlinear equation

$$\ddot{x} + f(\dot{x}) + x = w(t) \tag{14}$$

can be regarded as representing mass-spring-damper system containing nonlinear damping $f(\dot{x})$ and excitation force $w(t)$, where f is a first and third quadrant continuous nonlinear function such that

$$|f(y)| \leq 1, \quad -\infty < y < \infty$$

as illustrated below (Figure 4.12: A nonlinear damping function).



Example 4.23 (cont.)

The system is totally stable, because the E.P. of the unperturbed system is globally uniformly asymptotically stable using Lemma 2 of Chapter 3 again.

Is the output bounded for bounded input?

Consider the response of the system to the excitation force $w(t) = A\sin t$, $A > 8/\pi$. By writing (14) as

$$\ddot{x} + x = A\sin t - f(\dot{x}) \tag{15}$$

and solving this linear equation with zero initial condition with $u(t) = (A\sin t - f(\dot{x}))$ as input

$$\begin{aligned} x(t) &= \int_0^t \sin(t-\tau)(A\sin\tau - f(\dot{x}(\tau))d\tau \\ x(t) &= \frac{A}{2}(\sin t cos^2 t - \cos t(t - \frac{1}{2}\sin 2t)) - \int_0^t \sin(t-\tau)f(\dot{x})d\tau \\ &\geq \frac{A}{2}(\sin t cos^2 t - \cos t(t - \frac{1}{2}\sin 2t)) - \int_0^t |\sin(t-\tau)|d\tau \end{aligned}$$

Example 4.23 (cont.)

It can be seen that at $t = t_n = (2n+1)\pi$,

$$\sin t cos^2 t - \cos t(t - \frac{1}{2}\sin 2t) = (2n+1)\pi.$$

Also, it can be shown that, at $t = t_n = (2n+1)\pi$,

$$\int_0^{t_n} |\sin(t-\tau)| d\tau = 2(2n+1).$$

Thus

$$x(t_n) \geq (2n+1)\pi[\frac{A}{2} - \frac{2}{\pi}]$$

Therefore, if we take $A > 8/\pi$, then $x(t_n) \to \infty$.

Remark to Example 4.23

The frequency domain representation of (15) is

$$X(s) = \frac{1}{s^2 + 1} U(s)$$

where $X(s) = \mathcal{L}[x(t)]$ and $U(s) = \mathcal{L}[u(t)]$. Since $\mathcal{L}^{-1}[\frac{1}{s^2+1}] = \sin t$, we have

$$x(t) = \int_0^t \sin(t - \tau)(A \sin \tau - f(\dot{x}(\tau))d\tau$$

9 Positive Linear Systems

Introduction

- Positive systems is a class of linear systems arising from the passive circuits. It is closely related to passive systems.
- Positive systems have some special property that is crucial for adaptive control analysis.
- In the analysis and design of nonlinear systems, it is often possible and useful to decompose the system into a linear subsystem and a nonlinear subsystem. If the transfer function (or transfer matrix) of the linear subsystem is so-called positive real, then it has important properties which may lead to the generation of a Lyapunov function for the whole system.
- The properties of positive systems play a central role in proving the absolute stability result.

9.1 PR and SPR Transfer Functions

Relative degree.
Let a rational function of $n^{th}-$order single-input single-output linear systems be represented in the form

$$h(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_0}{p^n + a_{n-1} p^{n-1} + \cdots + a_0}$$

The coefficients of the numerator and denominator polynomials are assumed to be real numbers, $b_m \neq 0$, and $n \geq m$. The number $n - m$, which is the difference between the order of the denominator and that of the numerator, is called the **relative degree** of the system.

Definition 8.1

A transfer function $h(p)$ is positive real if

$$Re[h(p)] \geq 0 \quad for\ all \quad Re[p] \geq 0 \tag{16}$$

It is strictly positive real if $h(p - \epsilon)$ is positive real for some $\epsilon > 0$.

9.1 PR and SPR Transfer Functions

➢ **Remark:**

- Condition (16), called the positive real condition, means that $h(p)$ always has a positive (or zero) real part when p has positive (or zero) real part;
- Geometrically, it means that the rational function $h(p)$ maps every point in the closed right half (i.e., including the imaginary axis) of the complex plane into the closed right half of the $h(p)$ plane;
- The concept of positive real functions originally arose in the context of circuit theory, where the transfer function of a passive network (passive in the sense that no energy is generated in the network, e.g., a network consisting of only inductors, resistors, and capacitors) is rational and positive real.

➢ Example: A strictly positive real function

Consider the rational function

$$h(p) = \frac{1}{p+\lambda}$$

which is the transfer function of a first-order system, with $\lambda > 0$. Corresponding to the complex variable $p = \sigma + j\omega$,

$$h(p) = \frac{1}{(\sigma+\lambda)+j\omega} = \frac{\sigma+\lambda-j\omega}{(\sigma+\lambda)^2+\omega^2}$$

Obviously, $Re[h(p)] \geq 0$. Thus, $h(p)$ is a positive real function. In fact, one can easily see that $h(p)$ is strictly positive real.

9.1 PR and SPR Transfer Functions

For higher-order transfer functions, it is often difficult to use the definition directly in order to test the positive realness condition, because this involves checking the positivity condition over the entire right-half of the complex plane.

The following theorem can simplify the algebraic complexity.

Theorem 8.2

A transfer function $h(p)$ is strictly positive real (SPR) if and only if

1. *$h(p)$ is a strictly stable transfer function*
2. *the real part of $h(p)$ is strictly positive along the $j\omega$ axis, i.e.,*

$$\forall \omega \geq 0 \quad Re[h(j\omega)] > 0 \tag{17}$$

The proof of this theorem is presented in the next section, in connection with the so-called passive systems.

9.1 PR and SPR Transfer Functions

The above theorem implies simple necessary conditions for asserting whether a given transfer function $h(p)$ is SPR:

- $h(p)$ is strictly stable;
- The Nyquist plot of $h(j\omega)$ lies entirely in the right half complex plane. Equivalently, the phase shift of the system in response to sinusoidal inputs is always less than 90^o;
- $h(p)$ has relative degree 0 or 1;
- $h(p)$ is strictly minimum-phase (i.e., all its zeros are strictly in the left-half plane).

The first and second conditions are immediate from the theorem. The last two conditions can be derived from the second condition simply by recalling the procedure for constructing Bode or Nyquist frequency response plots (systems with relative degree larger than 1 and non-minimum phase systems have phase shifts larger than 90^o at high frequencies, or, equivalently have parts of the Nyquist plot lying in the left-half plane).

➢ Example: SPR and non-SPR transfer functions

$$h_1(p) = \frac{p-1}{p^2+ap+b}$$
$$h_2(p) = \frac{p+1}{p^2-p+1}$$
$$h_3(p) = \frac{1}{p^2+ap+b}$$
$$h_4(p) = \frac{p+1}{p^2+p+1}$$

The transfer functions h_1, h_2, and h_3 are not SPR, because h_1 is non-minimum phase, h_2 is unstable, and h_3 has relative degree larger than 1.

➢ Example: SPR and non-SPR transfer functions

Is the (strictly stable, minimum-phase, and of relative degree 1) function h_4 actually SPR? We have

$$h_4(j\omega) = \frac{j\omega + 1}{-\omega^2 + j\omega + 1} = \frac{[j\omega + 1][-\omega^2 - j\omega + 1]}{[1-\omega^2]^2 + \omega^2}$$

(where the second equality is obtained by multiplying numerator and denominator by the complex conjugate of the denominator) and thus

$$Re[h_4(j\omega)] = \frac{-\omega^2 + 1 + \omega^2}{[1-\omega^2]^2 + \omega^2} = \frac{1}{[1-\omega^2]^2 + \omega^2}$$

which shows that h_4 is SPR (since it is also strictly stable). Of course, condition (15) can also be checked directly on a computer.

9.1 PR and SPR Transfer Functions

The basic difference between PR and SPR transfer functions is that PR transfer functions may tolerate poles on the $j\omega$ axis, while SPR functions cannot.

➢ **Example:**

Consider the tranfer function of an integrator,

$$h(p) = \frac{1}{p}$$

Its value corresponding to $p = \sigma + j\omega$ is

$$h(p) = \frac{\sigma - j\omega}{\sigma^2 + \omega^2}$$

One easily sees that $h(p)$ is PR but not SPR.

9.1 PR and SPR Transfer Functions

More precisely, we have the following result, which complements Theorem 8.2.

Theorem 8.3

A transfer function $h(p)$ is positive real if, and only if,

1. *$h(p)$ is a stable transfer function;*
2. *The poles of $h(p)$ on the $j\omega$ axis are simple (i.e., distinct) and the associated residues are real and non-negative;*
3. *$Re[h(j\omega)] \geq 0$ for any $w \geq 0$ such that $j\omega$ is not a pole of $h(p)$.*

9.2 The Kalman-Yakubovich Lemma

If a transfer function of a system is SPR, there is an important mathematical property associated with its state-space representation, which is summarized in the celebrated Kalman-Yakubovich (KY) lemma.

Lemma 8.4

(Kalman-Yakubovich) Consider a controllable linear time-invariant system

$$\dot{x} = Ax + bu$$
$$y = c^T x$$

The transfer function

$$h(p) = c^T [pI - A]^{-1} b \tag{18}$$

is strictly positive real if, and only if, there exist positive definite matrices P *and* Q *such that*

$$\begin{aligned} A^T P + PA &= -Q \\ Pb &= c \end{aligned} \tag{19}$$

➢ Remark:

- The KY lemma can be easily extended to PR systems. For such systems, it can be shown that there exist a positive definite matrix P and a positive semi-definite matrix Q such that (17) are verified.
- The usefulness of this result is that it is applicable to transfer functions containing a pure integrator ($1/p$ in the frequency-domain), of which we shall see many in chapter 8 when we study adaptive controller design.
- The Kalman-Yakubovich lemma is also referred to as the positive real lemma.

9.2 The Kalman-Yakubovich Lemma

In the KY lemma, the involved system is required to be asymptotically stable and completely controllable. A modified version of the KY lemma, relaxing the controllability condition, can be stated as follows:

Lemma 8.5

(Meyer-Kalman-Yakubovich) Given a scalar $\gamma \geq 0$, vectors b and c, an asymptotically stable matrix A, and a symmetric positive definite matrix L, if the transfer function

$$H(p) = \frac{\gamma}{2} + c^T [pI - A]^{-1} b$$

is SPR, then there exist a scalar $\epsilon > 0$, a vector q, and a symmetric positive definite matrix P such that

$$\begin{aligned} A^T P + PA &= -qq^T - \epsilon L \\ Pb &= c + \sqrt{\gamma} q \end{aligned} \tag{20}$$

➢ **Remark:**

This lemma is different from Lemma 8.4 in two aspects.

- First, the involved system now has the output equation

$$y = c^T x + \frac{\gamma}{2} u$$

- Second, the system is only required to be stabilizable (but not necessarily controllable).

9.3 Positive Real Transfer Matrices

The concept of positive real transfer function can be generalized to rational positive real matrices. Such generalization is useful for the analysis and design of multi-input-multi-output nonlinear control systems.

Definition 8.6

An $m \times m$ transfer matrix $H(p)$ is called PR if

- $H(p)$ has elements which are analytic for $Re(p) > 0$;
- $H(p) + H^T(p^*)$ is positive semi-definite for $Re(p) > 0$.

where the asterisk $*$ denotes the complex conjugate transpose. $H(p)$ is SPR if $H(p - \epsilon)$ is PR for some $\epsilon > 0$.

The Kalman-Yakubovich lemma and Meyer-Kalman-Yakubovich lemma can be easily extended to positive real transfer matrices.

10 Absolute Stability

The absolute stability concerns systems shown in Figure 4.8. The forward path is a linear time-invariant system, and the feedback part is a memoryless nonlinearity, i.e., a nonlinear static mapping. The equations of such systems can be written as

$$\begin{aligned} \dot{x} &= Ax - b\phi(y) \\ y &= c^T x \end{aligned} \tag{21}$$

where ϕ is some nonlinear function and $G(p) = c^T[pI - A]^{-1}b$. Many systems of practical interest can be represented in this structure.

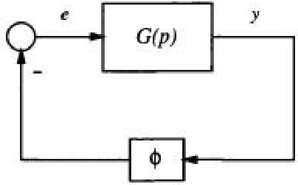


Figure 4.8 : System structure in absolute stability problems

Remark:

- If the feedback path simply contains a constant gain, i.e., if $\phi(y) = \alpha y$, then the stability of the whole system, a linear feedback system, can be simply determined by examining the eigenvalues of the closed-loop system matrix $A - \alpha b c^T$.

- What makes this absolute stability interesting is that it handles a class of systems with an arbitrary nonlinear feedback function ϕ.

In analyzing this kind of system using Lyapunov's direct method, we usually require the nonlinearity to satisfy a so-called sector condition, whose definition is given below.

Definition 8.7

A continuous function ϕ is said to belong to the sector $[k_1,\ k_2]$, if there exists two non-negative numbers k_1 and k_2 such that

$$y \neq 0 \quad \Rightarrow \quad k_1 \leq \frac{\phi(y)}{y} \leq k_2 \tag{22}$$

Geometrically, condition (20) implies that the nonlinear function always lies between the two straight lines $k_1 y$ and $k_2 y$, as shown in Figure 4.9.

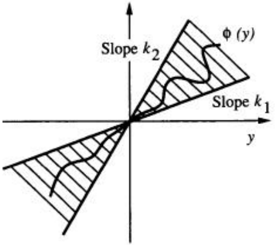


Figure 4.9 : The sector condition (4.49)

Remark: Two properties are implied by equation (20).

- First, it implies that that $\phi(0) = 0$.
- Secondly, it implies that $y\phi(y) \geq 0$, i.e., that the graph of $\phi(y)$ lies in the first and third quadrants.

Note that in many of later discussions, we will consider the special case of $\phi(y)$ belonging to the sector $[0, k]$, i.e., $\exists k > 0$, such that

$$0 \leq \phi(y) \leq ky \tag{23}$$

10.1 The Issue of Absolute Stability

Assume that the nonlinearity $\phi(y)$ is a function belonging to the sector $[k_1, k_2]$, and that the A matrix of the linear subsystem in the forward path is stable (i.e., Hurwitz).

What additional constraints are needed to guarantee the stability of the whole system?

A plausible but incorrect conjecture (M.A. Aizerman, 1949):
if the matrix $[A - bc^T k]$ is stable for all values of k in $[k_1, k_2]$, then the nonlinear system is globally asymptotically stable.

10.2 Popov's Criterion

- If the conjecture were true, it would allow us to deduce the stability of a nonlinear system by simply studying the stability of linear systems. However, several counter-examples show that this conjecure is false.
- After Aizerman, many researchers continued to seek conditions that guarantee the stability of the nonlinear system in Figure 4.8.
- Popov's criterion imposes additional conditions on the linear subsystem, leading to a sufficient condition for asymptotic stability reminiscent of Nyquist's criterion (a necessary and sufficient condition) in linear system analysis.

10.2 Popov's Criterion

A number of versions have been developed for Popov's criterion. The following basic version is fairly simple and useful.

Theorem 8.8

(Popov's Criterion) If the system described by (19) satisfies the conditions

- *the matrix A is Hurwitz (i.e., has all its eigenvalues strictly in the left-half plane) and the pair $[A, b]$ is controllable;*
- *the nonlinearity ϕ belongs to the sector $[0, k]$;*
- *there exists a strictly positive number α such that*

$$\forall \omega \geq 0 \qquad Re[(1 + j\alpha w)G(j\omega)] + \frac{1}{k} \geq \epsilon \tag{24}$$

for an arbitrary small $\epsilon > 0$, then the point 0 is globally asymptotically stable.

10.2 Popov's Criterion

Inequality (22) is called Popov's inequality. The criterion can be proven by constructing a Lyapunov function candidate based on the KY lemma.

Let us note the main features of Popov's criterion:

- It only applies to autonomous systems.
- It is restricted to a single memoryless nonlinearity.
- The stability of the nonlinear system may be determined by examining the frequency-response functions of a linear subsystem, without the need of searching for explicit Lyapunov functions.
- It only gives a sufficient condition.

10.2 Popov's Criterion

The criterion is most easy to apply by using its graphical interpretation. Let

$$G(j\omega) = G_1(\omega) + jG_2(\omega)$$

Then the expression (22) can be written

$$G_1(\omega) - \alpha w G_2(\omega) + \frac{1}{k} \geq \epsilon \tag{25}$$

Now let us construct associated transfer function $W(j\omega)$, with the same real part as $G(j\omega)$, but an imaginary part equal to $wIm(G(j\omega))$, i.e.,

$$W(j\omega) = x + jy = G_1(\omega) + j\omega G_2(\omega)$$

Then (23) implies that the nonlinear system is guaranteed to be globally asymptotically stable if, in the complex plane having x and y as coordinates, the polar plot of $W(j\omega)$ is (uniformly) below the line $x - \alpha y + (1/k) = 0$ (Figure 4.10).

10.2 Popov's Criterion

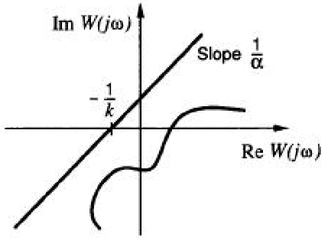


Figure 4.10 : A Popov plot

The polar plot of W is called a Popov plot. One easily sees the similarity of this criterion to the Nyquist criterion for linear systems.

- In the Nyquist criterion, the stability of a linear feedback system is determined by examining the position of the polar plot of $G(j\omega)$ relative to the point (0, -1);
- In the Popov criterion, the stability of a nonlinear feedback system is determined by checking the position of the associated transfer function $W(j\omega)$ with respect to a line.

➢ Example

Let us determine the stability of a nonlinear system of the form (19) where the linear subsystem is defined by

$$G(j\omega) = \frac{p+3}{p^2+7p+10}$$

and the nonlinearity satisfies condition (21).
First, the linear subsystem is strictly stable, because its pole are -2 and -5. Moreover, any dimension 2 realization of $G(j\omega)$ is controllable, because there is no pole-zero cancelation. Let us now check the Popov inequality. The frequency response function $G(j\omega)$ is

$$G(j\omega) = \frac{j\omega+3}{-\omega^2+7wj+10}$$

➢ Example

Therefore,

$$G_1(j\omega) = \frac{4\omega^2 + 30}{\omega^4 + 29\omega^2 + 100}$$
$$G_2(j\omega) = \frac{-w(\omega^2 + 11)}{\omega^4 + 29\omega^2 + 100}$$

Substituting the above into (23) leads to

$$4\omega^2 + 30 + \alpha\omega^2(\omega^2 + 11) + (\frac{1}{k} - \epsilon)(\omega^4 + 29\omega^2 + 100) > 0$$

It is clear that this inequality can be satisfied by any strictly positive number α, and any strictly positive number k, i.e., $0 < k < \infty$. Thus the nonlinear system is globally asymptotically stable as long as the nonlinearity belongs to the first and third quadrants.

10.3 The Circle Criterion

Theorem 8.9

(Circle Theorem) If the system (19) satisfies the conditions

- *the matrix A has no eigenvalue on the $j\omega$ axis, and has ρ eigenvalues strictly in the right-half plane;*
- *the nonlinearity ϕ belongs to the sector $[k_1, k_2]$;*
- *one of the following is true*
    - $0 < k_1 \leq k_2$, *the Nyquist plot of $G(j\omega)$ does not enter the disk $\mathbf{D}(k_1, k_2)$ and encircles it ρ times counter-clockwise;*
    - $0 = k_1 < k_2$, *and the Nyquist plot of $G(j\omega)$ stays in the half-plane $Re\ p > -1/k_2$;*
    - $k_1 < 0 < k_2$, *and the Nyquist plot of $G(j\omega)$ stays in the interior of the disk $\mathbf{D}(k_1, k_2)$;*
    - $k_1 < k_2 < 0$, *the Nyquist plot of $-G(j\omega)$ does not enter the disk $\mathbf{D}(-k_1, -k_2)$ and encircles it ρ times counter-clockwise;*

then the equilibrium point 0 of the system is globally asymptotically stable.

10.3 The Circle Criterion

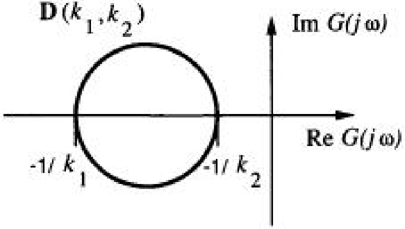


Figure 4.11 : The circle criterion

Essentially, the critical point $-1/k$ in Nyquist's criterion is replaced in the circle criterion by the circle of Figure 4.11 (which tends towards the point $-1/k_1$ as k_2 tends to k_1, i.e., as the conic sector gets thinner). No that the circle criterion states sufficient but not necessary conditions.
The circle criterion can be extended to non-autonomous systems.

11 Establishing Boundedness of Signals

In the stability analysis or convergence analysis of nonlinear systems, a frequently encountered problem is that of establishing the boundedness of certain signals.

For instance, in order to use Barbalat's lemma, one has to show the uniform continuity of $\dot{f}$, which can be most conveniently shown by proving the boundedness of $\ddot{f}$. Similarly, in studying the effects of disturbances, it is also desirable to prove the boundedness of system signals in the presence of disturbances.

In this section, we provide two useful results for such purposes.

11.1 The Bellman-Gronwall Lemma

In system analysis, one can often manipulate the signal relations into an integral inequality of the form

$$y(t) \leq \int_0^t a(\tau)y(\tau)d\tau + b(t) \tag{26}$$

where $y(t)$, the variable of concern, appears on both sides of the inequality. The problem is to gain an explicit bound on the magnitude of y from the above inequality. The Bellman-Gronwall lemma can be used for this purpose.

11.1 The Bellman-Gronwall Lemma

Lemma 8.10

(Bellman-Gronwall) Let a variable $y(t)$ satisfy (26), where $a(t)$, $b(t)$ are known real functions with $a(t)$ nonnegative. Then

$$y(t) \leq \int_0^t a(\tau)b(\tau)\exp[\int_\tau^t a(r)dr]d\tau + b(t) \tag{27}$$

If $b(t)$ is differentiable, then

$$y(t) \leq b(0)\exp[\int_0^t a(\tau)d\tau] + \int_0^t \dot{b}(\tau)\exp[\int_\tau^t a(r)dr]d\tau \tag{28}$$

In particular, if $b(t)$ is a constant, we simply have

$$y(t) \leq b(0)\exp[\int_0^t a(\tau)d\tau] \tag{29}$$

11.1 The Bellman-Gronwall Lemma

Proof: The proof is based on defining a new variable and transforming the integral inequality into a differential equation, which can be easily solved. Let

$$v(t) = \int_0^t a(\tau)y(\tau)d\tau \tag{30}$$

Then differentiation of v and use of (26), i.e., $y \leq v + b$, leads to

$$\dot{v}(t) = a(t)y(t) \leq a(t)v(t) + a(t)b(t)$$

Let

$$s(t) = a(t)y(t) - a(t)v(t) - a(t)b(t)$$

which is a non-positive function. Then $v(t)$ satisfies

$$\dot{v}(t) - a(t)v(t) = a(t)b(t) + s(t) \tag{31}$$

11.1 The Bellman-Gronwall Lemma

(31) is a first order linear ODE. Consider the identity

$$\begin{aligned}(v(\tau)\exp[\int_0^\tau -a(r)dr])' &= \exp[\int_0^\tau -a(r)dr](\dot{v}(\tau) - a(\tau)v(\tau)) \\ &= \exp[\int_0^\tau -a(r)dr](a(\tau)b(\tau) + s(\tau))\end{aligned} \tag{32}$$

Integrating both sides of (32) from $\tau = 0$ to $\tau = t$ with the initial condition $v(0) = 0$ yields

$$v(t)\exp[\int_0^t -a(r)dr] = \int_0^t \exp[\int_0^\tau -a(r)dr][a(\tau)b(\tau) + s(\tau)]d\tau \tag{33}$$

which gives

$$v(t) = \int_0^t \exp[\int_\tau^t a(r)dr][a(\tau)b(\tau) + s(\tau)]d\tau \tag{34}$$

Since $s(.)$ is a non-positive function,

$$v(t) \leq \int_0^t \exp[\int_\tau^t a(r)dr]a(\tau)b(\tau)d\tau \tag{35}$$

11.1 The Bellman-Gronwall Lemma

This together with the definition of v and the inequality (26), leads to

$$y(t) \leq \int_0^t \exp[\int_\tau^t a(r)dr]a(\tau)b(\tau)d\tau + b(t) \tag{36}$$

If $b(t)$ is differentiable, we obtain, by partial integration

$$\begin{aligned}
&\int_0^t \exp[\int_\tau^t a(r)dr]a(\tau)b(\tau)d\tau \\
&= -\int_0^t b(\tau)d\exp[\int_\tau^t a(r)dr] \\
&= -b(\tau)\exp[\int_\tau^t a(r)dr] \,|_{\tau=0}^{\tau=t} + \int_0^t \dot{b}(\tau)\exp[\int_\tau^t a(r)dr]d\tau \\
&= -b(t) + b(0)\exp[\int_0^t a(r)dr] + \int_0^t \dot{b}(\tau)\exp[\int_\tau^t a(r)dr]d\tau
\end{aligned} \tag{37}$$