Chapter 5 Nonlinear control systems design

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Outline

Given a nonlinear control system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \qquad x(0) = x_0 \\ y &= h(x), \end{aligned} \tag{5.1}$$

find a feedback control law so that the closed-loop system has some desirable behaviors.

System (5.1) is linear in the input u and is called affine system. Most practical systems can be put in the form (5.1).

1.2 Two fundamental control problems

1.2.1 Stabilization problem

Given (5.1), find a state feedback control law

$$u = k(x), \qquad k(0) = 0 \tag{5.2}$$

such that the E.P. of the closed-loop system

$$\dot{x} = f(x) + g(x)k(x)$$

at $x = 0$ is (globally) asymptotically stable.

Remarks:
(a) Lyapunov linearization cannot guarantee global A.S. Thus, $k(x)$ is obtained based on Lyapunov's direct method. $k(x)$ is usually nonlinear.

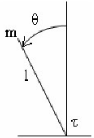
(b) **Example 1:**
An inverted pendulum

$$J\ddot{\theta} - mgl\sin\theta = \tau \quad \Rightarrow \quad \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{u + mgl\sin x_1}{J} \end{array}$$

where $u = \tau,\ x_1 = \theta$ and $x_2 = \dot{\theta}$.
Let

$$u = J(-k_d x_2 - k_p x_1) - mgl\sin x_1 \quad \Rightarrow \quad \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_d x_2 - k_p x_1. \end{array}$$

Therefore, the E.P. of the closed-loop system is G.A.S. if $k_d > 0$, $k_p > 0$.



Note: The closed-loop system is a linear system.

1.2.2 Tracking problem

Given (5.1) and a (sufficiently smooth) reference input (desirable trajectory) $y_d(t)$, find a control law

$$u = k(x, y_d(t), ..., y_d^{(\rho)}(t)) \tag{5.3}$$

where $k(0, 0, ..., 0) = 0$, and ρ is some integer such that
(i) the E.P. of

$$\dot{x} = f(x) + g(x)k(x, 0, ..., 0)$$

is (globally) A.S. and
(ii)

$$\lim_{t\to\infty} (y(t) - y_d(t)) = 0.$$

Remark: Condition (i) guarantees that the state of the closed-loop system is bounded for sufficiently small reference input and initial state (Total Stability).

1.2.2 Tracking problem (cont.)

Example 2: Consider the inverted pendulum with $y_d(t) = \sin \omega t$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{u + mgl \sin x_1}{J} \\ y &= x_1. \end{aligned}$$

Letting $e = y - y_d(t)$ and

$$u = J(\ddot{y}_d - k_d \dot{e} - k_p e) - mgl \sin x_1$$

gives

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{y}_d(t) - k_d \dot{e} - k_p e \\ y &= x_1. \end{aligned}$$

Therefore e satisfies

$$\ddot{e} + k_d \dot{e} + k_p e = 0.$$

Since $s^2 + k_d s + k_p$ is stable when $k_d > 0$, $k_p > 0$, $\lim_{t \to \infty} e(t) = 0$.

1.2.2 Tracking problem (cont.)

Remarks:

(a) Problem 1 is the special case of problem 2 when $y_d(t) = 0$.

(b) The two examples have shown that it is possible to use nonlinear control laws to obtain a closed-loop system with a desirable behavior. How to systematically obtain the control laws is the topic of the nonlinear control system design.

1.3 Other desirable behaviors

- Disturbance rejection / Attenuation
- Speed of response
- Control effort
- Robustness
- Cost...

1.4 Approaches

- Backstepping for strictly feedback system.
- Input-output linearization for minimum phase systems.
- Variable structure control / sliding control for uncertain minimum phase systems.
- Adaptive control for parametric uncertain systems.
- For nonlinear systems, not a single method works for all systems.
- Different methods have been developed to handle different classes of nonlinear systems with various special forms.

2 Stabilization by backstepping 2.1 Introduction

- Backstepping is a systematic nonlinear control design method devoloped in the 1990s.
- It can be used to solve the stabilization problem and the asymptotic tracking problem for nonlinear systems in strict feedback form to be described shortly.
- In what follows, we will focus on the stabilization problem.
- The exposition of the rest of this chapter is mainly based on

  Hassan K. Khalil, "Nonlinear Systems," Third Edition, Prentice Hall, 2002.

2.2 Review of Lyapunov's second method

Given $\dot{x} = f(x)$, $f(0) = 0$, $x \in \Re^n$, $x = 0$ is G.A.S. if there exists a radially unbounded P.D. function $V(x)$ such that

$$\dot{V} = \frac{\partial V(x)}{\partial x} f(x)$$

is N.D. for all $x \in \Re^n$.

2.2 Review of Lyapunov's second method (cont.)

Consider

$$\dot{x} = f(x) + g(x)u, \qquad x \in \Re^n, \qquad u \in \Re^1.$$

Want to find $u = k(x)$, $k(0) = 0$ such that there exists a radially unbounded global Lyapunov function $V(x)$ for

$$\dot{x} = f_c(x) = f(x) + g(x)k(x)$$

satisfying

$$\frac{\partial V(x)}{\partial x}\left(f(x) + g(x)k(x)\right) < 0, \qquad \forall\, x \neq 0.$$

Remark: Unlike the stability analysis of a nonlinear system $\dot{x} = f(x)$ where a single function $V(x)$ needs to be found, the stabilization problem studied here involves two functions $k(x)$ and $V(x)$ for $\dot{x} = f(x) + g(x)u$.

2.3 systems in strict-feedback form or lower triangular form

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
&\ldots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, ..., x_{n-1}) + g_{n-1}(x_1, ..., x_{n-1})x_n \\
\dot{x}_n &= f_n(x_1, ..., x_n) + g_n(x_1, ..., x_n)u.
\end{aligned}
$$

where $x_i \in \Re$, $i = 1, \cdots, n$, and $u \in \Re$.

2.3 systems in strict-feedback form or lower triangular form (cont.)

The system can be put in the standard form (5.1) with

$$
f(x) = \begin{bmatrix} f_1(x_1) + g_1(x_1)x_2 \\ f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dots \\ f_{n-1}(x_1, ..., x_{n-1}) + g_{n-1}(x_1, ..., x_{n-1})x_n \\ f_n(x_1, ..., x_n) \end{bmatrix},
$$

$$
g(x) = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ g_n(x_1, ..., x_n) \end{bmatrix}.
$$

2.4 Backstepping

2.4.1 Control Lyapunov function (C.L.F.)

Consider

$$\dot{\eta} = f(\eta) + g(\eta)u, \quad \eta \in \Re^n, \quad u \in \Re^1 \tag{5.4}$$

where $f(0) = 0$, and $f(\eta)$ and $g(\eta)$ are smooth functions of $\eta \in \Re^n$.

A P.D. function $V(\eta)$ is a control Lyapunov function of (5.4) with respect to state feedback $u = \phi(\eta)$ with $\phi(0) = 0$ if

$$\dot{V}(\eta)\Big|_{u=\phi(\eta)} = \frac{\partial V(\eta)}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] < 0, \qquad \forall\ \eta \neq 0 \tag{5.5}$$

Remarks

(a) If (5.4) has a C.L.F. w.r.t. $u = \phi(\eta)$, then the E.P. of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) \tag{5.6}$$

at $\eta = 0$ is A.S. Moreover, if $V(\eta)$ is radially unbounded, then the E.P. is G.A.S.

(b) Existence of C.L.F. for one dimensional system

If $n = 1$, $g(\eta) \neq 0$, $\forall\ \eta$, then $V(\eta) = \frac{1}{2}\eta^2$ is a C.L.F. for (5.4) w.r.t

$$u = \phi(\eta) = \frac{-\alpha\eta - f(\eta)}{g(\eta)}, \qquad \forall\ \alpha > 0 \tag{5.7}$$

In fact, the closed-loop system is

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) = -\alpha\eta$$

and

$$\dot{V}(\eta) = \eta\dot{\eta} = -\alpha\eta^2 < 0,\ \ \eta \neq 0.$$

But $g(\eta) \neq 0$ is not necessary. For example, consider

$$\dot{\eta} = \eta u$$

It can be verified that $V(\eta) = \frac{1}{2}\eta^2$ is a C.L.F. for $\dot{\eta} = \eta u$ w.r.t. $\phi(\eta) = -\eta^2$ since $\dot{V}(\eta)\Big|_{u=\phi(\eta)} = \frac{\partial V(\eta)}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] = -\eta^4$.
The C.L.F. defined here is not unique and it depends on the control function ϕ. It is possible to define C.L.F. uniquely solely based on the given system (5.4).

Lemma 1

Consider

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\zeta, \quad \eta \in \Re^n && (5.8) \\ \dot{\zeta} &= u \qquad\qquad \zeta \in \Re^1, \ u \in \Re^1 && (5.9) \end{aligned}$$

where $f(0) = 0$, $f(\eta)$ and $g(\eta)$ are smooth.

If (5.8) has a control Luapunov function $V(\eta)$ w.r.t. state feedback $\zeta = \phi(\eta)$, then the system described by (5.8) and (5.9) has a control Lyapunov function

$$V_a(\eta, \zeta) = V(\eta) + \frac{1}{2}(\zeta - \phi(\eta))^2 \tag{5.10}$$

w.r.t. state feedback

$$\begin{aligned} u &= \phi_a(\eta, \zeta) \\ &= \frac{\partial \phi(\eta)}{\partial \eta}[f(\eta) + g(\eta)\zeta] - \frac{\partial V(\eta)}{\partial \eta} g(\eta) - k(\zeta - \phi(\eta)) && (5.11) \end{aligned}$$

where $k > 0$.

2.4.2 Lemma 1 (cont.)

Thus the E.P. of the closed-loop system

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\zeta \\ \dot{\zeta} &= \phi_a(\eta, \zeta) \end{aligned} \tag{5.12}$$

is A.S. Moreover, if $V(\eta)$ is radially unbounded, so is $V_a(\eta, \zeta)$. Thus, the E.P. of (5.12) is G.A.S.

2.4.3 Example 1

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 \quad (5.13)$$

$$\dot{x}_2 = u \quad (5.14)$$

which is in the form of (5.8) and (5.9) with $\eta = x_1$ and $\zeta = x_2$

$$f(\eta) = \eta^2 - \eta^3, \qquad g(\eta) = 1.$$

Consider (5.13). By (5.7), under the control $\phi(x_1) = -x_1 - x_1^2 + x_1^3$, the closed-loop system has a control Lyapunov function

$$V(x_1) = \frac{1}{2}x_1^2$$

(In fact, under $x_2 = \phi(x_1)$, (5.13) becomes $\dot{x}_1 = -x_1$).

2.4.3 Example 1 (cont.)

By Lemma 1, (5.13) and (5.14) have a control Lyapunov function

$$\begin{aligned} V_a(x_1, x_2) &= V(x_1) + \frac{1}{2}(x_2 - \phi(x_1))^2 \\ &= \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2 - x_1^3)^2 \end{aligned} \tag{5.15}$$

w.r.t.

$$\begin{aligned} u &= \phi_a(x_1, x_2) = \frac{\partial \phi(x_1)}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial V(x_1)}{\partial x_1} - k(x_2 - \phi(x_1)) \\ &= (-1 - 2x_1 + 3x_1^2)(x_1^2 - x_1^3 + x_2) - x_1 - k(x_2 + x_1 + x_1^2 - x_1^3) \end{aligned}$$

where $k > 0$. Therefore, the E.P. of

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= \phi_a(x_1, x_2) \end{aligned}$$

is A.S. (In fact, G.A.S. since $V(x_1)$ is radially unbounded).

2.4.4 Proof of Lemma 1

Proof: Since $V(\eta)$ is C.L.F. for (5.8) w.r.t. $\zeta = \phi(\eta)$, we have

$$\dot{V}(\eta)\Big|_{\zeta=\phi(\eta)} = \frac{\partial V(\eta)}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] < 0, \qquad \forall\ \eta \neq 0$$

Let

$$V_a(\eta, \zeta) = V(\eta) + \frac{1}{2}(\zeta - \phi(\eta))^2,$$

which is globally positive definite.

2.4.4 Proof of Lemma 1 (cont.)

Then

$$
\begin{aligned}
\dot{V}_a(\eta,\zeta) &= \frac{\partial V_a(\eta,\zeta)}{\partial \eta}\dot{\eta} + \frac{\partial V_a(\eta,\zeta)}{\partial \zeta}\dot{\zeta} \\
&= \frac{\partial V(\eta)}{\partial \eta}[f(\eta)+g(\eta)\zeta] - (\zeta-\phi(\eta))\frac{\partial \phi(\eta)}{\partial \eta}[f(\eta)+g(\eta)\zeta] \\
&\quad +(\zeta-\phi(\eta))u \\
&= \frac{\partial V(\eta)}{\partial \eta}[f(\eta)+g(\eta)(\zeta-\phi(\eta)+\phi(\eta))] \\
&\quad -(\zeta-\phi(\eta))\left[\frac{\partial \phi(\eta)}{\partial \eta}\{f(\eta)+g(\eta)\zeta\}-u\right] \\
&= \frac{\partial V(\eta)}{\partial \eta}[f(\eta)+g(\eta)\phi(\eta)] \\
&\quad -(\zeta-\phi(\eta))\left[\frac{\partial \phi(\eta)}{\partial \eta}\{f(\eta)+g(\eta)\zeta\}-u-\frac{\partial V(\eta)}{\partial \eta}g(\eta)\right]
\end{aligned}
$$

2.4.4 Proof of Lemma 1 (cont.)

Choosing u such that

$$\left[\frac{\partial \phi(\eta)}{\partial \eta}\{f(\eta)+g(\eta)\zeta\} - u - \frac{\partial V(\eta)}{\partial \eta}g(\eta)\right] = -k(\zeta - \phi(\eta))$$

for some $k > 0$ gives

$$u = \phi_a(\eta,\zeta) = \frac{\partial \phi(\eta)}{\partial \eta}[f(\eta)+g(\eta)\zeta] - \frac{\partial V(\eta)}{\partial \eta}g(\eta) - k(\zeta - \phi(\eta))$$

. Then, $\forall\ (\eta,\zeta) \neq (0,0)$,

$$\dot{V}_a(\eta,\zeta) = \frac{\partial V(\eta)}{\partial \eta}[f(\eta)+g(\eta)\phi(\eta)] - k(\zeta - \phi(\eta))^2 < 0$$

Therefore, $V_a(\eta,\zeta)$ is a C.L.F. of (5.8) and (5.9) w.r.t. $\phi_a(\eta,\zeta)$. Moreover, if $V(\eta)$ is radially unbounded, so is $V_a(\eta,\zeta)$.

Thus, under $u = \phi_a(\eta, \zeta)$, the E.P. of

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\zeta \\ \dot{\zeta} &= \phi_a(\eta, \zeta) \end{aligned}$$

is G.A.S.

2.5 Applying Lemma 1 to dimension 3 system

Example 2

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases} \tag{5.16}$$

Let

$$\eta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(\eta) = \begin{bmatrix} x_1^2 - x_1^3 + x_2 \\ 0 \end{bmatrix}, \quad \zeta = x_3, \quad g(\eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the system is in the form

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)\zeta \\ \dot{\zeta} &= u \end{aligned} \tag{5.17}$$

2.5 Applying Lemma 1 to dimension 3 system (cont.)

To find a C.L.F. for system (5.17), all we need to do is to find a C.L.F. for the η subsystem

$$\dot{\eta} = f(\eta) + g(\eta)\zeta \tag{5.18}$$

or

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \end{cases} \tag{5.19}$$

viewing $\zeta = x_3$ as a control.

2.5 Applying Lemma 1 to dimension 3 system (cont.)

But, by Example 1, we have already known

$$V_a(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2 - x_1^3)^2 \tag{5.20}$$

is a C.L.F. for (5.19) w.r.t.

$$\begin{aligned} \phi_a(x_1, x_2) &= (-1 - 2x_1 + 3x_1^2)(x_1^2 - x_1^3 + x_2) \\ &\quad -x_1 - k_1(x_2 + x_1 + x_1^2 - x_1^3), \quad k_1 > 0 \end{aligned} \tag{5.21}$$

2.5 Applying Lemma 1 to dimension 3 system (cont.)

Thus, by Lemma 1, a C.L.F. for (5.16) is

$$V(x_1, x_2, x_3) = V_a(x_1, x_2) + \frac{1}{2}(x_3 - \phi_a(x_1, x_2))^2 \tag{5.22}$$

w.r.t.

$$\begin{aligned} u &= \phi(x_1, x_2, x_3) \\ &= \frac{\partial \phi_a(x_1, x_2)}{\partial(x_1, x_2)}[f(x_1, x_2) + g(x_1, x_2)x_3] \\ &\quad - \frac{\partial V_a(x_1, x_2)}{\partial(x_1, x_2)} g(x_1, x_2) - k_2(x_3 - \phi_a(x_1, x_2)) \end{aligned} \tag{5.23}$$

where $k_2 > 0$.

2.6 Extension of Lemma 1

Consider

$$\dot{\eta} = f(\eta) + g(\eta)\zeta, \qquad \eta \in \Re^n \tag{5.24}$$

$$\dot{\zeta} = f_a(\eta,\zeta) + g_a(\eta,\zeta)u, \qquad \zeta \in \Re^1, \quad u \in \Re^1 \tag{5.25}$$

where $f_a(0,0) = 0$ and $g_a(\eta,\zeta) \neq 0$, $\forall\ \eta,\ \zeta$.
Using the input transformation

$$u = \frac{1}{g_a(\eta,\zeta)}[u_a - f_a(\eta,\zeta)] \tag{5.26}$$

gives

$$\dot{\eta} = f(\eta) + g(\eta)\zeta \tag{5.27}$$

$$\dot{\zeta} = u_a \tag{5.28}$$

2.6 Extension of Lemma 1 (cont.)

Therefore, by Lemma 1, if (5.27) has a C.L.F. $V(\eta)$ w.r.t. state feedback $\zeta = \phi(\eta)$, then (5.27) and (5.28) have a C.L.F.

$$V_a(\eta, \zeta) = V(\eta) + \frac{1}{2}(\zeta - \phi(\eta))^2 \tag{5.29}$$

w.r.t.

$$u_a = \phi_a(\zeta, \eta) = \frac{\partial \phi(\eta)}{\partial \eta}[f(\eta) + g(\eta)\zeta] - \frac{\partial V(\eta)}{\partial \eta} g(\eta) - k(\zeta - \phi(\eta))$$

where $k > 0$. Therefore, (5.24) and (5.25) have the same C.L.F. w.r.t.

$$u = \frac{1}{g_a(\eta, \zeta)}[\phi_a(\eta, \zeta) - f_a(\eta, \zeta)] \tag{5.30}$$

Example 3

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_2 \sin x_1 + e^{-x_3} u \end{cases} \tag{5.31}$$

Letting $u = e^{x_3}(u_a - x_2 \sin x_1)$ gives

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_a \end{aligned} \tag{5.32}$$

From previous example, it is known that a C.L.F. for (5.32) is given by $V(x_1, x_2, x_3)$ described by (5.22) w.r.t. $u_a = \phi(x_1, x_2, x_3)$ described by (5.23).

According to (5.30), a C.L.F. for (5.31) is also given by $V(x_1, x_2, x_3)$ w.r.t.

$$u = \frac{1}{e^{-x_3}} [\phi(x_1, x_2, x_3) - x_2 \sin x_1] \tag{5.33}$$

2.7 General case

$$\begin{cases} \dot{x} = f_0(x) + g_0(x)z_1 \\ \dot{z}_1 = f_1(x, z_1) + g_1(x, z_1)z_2 \\ \dot{z}_2 = f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\ \ldots \\ \dot{z}_{l-1} = f_{l-1}(x, z_1, ..., z_{l-1}) + g_{l-1}(x, z_1, ..., z_{l-1})z_l \\ \dot{z}_l = f_l(x, z_1, ..., z_l) + g_l(x, z_1, ..., z_l)u \quad (l = 1 \Rightarrow (5.24)\&(5.25)) \end{cases} \tag{5.34}$$

where

$$x \in \Re^n, \qquad z_1, ..., z_l \in \Re^1,$$
$$f_0(0) = f_1(0) = ... = f_l(0) = 0,$$
$$g_i(x, z_1, ..., z_i) \neq 0, \qquad \text{for } 0 \leq i \leq l$$

2.7 General case (cont.)

Assume $V_0(x)$ is C.L.F. for

$$\dot{x} = f_0(x) + g_0(x)z_1 \tag{5.35}$$

w.r.t. $z_1 = \phi_0(x)$. Then a feedback control law

$$u = \phi(x, z_1, ..., z_l)$$

for stabilizing (5.34) can be recursively constructed as follows:

Step 1

Applying the extension of Lemma 1 to

$$\begin{cases} \dot{x} = f_0(x) + g_0(x)z_1 \\ \dot{z}_1 = f_1(x_1, z_1) + g_1(x_1, z_1)z_2 \end{cases} \tag{5.36}$$

and using (5.29) and (5.30) to obtain a C.L.F. $V_1(x, z_1)$ for (5.36) as follows

$$V_1(x, z_1) = V_0(x) + \frac{1}{2}(z_1 - \phi_0(x))^2 \tag{5.37}$$

w.r.t.

$$\begin{aligned} z_2 &= \phi_1(x, z_1) \\ &= \frac{1}{g_1(x, z_1)}[\frac{\partial \phi_0(x)}{\partial x}(f_0(x) + g_0(x)z_1) - \frac{\partial V_0(x)}{\partial x} g_0(x) \\ &\quad -k_1(z_1 - \phi_0(x)) - f_1(x, z_1)], \qquad k_1 > 0 \end{aligned} \tag{5.38}$$

2.7 General case (cont.)

Step 2

Applying the extension of Lemma 1 again to

$$\begin{cases} \dot{x} = f_0(x) + g_0(x) z_1 \\ \dot{z}_1 = f_1(x_1, z_1) + g_1(x_1, z_1) z_2 \\ \dot{z}_2 = f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \end{cases} \tag{5.39}$$

with

$$\begin{aligned} \eta &= \begin{bmatrix} x \\ z_1 \end{bmatrix}, \quad \zeta = z_2, \quad u = z_3, \\ f(\eta) &= \begin{bmatrix} f_0(x) + g_0(x) z_1 \\ f_1(x_1, z_1) \end{bmatrix}, \quad g(\eta) = \begin{bmatrix} 0 \\ g_1(x_1, z_1) \end{bmatrix} \\ f_a(\eta, \zeta) &= f_2(x, z_1, z_2), \quad g_a(\eta, \zeta) = g_2(x, z_1, z_2) \end{aligned}$$

2.7 General case (cont.)

and using (5.29) and (5.30) again to obtain a C.L.F. $V_2(x, z_1, z_2)$ for (5.39) as follows

$$V_2(x, z_1, z_2) = V_1(x, z_1) + \frac{1}{2}(z_2 - \phi_1(x, z_1))^2 \tag{5.40}$$

w.r.t.

$$\begin{aligned} z_3 &= \phi_2(x, z_1, z_2) \\ &= \frac{1}{g_2(x, z_1, z_2)} \frac{\partial \phi_1(x, z_1)}{(x, z_1)} \begin{bmatrix} f_0(x) + g_0(x) z_1 \\ f_1(x_1, z_1) + g_1(x_1, z_1) z_2 \end{bmatrix} \\ &\quad - \frac{1}{g_2(x, z_1, z_2)} \frac{\partial V_1(x, z_1)}{\partial (x, z_1)} \begin{bmatrix} 0 \\ g_1(x_1, z_1) \end{bmatrix} \\ &\quad - \frac{1}{g_2(x, z_1, z_2)} k_2 (z_2 - \phi_1(x, z_1)) \\ &\quad - \frac{1}{g_2(x, z_1, z_2)} f_2(x, z_1, z_2), \qquad k_2 > 0 \end{aligned} \tag{5.41}$$

2.7 General case (cont.)

Step I

Assume $V_{l-1}(x, z_1, ..., z_{l-1})$ is a C.L.F. for

$$\begin{cases} \dot{x} = f_0(x) + g_0(x)z_1 \\ \dot{z}_1 = f_1(x, z_1) + g_1(x, z_1)z_2 \\ \dot{z}_2 = f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\ \dots \\ \dot{z}_{l-1} = f_{l-1}(x, z_1, ..., z_{l-1}) + g_{l-1}(x, z_1, ..., z_{l-1})z_l \end{cases} \tag{5.42}$$

w.r.t.

$$z_l = \phi_{l-1}(x, z_1, ..., z_{l-1}) \tag{5.43}$$

2.7 General case (cont.)

Let

$$\begin{aligned}
\eta &= \begin{bmatrix} x \\ z_1 \\ \ldots \\ z_{l-1} \end{bmatrix}, \quad \zeta = z_l \\
f(\eta) &= \begin{bmatrix} f_0(x) + g_0(x) z_1 \\ f_1(x, z_1) + g_1(x, z_1) z_3 \\ \ldots \\ f_{l-1}(x, z_1, \ldots, z_{l-1}) \end{bmatrix}, g(\eta) = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ g_{l-1}(x, z_1, \ldots, z_{l-1}) \end{bmatrix} \\
f_a(\eta, \zeta) &= f_l(x, z_1, \ldots, z_l), \qquad g_a(\eta, \zeta) = g_l(x, z_1, \ldots, z_l).
\end{aligned}$$

2.7 General case (cont.)

Then using (5.29) and (5.30) gives a C.L.F. for (5.34)

$$\begin{aligned} V(x, z_1, ..., z_{l-1}, z_l) &= V_{l-1}(x, z_1, ..., z_{l-1}) \\ &\quad + \frac{1}{2}(z_l - \phi_{l-1}(x, z_1, ..., z_{l-1}))^2 \end{aligned}$$

w.r.t. the control

$$\begin{aligned} u &= \frac{1}{g_l(x, z_1, ..., z_l)} \frac{\partial \phi_{l-1}(x, z_1, ..., z_{l-1})}{\partial u}(f(u) + g(u)z_l) \\ &\quad - \frac{1}{g_l(x, z_1, ..., z_l)} \frac{\partial V_{l-1}(x, z_1, ..., z_{l-1})}{\partial u} g(u) \\ &\quad - \frac{1}{g_l(x, z_1, ..., z_l)} k_l(z_l - \phi_{l-1}(x, z_1, ..., z_{l-1})) \\ &\quad - \frac{1}{g_l(x, z_1, ..., z_l)} f_l(x, z_1, ..., z_l), \qquad k_l > 0 \end{aligned}$$

2.7 General case (cont.)

Remarks:

(i) The key to the success of the above procedure is to be able to find $V_0(x)$ and $\phi_0(x)$.

(ii) $V_0(x)$ and $\phi_0(x)$ are not unique.

(iii) $k_1, k_2, ...k_l$ are design parameters.