Chapter 6 Input-Output linearization

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Outline

1 Introduction

Input-output linearization is one of the fundamental nonlinear control methods.

It can be used to solve the asymptotic tracking problem for a class of nonlinear systems of the following form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \qquad x \in \Re^n, \qquad u \in \Re^m \\ y &= h(x) \qquad\qquad\qquad y \in \Re^p. \end{aligned} \tag{6.1}$$

It is also the basis of some other more advanced nonlinear control methods such as sliding control and adaptive control.

2 Asymptotic tracking by input-output linearization

Given

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad (6.2)$$

and a reference input (desired output) $y_d(t)$, find a state feedback control

$$u = k(x, y_d(t), ..., y_d^{(\rho)}(t)) \quad (6.3)$$

where $k(0, .., 0) = 0$ and ρ is an integer called relative degree of (6.2) s.t.
(i) The E.P. of $\dot{x} = f(x) + g(x)k(x, 0, ..., 0)$ is asymptotically stable.
(ii)

$$\lim_{t\to\infty}(y(t) - y_d(t)) = 0$$

Remark: Condition (i) guarantees (local) BIBO stability (Total Stability).

2 Asymptotic tracking by input-output linearization (cont.)

Idea: Let

$$e(t) = y(t) - y_d(t) \tag{6.4}$$

and try to find an integer $\rho > 0$ and a state feedback (6.3) such that $e(t)$ satisfies

$$e^{(\rho)}(t) + \alpha_1 e^{(\rho-1)}(t) + ... + \alpha_\rho e(t) = 0 \tag{6.5}$$

where $\alpha_1, ..., \alpha_\rho$ are such that

$$s^\rho + \alpha_1 s^{(\rho-1)} + ... + \alpha_{\rho-1} s + \alpha_\rho \tag{6.6}$$

is a stable polynomial.

2 Asymptotic tracking by input-output linearization (cont.)

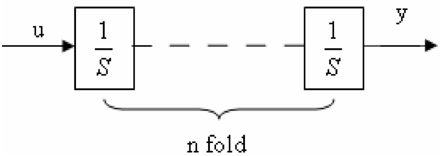
As a result, e satisfies

$$\lim_{t \to \infty} e(t) = 0 \tag{6.7}$$

(6.5) can be achieved if (6.2) is input-output linearizable in a sense to be clarified later.

Note: The integer ρ is defined by the given system and is called the *relative degree of the system*. In the special case where (6.2) is linear, ρ is the relative degree of the transfer function of (6.2).

2.1 Chain integrator system



Consider the following so-called chain integrator system

$$y^{(n)} = u \tag{6.8}$$

which is a linear system with the relative degree n. Let $x_1 = y$, $x_2 = \dot{y}, \ \cdots, \ x_n = y^{(n-1)}$. Then the state space realization of (6.8) is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

2.1 Chain integrator system (cont.)

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix},$$

$$C = [1 \ \dots \ \dots \ \dots \ 0], \quad x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

Therefore (6.8) can be converted into the standard form (6.2).

2.1 Chain integrator system (cont.)

Let $e = y(t) - y_d(t)$. In order to achieve (6.5) with $\rho = n$, i.e.

$$(y^{(n)} - y_d^{(n)}(t)) + \alpha_1(y^{(n-1)} - y_d^{(n-1)}) + ... + \alpha_n(y - y(t)) = 0. \quad (6.9)$$

substituting (6.8) into (6.9) gives

$$\begin{aligned} & (u - y_d^{(n)}(t)) + \alpha_1(y^{(n-1)} - y_d^{(n-1)}(t)) + ... + \alpha_n(y - y_d) = 0 \\ \Rightarrow \quad & u = y_d^{(n)} - \alpha_1(y^{(n-1)} - y_d^{(n-1)}) - ... - \alpha_n(y - y_d) \\ & \quad = y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)}. \end{aligned} \quad (6.10)$$

(6.10) can be written as

$$\begin{aligned} u &= -\alpha_1 x_n - \alpha_2 x_{n-1} - ... - \alpha_n x_1 + y_d^{(n)} + \alpha_1 y_d^{(n-1)} + ... + \alpha_1 y_d \\ &= k(x, y_d, ..., y_d^{(n)}) \end{aligned}$$

which is in the **standard form**.

2.1 Chain integrator system (cont.)

When $y_d = 0$, the closed-loop system is

$$\begin{aligned} \dot{x} &= Ax + Bk(x, 0, ..., 0) = (A - B[\alpha_n, ..., \alpha_1])x \\ &= \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & 1 \\ -\alpha_n & -\alpha_{n-1} & 0 & ... & -\alpha_1 \end{bmatrix} x. \end{aligned} \tag{6.11}$$

Clearly, $(A - B[\alpha_n, ..., \alpha_1])$ is a companion matrix with its characteristic polynomial being

$$s^n + \alpha_1 s^{(n-1)} + ... + \alpha_{n-1} s + \alpha_n. \tag{6.12}$$

Thus, if $\alpha_1, ..., \alpha_n$ are such that (6.12) is a stable polynomial, the E.P. of (6.11) at the origin is G.A.S.

Note: For this simple system, the integer ρ is equal to n, which is the dimension of the system.

2.2 Companion form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \alpha(x) + \beta(x)u \\ y &= x_1 \end{aligned} \tag{6.13}$$

where $\beta(0) \neq 0$. (6.13) can be equivalently put into the form

$$y^{(n)} = \alpha(x) + \beta(x)u, \qquad \text{where} \quad x = \begin{bmatrix} y \\ \dot{y} \\ \cdots \\ y^{(n-1)} \end{bmatrix} \tag{6.14}$$

2.2 Companion form (cont.)

To achieve asymptotic tracking for (6.13), note that using an input transformation

$$\alpha(x) + \beta(x)u = u_a \tag{6.15}$$

or

$$u = \frac{u_a - \alpha(x)}{\beta(x)} \tag{6.16}$$

gives

$$y^{(n)} = u_a \tag{6.17}$$

which is in the chain integrator form.

2.2 Companion form (cont.)

Thus, by (6.10), letting

$$u_a = y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)} \tag{6.18}$$

gives

$$e^{(n)} + \alpha_1 e^{(n-1)} + \dots + \alpha_n e = 0 \tag{6.19}$$

Therefore, substituting (6.18) into (6.16) gives

$$u = \frac{y_d^{(n)} - \sum_{i=1}^{n} \alpha_i e^{(n-i)} - \alpha(x)}{\beta(x)} \tag{6.20}$$

which achieves asymptotic tracking.

2.2 Companion form (cont.)

Note:

(i) Again for this simple system, the integer ρ is equal to n, the dimension of the system.

(ii) It can be verified that

$$\dot{x} = f(x) + g(x)k(x, 0, ..., 0) = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & 1 \\ -\alpha_n & -\alpha_{n-1} & 0 & ... & -\alpha_1 \end{bmatrix} x \quad (6.21)$$

which is a stable linear system since its characteristic polynomial $s^n + \alpha_1 s^{(n-1)} + ... + \alpha_{n-1} s + \alpha_n$ is stable.

(iii) The closed-loop system (6.19) is a linear system. Thus the control law (6.20) stabilizes the given system by making the closed-loop system a stable linear system.

2.2 Companion form (cont.)

Example: The inverted pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{mgl}{J}\sin x_1 + \frac{u}{J} \\ y &= x_1 \end{aligned}$$

By (6.20),

$$u = \left(\ddot{y}_d(t) - \alpha_1 \dot{e} - \alpha_2 e - \frac{mgl}{J}\sin x_1\right) J$$

results in $\ddot{e} + \alpha_1 \dot{e} + \alpha_2 e = 0$.

2.3 General case

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x \in \Re^n \\ y &= h(x), \qquad\quad u \in \Re,\ y \in \Re \end{aligned} \tag{6.22}$$

Idea: convert it into the (partial) companion form

Step 1

Differentiating y until there exists an integer ρ such that

$$y^{(\rho)} = \alpha(x) + \beta(x)u \tag{6.23}$$

where $\beta(x) \neq 0$ or at least $\beta(0) \neq 0$.

2.3 General case (cont.)

Step 2

Using the input transformation

$$u = \frac{u_a - \alpha(x)}{\beta(x)} \tag{6.24}$$

to obtain

$$y^{(\rho)} = u_a \tag{6.25}$$

2.3 General case (cont.)

Step 3
Using

$$u_a = y_d^{(\rho)} - \sum_{i=1}^{\rho} \alpha_i e^{(n-i)} \tag{6.26}$$

gives

$$e^{(\rho)} + \alpha_1 e^{(\rho-1)} + ... + \alpha_{\rho-1}\dot{e} + \alpha_\rho e = 0. \tag{6.27}$$

Thus, by (6.20), the control law that achieves (6.27) is

$$u = \frac{y_d^{(\rho)} - \sum_{i=1}^{\rho} \alpha_i e^{(\rho-i)} - \alpha(x)}{\beta(x)} = k(x, y_d, ..., y_d^{(\rho)}). \tag{6.28}$$

Remark:

The closed-loop system is

$$\dot{x} = f(x) + g(x)k(x, y_d, ..., y_d^{(\rho)}).$$

Need to check the stability of

$$\dot{x} = f(x) + g(x)k(x, 0, ..., 0).$$

The stability is not guaranteed unless $\rho = n$.

2.3 General case (cont.)

Example:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} ax_1 \\ x_1x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} u \\ y &= h(x) = x_3\end{aligned}$$

where a is some constant.

2.3 General case (cont.)

Simple calculations gives

$$\begin{aligned} \dot{y} &= \dot{x}_3 = x_2 \\ \ddot{y} &= x_1 x_2 + u. \end{aligned}$$

Thus, $\rho = 2$. Using the control law (6.28), that is,

$$u = -x_1 x_2 + \ddot{y}_d - \alpha_1 \dot{e} - \alpha_2 e = k(x, y_d, \dot{y}_d, \ddot{y}_d)$$

gives

$$\ddot{e} + \alpha_1 \dot{e} + \alpha_2 e = 0.$$

Note that

$$k(x,0,0,0) = -x_1x_2 - \alpha_1\dot{y} - \alpha_2 y = -x_1x_2 - \alpha_1 x_2 - \alpha_2 x_3$$

and

$$\begin{aligned} \dot{x} &= f(x) + g(x)k(x,0,0,0) \\ &= \begin{bmatrix} ax_1 + \exp(x_2)(-x_1x_2 - \alpha_1x_2 - \alpha_2x_3) \\ x_1x_2 - x_1x_2 - \alpha_1x_2 - \alpha_2x_3 \\ x_2 \end{bmatrix} = f_c(x). \end{aligned}$$

2.3 General case (cont.)

The Jacobian matrix of $f_c(x)$ at the origin is:

$$
\begin{aligned}
J &= \left.\frac{\partial f_c}{\partial x}\right|_{x=0} = \begin{bmatrix} a & -\alpha_1 & -\alpha_2 \\ 0 & -\alpha_1 & -\alpha_2 \\ 0 & 1 & 0 \end{bmatrix} \\
\det(sI - J) &= \begin{vmatrix} s-a & \alpha_1 & \alpha_2 \\ 0 & s+\alpha_1 & \alpha_2 \\ 0 & -1 & s \end{vmatrix} = (s-a)(s^2+\alpha_1 s+\alpha_2)
\end{aligned}
$$

Therefore,

$a < 0$	A.S.
$a > 0$	unstable
$a = 0$	cannot conclude.

2.3 General case (cont.)

Note:

(i) (6.23) can be viewed as an informal definition of the relative degree ρ of the given system. As long as such an integer exists, one can find an input transformation (6.24) such that the output y and the new input u_a satisfy the linear relation (6.25) (chain integrator). The control law (6.26) is designed based on the linear system (6.25). That is why the approach is called input-output linearization.

(ii) Unfortunately, for this general case, the integer ρ may not exist, and even if it exists, the closed-loop system under the control law (6.28) with $y_d(t)$ set to zero may not be asymptotically stable or even stable. Thus whether or not ρ exists and whether or not the control law (6.28) may stabilize the system need to be further studied next.

(iii) If $\rho = n$, the closed-loop system can always be made an asymptotically stable linear system.

3 Solution of input-output feedback linearization
3.1 Notation

Suppose that $h : \Re^n \to \Re^1$ is a smooth function and $f : \Re^n \to \Re^n$, $g : \Re^n \to \Re^n$ are vector fields. Then we introduce the following notation:

$$\begin{aligned} \frac{\partial h}{\partial x} &= \left[\frac{\partial h}{\partial x_1}, ..., \frac{\partial h}{\partial x_n}\right] \\ L_f^0 h &= h \\ L_f h &= \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x) = \left[\frac{\partial h}{\partial x_1}, ..., \frac{\partial h}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ ... \\ f_n(x) \end{bmatrix} = \frac{\partial h}{\partial x} f(x) \\ & ... \end{aligned}$$

We call $\frac{\partial h}{\partial x}$ the gradient of h and $L_f h$ the lie derivative of h along the vector field $f(x)$.

3.1 Notation (cont.)

Since $L_f h$ is still a smooth function from $\Re^n \to \Re^1$, we can define

$$L_f^2 h = L_f(L_f h) = L_f\left(\frac{\partial h}{\partial x} f(x)\right) = \frac{\partial\left(\frac{\partial h}{\partial x} f(x)\right)}{\partial x} f(x)$$

In general,

$$L_f^k h(x) = L_f\left(L_f^{k-1} h(x)\right) = \frac{\partial L_f^{k-1} h}{\partial x} f(x)$$

Also, for $k = 0, 1, \ldots$

$$L_g L_f^k h(x) = L_g\left(L_f^k h(x)\right) = \frac{\partial L_f^k h}{\partial x} g(x).$$

3.2 Relative degree

Consider the relation of the derivatives of y and u in a neighborhood of some given point $x_0 \in \Re^n$ (x_0 is called an "analysis point" and is often taken to be an equilibrium point).

$$\begin{aligned} y &= h(x) \\ \dot{y} &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) u = L_f h(x) + L_g h(x) u. \end{aligned}$$

Suppose $L_g h(x) = 0$ in a neighborhood of x_0. Then

$$\dot{y} = L_f h(x).$$

Since $\dot{y}$ is independent of u, further differentiating $\dot{y}$ gives

$$\begin{aligned} \ddot{y} = \frac{\partial L_f h(x)}{\partial x}\dot{x} &= \frac{\partial L_f h(x)}{\partial x} f(x) + \frac{\partial L_f h}{\partial x} g(x) u \\ &= L_f^2 h(x) + L_g L_f h(x) u \end{aligned}$$

3.2 Relative degree (cont.)

If $L_g L_f h(x) = 0$ for x in a neighborhood of x_0, then

$$\ddot{y}(t) = L_f^2 h(x).$$

Repeat this procedure until we find an integer ρ such that
(i)

$$L_g L_f^k h(x) = 0$$

for all $k < \rho - 1$, and for all x in a neighborhood of x_0.
(ii)

$$L_g L_f^{(\rho-1)} h(x_0) \neq 0.$$

3.2 Relative degree (cont.)

Then the system's input-output relation at x_0 is given by

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u = \alpha(x) + \beta(x) u.$$

Then, we say the system has a relative degree ρ at the point x_0. For example,

$$\dot{x} = f(x) + g(x)u = \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} u$$

$$\begin{aligned} y &= h(x) = x_3 \\ \dot{y} &= L_f^1 h(x) = x_2 \\ \ddot{y} &= L_f^2 h(x) + L_g L_f h(x) u \end{aligned}$$

relative degree=2

$$\begin{aligned} h &= x_3 \\ L_f h &= x_2 \\ L_f^2 h &= x_1 x_2 \\ L_g h &= 0 \\ L_g L_f h &= 1 \end{aligned}$$

3.3 Solvability of input-output linearization

The following statements are equivalent:

(i) The system is input-output linearizable at x_0

(ii) The system has a relative degree at x_0

(iii) There exists an integer ρ such that

(a) $L_g L_f^k h(x) = 0$ for all $k < \rho - 1$ and for all x in a neighborhood of x_0, and

(b) $L_g L_f^{(\rho-1)} h(x_0) \neq 0$

(iv) The system has an input-output relationship at x_0 in the sense that

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u$$

where $L_g L_f^{\rho-1} h(x_0) \neq 0$

3.3 Solvability of input-output linearization (cont.)

Remarks: (i) If the system has a relative degree at x_0, then, for $j = 0, 1, \cdots, (\rho - 1)$, and sufficiently small t,

$$y^{(j)}(t) = L_f^j h(x(t))$$

with $x(0) = x_0$.

(ii) There exists a state-feedback control law

$$u = \frac{-L_f^\rho h(x) + y_d^{(\rho)}(t) - \sum_{i=1}^{\rho} \alpha_i e^{(\rho-i)}}{L_g L_f^{(\rho-1)} h(x)} = k(x, y_d(t), ..., y_d^{(\rho)}(t))$$

where $e = y(t) - y_d(t)$ such that u achieves

$$e^{(\rho)}(t) + \alpha_1 e^{(\rho-1)}(t) + ... + \alpha_\rho e(t) = 0$$

3.3 Solvability of input-output linearization (cont.)

(iii) If there exists a positive integer ρ such that

$$L_g L_f^{(\rho-1)} h(x) \neq 0, \qquad \forall\ x$$
$$L_g L_f^k h(x) = 0, \qquad \forall\ k < \rho - 1, \qquad \forall\ x$$

then ρ is called a global relative degree or uniform relative degree of the system and the system is said to be globally input-output linearizable.

3.3 Solvability of input-output linearization (cont.)

(iv) For linear systems

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

we have $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$. It can be easily verified that

$$L_f^k h(x) = CA^k x, \qquad L_g L_f^k h(x) = CA^k B$$

The relative degree ρ is characterized by an integer ρ such that

$$\begin{aligned}CA^k B &= 0 \qquad \forall\ k < \rho - 1 \\ CA^{\rho-1} B &\neq 0\end{aligned}$$

3.3 Solvability of input-output linearization (cont.)

(v) Let

$$H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)}$$

where, from linear system theory,

$$\begin{aligned}
a(s) &= \det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n \\
b(s) &= b_1 s^{n-1} + \ldots + b_{n-1}s + b_n \\
b_1 &= CB \\
b_2 &= CAB + a_1 CB \\
&\ldots \\
b_n &= CA^{n-1}B + \ldots + a_{n-1}CB
\end{aligned}$$

Therefore, ρ is the difference between the degree of $a(s)$ and the degree of $b(s)$. That is why ρ is called relative degree.

3.3 Solvability of input-output linearization (cont.)

(vi) Some systems may not have well defined relative at some points, e.g., the ball and beam system described by the following equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= Bx_1x_4^2 - BG\sin x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \\ y &= x_1 \end{aligned}$$

does not have a relative degree at $x_0 = 0$ since

$$\begin{aligned} \dot{y} &= x_2 \\ \ddot{y} &= Bx_1x_4^2 - BG\sin x_3 \\ y^{(3)} &= Bx_2x_4^2 - BGx_4\cos x_3 + 2Bx_1x_4u. \end{aligned}$$

Why?

3.3 Solvability of input-output linearization (cont.)

(vii) In general, the system does not have a well defined relative degree at a point x_0 if the first function of the following sequence

$$L_g h(x), L_g L_f h(x), \cdots, L_g L_f^k g(x), \cdots$$

which is equal to zero at the point $x = x_0$, but is not identically zero in any neighborhood of x_0.

3.4 Stability of the closed-loop system

Assume the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

has relative degree ρ at $x_0 = 0$, that is, $\exists\ \rho$ such that
(a) $L_g L_f^k h(x) = 0$ for all $k < \rho - 1$ and all x in a neighborhood of $x_0 = 0$, and
(b) $L_g L_f^{(\rho-1)} h(x_0) \neq 0$.

Then we have

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{(\rho-1)} h(x) u.$$

3.4 Stability of the closed-loop system (cont.)

Thus the control law

$$u = \frac{-L_f^{\rho} h(x) + y_d^{(\rho)} - \sum_{i=1}^{\rho} \alpha_i e^{(\rho-i)}}{L_g L_f^{(\rho-1)} h(x)}$$

achieves

$$e^{(\rho)} + \alpha_1 e^{(\rho-1)} + ... + \alpha_{\rho-1} \dot{e} + \alpha_\rho e = 0$$

where $e = y - y_d$.
Note that condition (a) implies, for all x in a neighborhood of $x_0 = 0$,

$$\dot{y} = L_f h(x), \ \cdots, \ y^{(k)} = L_f^k h(x).$$

Therefore $e^{(i)} = L_f^i h(x) - y_d^{(i)}$, $i = 1, \cdots, \rho$, and u takes the following form $u = k(x, y_d, ..., y_d^{(\rho)})$.

3.4 Stability of the closed-loop system (cont.)

We will consider the stability of the closed-loop system

$$\dot{x} = f(x) + g(x)k(x, 0, ..., 0) \tag{6.29}$$

where

$$k(x, 0, ..., 0) = \frac{-L_f^{\rho} h(x) - \sum_{i=1}^{\rho} \alpha_i L_f^{(\rho-i)} h(x)}{L_g L_f^{\rho-1} h(x)}. \tag{6.30}$$

3.4 Stability of the closed-loop system (cont.)

(i) Local A.S. of the E.P. of (6.29)

Let

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}, \quad B = g(0), \quad C = \left.\frac{\partial h}{\partial x}\right|_{x=0}, \quad K = \left.\frac{-\partial k(x, 0, ..., 0)}{\partial x}\right|_{x=0}$$

Then the Jacobian linearization of

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{6.31}$$

at the origin is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6.32}$$

The Jacobian linearization of (6.29) at the origin is

$$\dot{x} = (A - BK)x \tag{6.33}$$

Clearly the E.P. of (6.29) is L.A.S. if all the eigenvalues of $(A - BK)$ have negative real parts.

3.4 Stability of the closed-loop system (cont.)

(ii) **Lemma:** Assume (6.31) has a relative degree ρ at $x = 0$. Then (6.32) also has a relative degree ρ, i.e.,

$$\begin{cases} CA^k B = 0 & \forall\ k < \rho - 1 \\ CA^{\rho-1} B \neq 0 \end{cases} \tag{6.34}$$

Sketch of the proof:
It can be shown that

$$\begin{aligned} \left.\frac{\partial L_f h(x)}{\partial x}\right|_{x=0} &= \left.\left(\frac{\partial^2 h}{\partial x^2} f\right)\right|_{x=0} + \left.\left(\frac{\partial h}{\partial x}\frac{\partial f}{\partial x}\right)\right|_{x=0} = CA \\ \left.\frac{\partial L_f^k h(x)}{\partial x}\right|_{x=0} &= \left.\frac{\partial L_f^{k-1} h(x)}{\partial x}\frac{\partial f}{\partial x}\right|_{x=0} = CA^k, \qquad k = 2, 3, \ldots \end{aligned}$$

3.4 Stability of the closed-loop system (cont.)

Therefore,

$$L_g L_f^k h(x)\Big|_{x=0} = \frac{\partial L_f^k h(x)}{\partial x} g(x)\Big|_{x=0} = CA^k B, \qquad k = 0, 1, 2, ... \tag{6.35}$$

Thus, the fact that $L_g L_f^k h(x) = 0$ for all $0 \leq k < \rho - 1$ and for all x in a neighborhood of $x_0 = 0$ implies $CA^k B = 0$ for all $0 \leq k < \rho - 1$, and the fact that $L_g L_f^{\rho-1} h(x)\Big|_{x=0} \neq 0$ implies $CA^{\rho-1} B \neq 0$.

Exercise: Show that (6.34) implies that the row vectors $C, CA, \cdots, CA^{\rho-1}$ are linearly independent.

(iii) **Theorem:** Assume (6.31) has a relative degree ρ at the origin. Then the transfer function of (6.32) is

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B = \frac{b(s)}{a(s)} \\ &= \frac{b_\rho s^{n-\rho} + ... + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n} \end{aligned} \tag{6.36}$$

and

$$\begin{aligned} &\det(sI - (A - BK)) \\ &= \frac{1}{CA^{\rho-1}B} b(s)(s^\rho + \alpha_1 s^{\rho-1} + ... + \alpha_{\rho-1}s + \alpha_\rho). \end{aligned} \tag{6.37}$$

As a result, the E.P. of (6.29) can be made L.A.S. if all the roots of $b(s)$ have negative real parts, and is unstable if some roots of $b(s)$ have positive real part.

Sketch of the proof:

(a) (6.36) holds since the relative degree of (6.32) is also ρ.

(b) To prove (6.37), we need to calculate $\det(sI-(A-BK))$. This can be done as follows:

Consider the control law

$$u = k(x,0,...,0) + \frac{v}{L_g L_f^{\rho-1} h(x)} \tag{6.38}$$

which gives the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\left(k(x,0,...,0) + \frac{v}{L_g L_f^{\rho-1} h(x)}\right) \\ &= f(x) + g(x)k(x,0,...,0) + \frac{g(x)}{L_g L_f^{\rho-1} h(x)} v \\ y &= h(x) \end{aligned} \tag{6.39}$$

3.4 Stability of the closed-loop system (cont.)

The linearization of (6.39) is

$$\begin{aligned} \dot{x} &= (A - BK)x + \frac{B}{CA^{\rho-1}B}v \\ y &= Cx \end{aligned} \tag{6.40}$$

The transfer function from v to y is

$$\begin{aligned} \frac{Y(s)}{V(s)} &= C(sI - (A - BK))^{-1}\frac{B}{CA^{\rho-1}B} \\ &= \frac{b(s)}{\det(sI - (A - BK))}\frac{1}{CA^{\rho-1}B} \end{aligned} \tag{6.41}$$

because the numerator of $C(sI - A)^{-1}B =$ the numerator of $C(sI - (A - BK))^{-1}B$ for any K since

$$\begin{aligned} \det\begin{bmatrix} sI - (A - BK) & -B \\ C & 0 \end{bmatrix} &= \det\left(\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\right) \\ &= \det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = b(s). \end{aligned}$$

➢ **Remark:** Given $\dot{x} = Ax + Bu, \quad y = Cx + Du$, its transfer function is

$$P(s) = C(sI - A)^{-1}B + D = \frac{\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}}{\det(sI - A)}$$

Proof: Note that

$$\begin{pmatrix} I_n & 0_{n\times 1} \\ -C & 1 \end{pmatrix} \begin{pmatrix} (sI - A)^{-1} & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{pmatrix} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & -(sI - A)^{-1}B \\ 0_{n\times 1} & C(sI - A)^{-1}B + D \end{pmatrix}$$

Thus

$$\det[(sI - A)^{-1}] \det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = C(sI - A)^{-1}B + D$$

3.4 Stability of the closed-loop system (cont.)

On the other hand, note that

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u. \tag{6.42}$$

Thus under the control law (6.38), we have

$$\begin{aligned} y^{(\rho)} &= -\sum_{i=1}^{\rho} \alpha_i y^{(\rho-i)} + v. \\ \Rightarrow \quad \frac{Y(s)}{V(s)} &= \frac{1}{s^{\rho} + \alpha_1 s^{\rho-1} + \ldots + \alpha_{\rho-1} s + \alpha_{\rho}} \\ &= \frac{b(s)}{b(s)(s^{\rho} + \alpha_1 s^{\rho-1} + \ldots + \alpha_{\rho})}. \end{aligned} \tag{6.43}$$

Comparing (6.41) and (6.43) gives

$$\det(sI - (A - BK)) = \frac{1}{CA^{\rho-1}B} b(s)(s^{\rho} + \alpha_1 s^{\rho-1} + \ldots + \alpha_{\rho}).$$

3.4 Stability of the closed-loop system (cont.)

(iv) **Example**

$$\dot{x} = \begin{bmatrix} ax_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} u \quad . \qquad (6.44)$$
$$y = x_3$$

We have shown before that $\rho = 2$, and under the control law

$$u = k(x, 0, 0, 0) = -x_1 x_2 - \alpha_1 x_2 - \alpha_2 x_3,$$

the E.P. of the closed-loop system is L.A.S. if $a < 0$, and is unstable if $a > 0$.

3.4 Stability of the closed-loop system (cont.)

Now consider the Jacobian linearization of (6.44) at the origin

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Simple calculation gives

$$
H(s) = C(sI - A)^{-1}B = \frac{(s-a)}{\det(sI - A)}
$$

i.e., $b(s) = s - a$. Thus, the E.P. of the closed-loop system is L.A.S. if $a < 0$, and is unstable if $a > 0$ since $b(s)$ is unstable if $a > 0$ and stable if $a < 0$.

3.5 Minimum phase systems

(i) Consider linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6.45}$$

Let $H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)}$

- (6.45) is said to be *minimum phase* if all the roots of $b(s)$ have negative real part and *nonminimum phase* if otherwise.
- The system is said to be *strictly nonminumum phase* if at least one root of $b(s)$ has positive real part.

3.5 Minimum phase systems (cont.)

(ii) Nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \qquad (6.46)$$

- (6.46) is said to be *(locally) minimum phase* if its linearization (6.45) is minimum phase, and is said to be *strictly nonminimum phase* if its linearization (6.45) is.
- Input-output linearization method only applies to minimum phase system. When $\rho = n$, (6.46) is minimum phase.

(iii) The case where the linearization of nonlinear system (6.46) is neither minumum phase nor strictly nonminumum phase is called *critical case*. The stability property of the closed-loop system resulting from input-output linearization control law for this case cannot be ascertained based on the Lyapunov's linearization method.

3.6 Design example for a robot system

A flexible-joint mechanism : Figure 6.6 illustrates a mechanism representing a link driven by a motor through a torsional spring (a single-link flexible-joint robot).

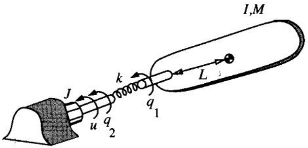


Figure 6.6 : A flexible-joint mechanism

Motion equation:

$$\begin{aligned} I\ddot{q}_1 + Mg\sin q_1 + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - k(q_1 - q_2) &= u \end{aligned}$$

3.6 Design example for a robot system (cont.)

State equation:

Letting $x = [q_1,\ \dot{q}_1,\ q_2,\ \dot{q}_2]^T$ gives

$$\dot{x} = f(x) + g(x)u = \begin{bmatrix} x_2 \\ -\frac{Mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix} u.$$

Assume q_1 is the output. Let $y = x_1$. Then

$$\begin{aligned} \dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -\frac{Mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ y^{(3)} &= -\frac{Mgl}{I}x_2 \cos x_1 - \frac{k}{I}(x_2 - x_4) \end{aligned}$$

3.6 Design example for a robot system (cont.)

$$
\begin{aligned}
y^{(4)} &= -\frac{Mgl}{I}(\dot{x}_2 \cos x_1 - \dot{x}_1 x_2 \sin x_1) - \frac{k}{I}(\dot{x}_2 - \dot{x}_4) \\
&= -\frac{Mgl}{I}\left\{\left(\frac{-Mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)\right)\cos x_1 - x_2^2 \sin x_1\right\} \\
&\quad -\frac{k}{I}\left(-\frac{Mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) - \frac{k}{J}(x_1 - x_3) - \frac{1}{J}u\right) \\
&= \frac{Mgl}{I}\sin x_1\left(x_2^2 + \frac{Mgl}{I}\cos x_1 + \frac{k}{I}\right) \\
&\quad +\frac{k}{I}(x_1 - x_3)\left(\frac{k}{I} + \frac{k}{J} + \frac{Mgl}{I}\cos x_1\right) + \frac{k}{IJ}u \\
&= L_f^4 h(x) + L_g L_f^3 L u = \alpha(x) + \beta(x) u
\end{aligned}
\tag{6.47}
$$

Clearly $\frac{k}{IJ} \neq 0$. The system is globally input-output linearizable.

3.6 Design example for a robot system (cont.)

Given $y_d(t)$, let $e = y - y_d(t)$.
Then the control law

$$u = \frac{y_d^{(4)}(t) - \alpha_1 e^{(3)} - \alpha_2 \ddot{e} - \alpha_3 \dot{e} - \alpha_4 e - \alpha(x)}{\beta(x)}$$

will achieve

$$e^{(4)} + \alpha_1 e^{(3)} + \alpha_2 \ddot{e} + \alpha_3 \dot{e} + \alpha_4 e = 0.$$

Thus asymptotic tracking can be achieved.
In particular, when $y_d(t) = 0$, global A.S. can be achieved.

4.1 Nonlinear State Transformations

The concept of diffeomorphism can be viewed as a generalization of the familiar concept of coordinate transformation.

Let V be an open neighborhood of a point $x_0 \in \Re^n$ and $f : V \mapsto \Re^n$ be continuous. Let $W = f(V) = \{y \in \Re^n \mid y = f(x) \ \& \ x \in V\}$. Then W is open and contains $f(x_0)$. If there exists a continuous function $g : W \mapsto \Re^n$ such that $g(f(x)) = x$ for all $x \in V$, then the function g is said to be a (local) inverse function of f and is denoted as f^{-1}. If $V = W = \Re^n$, then g is said to be a global inverse function of f.

Examples:

(a) Let $f(x) = x^3$ where $x \in \Re$. Then f has a global inverse $f^{-1}(y) = y^{\frac{1}{3}}$.

(b) Let $f(x) = Ax$ where $x \in \Re^n$ and $A \in \Re^{n\times n}$ is nonsingular. Then f has a global inverse $f^{-1}(y) = A^{-1}y$

(c) $f(x) = x^2$ does not have a global inverse.

4.1 Nonlinear State Transformations (cont.)

Diffeomorphism: Let V be an open neighborhood of a point $x_0 \in \Re^n$ and $\phi : V \mapsto \Re^n$. ϕ is called a (local) diffeomorphism if it is smooth on V, and its inverse $\phi^{-1} : \phi(V) \mapsto V$ exists and is smooth. If $V = \phi(V) = \Re^n$, then ϕ is called a global diffeomorphism on $\Re^n$.

Remark: Global diffeomorphisms are rare, and therefore one often looks for local diffeomorphism, *i.e.*, for transformations defined only in an open neighborhood of a given point.

4.1 Nonlinear State Transformations (cont.)

Given a nonlinear function $\phi(x)$, it is easy to check whether it is a local diffeomorphism by using the following well-known inverse function theorem.

Inverse function theorem: Let V be an open set of $\Re^n$ and $\phi : V \mapsto \Re^n$ be smooth. If the Jacobian matrix $\frac{\partial \phi}{\partial x}$ is nonsingular at some point $x_0 \in V$, then there exists an open neighborhood U of x_0 in V such that $\phi : U \mapsto \phi(U)$ is a diffeomorphism.

Remark: If $\phi : V \mapsto \Re^n$ is C^1, and the Jacobian matrix $\frac{\partial \phi}{\partial x}$ is nonsingular at some point $x_0 \in V$, then there exists an open neighborhood U of x_0 in V such that the inverse of ϕ denoted by $\phi^{-1} : \phi(U) \mapsto U$ exists and is C^1.

A diffeomorphism can be used to transform a nonlinear system into another nonlinear system in terms of a new set of states, similarly to what is commonly done in the analysis of linear systems.

4.1 Nonlinear and State Transformations (cont.)

Consider the dynamic system described by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned}$$

Let $\phi : \Re^n \mapsto \Re^n$ be a global diffeomorphism. Then the function $z = \phi(x)$ defines a new set of states with its inverse function denoted by
$x = \phi^{-1}(z)$.
Differentiation of z yields

$$\dot{z} = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}(f(x) + g(x)u) = L_f\phi(x) + L_g\phi(x)u.$$

In terms of z, the new state-space representation is

$$\begin{aligned} \dot{z} &= L_f\phi(\phi^{-1}(z)) + L_g\phi(\phi^{-1}(z))u = f^*(z) + g^*(z)u \\ y &= h(x) = h(\phi^{-1}(z)) = h^*(z) \end{aligned}$$

where the functions f^*, g^* and h^* are defined obviously.

An Example of Global Diffeomorphism:

For the robot system, let $z = \phi(x)$ be defined as follows

$$\begin{aligned} z_1 &= h(x) = x_1 \\ z_2 &= L_f h(x) = x_2 \\ z_3 &= L_f^2 h(x) = -\frac{Mgl}{I} \sin x_1 - \frac{k}{I}(x_1 - x_3) \\ z_4 &= L_f^3 h(x) = -\frac{Mgl}{I} x_2 \cos x_1 - \frac{k}{I}(x_2 - x_4). \end{aligned} \tag{6.48}$$

4.1 Nonlinear State Transformations (cont.)

Then (6.48) defines a global deffeomorphism in the sense that we can solve x_1, x_2, x_3 and x_4 from the above equations in terms of z_1, z_2, z_3, z_4 as follows:

$$
\begin{aligned}
x_1 &= z_1 \\
x_2 &= z_2 \\
x_3 &= z_1 + \frac{I}{k}\left(z_3 + \frac{Mgl}{I}\sin z_1\right) \qquad \mathit{or} \qquad x = \phi^{-1}(z) \\
x_4 &= z_2 + \frac{I}{k}\left(z_4 + \frac{Mgl}{I}\cos z_1\right).
\end{aligned}
\tag{6.49}
$$

4.1 Nonlinear State Transformations (cont.)

Moreover, under the new coordinates, we have

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= L_f^4 h(x) + L_g L_f^3 h(x) u \\
&= L_f^4 h(\phi^{-1}(z)) + L_g L_f^3 h(\phi^{-1}(z)) u
\end{aligned} \tag{6.50}$$

4.1 Nonlinear State Transformations (cont.)

Remark:
The robots system is globally diffeomorphic to a nonlinear system in companion form. All systems with $\rho = n$ can be put into the companion form by state transformation. A further input transformation

$$L_f^4 h(x) + L_g L_f^3 h(x) u = L_f^4 h(\phi^{-1}(z)) + L_g L_f^3 h(\phi^{-1}(z)) u = v$$

gives a linear system as follows:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v. \end{aligned} \qquad (6.51)$$

Thus the robot system can be put into a linear system by state and input transformation.

4.1 Nonlinear State Transformations (cont.)

Example : A non-global diffeomorphism

The nonlinear vector function

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \phi(x) = \begin{bmatrix} 2x_1 + 5x_1x_2^2 \\ 3\sin x_2 \end{bmatrix} \tag{6.52}$$

is well defined for all x_1 and x_2. Its Jacobian matrix is

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} 2 + 5x_2^2 & 10x_1x_2 \\ 0 & 3\cos x_2 \end{bmatrix}$$

which has rank 2 at $x = (0, 0)$. Therefore, the inverse function theorem indicates that the function (6.52) defines a local diffeomorphism around the origin.

In fact, the diffeomorphism is valid in the region $\Omega = \{(x_1, x_2), |x_2| < \frac{\pi}{2}\}$ because the inverse exists and is smooth for x in this region.

However, outside this region, say, at $|x_2| = \frac{\pi}{2}$, ϕ does not define a diffeomorphism, because the inverse does not uniquely exist.

4.2 A Special Version of The Frobenius Theorem

- The Frobenius theorem provides a necessary and sufficient condition for the solvability of a special class of partial differential equations, and it is an important tool in the formal treatment of feedback linearization for nth-order nonlinear systems.
- This section will present a special version of the Frobenius Theorem.

Frobenius Theorem (Special version): Let $g : \Re^n \mapsto \Re^n$ be a vector field and $g(x_0) \neq 0$ for some $x_0 \in \Re^n$. Then there exist $n-1$ smooth functions $\phi_i : \Re^n \mapsto \Re$, $i = 1, \cdots, n-1$, such that, for x in a neighborhood of x_0,

$$L_g \phi_i(x) = 0,\ i = 1, \cdots, n-1$$

and the row vectors $\frac{\partial \phi_1}{\partial x}(x_0)$, $\cdots$, $\frac{\partial \phi_{n-1}}{\partial x}(x_0)$ are linearly independent.

4.3 Normal Form and Zero Dynamics

In general, suppose system (6.46) has a relative degree ρ at $x_0 = 0$ with $\rho \leq n$. Let $\phi_i(x) = L_f^{i-1} h(x),\ i = 1, \cdots, \rho$. Then, by the exercise in page 42, the ρ row vectors $\frac{\partial \phi_i(x)}{\partial x}|_{x=0}$, $i = 1, \cdots, \rho$, are linearly independent. Therefore, there exist $n - \rho$ smooth functions $\phi_i(x),\ i = (\rho + 1), \cdots, n$ vanishing at the origin such that $\frac{\partial \phi(x)}{\partial x}|_{x=0}$ is nonsingular where

$$\phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}.$$

By inverse function theorem, $\phi(x)$ is invertible in a neighborhood of x_0.

4.3 Normal Form and Zero Dynamics (cont.)

Let $z = \phi(x)$. Then

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
&\cdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= \left(L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u\right)\Big|_{x=\phi^{-1}(z)} \\
\dot{z}_{\rho+1} &= (L_f \phi_{\rho+1}(x) + L_g \phi_{\rho+1}(x) u)|_{x=\phi^{-1}(z)} \\
&\vdots \\
\dot{z}_n &= (L_f \phi_n(x) + L_g \phi_n(x) u)|_{x=\phi^{-1}(z)} \\
y &= z_1
\end{aligned}
\tag{6.53}
$$

We call (6.53) the *normal form* of the system (6.46).

4.3 Normal Form and Zero Dynamics (cont.)

Remark: Since system (6.46) has a relative degree ρ at $x_0 = 0$, with $\phi_i(x) = L_f^{i-1} h(x),\ i = 1, \cdots, \rho$, we have

$$L_g \phi_i(x) = 0, \quad i = 1, \cdots, \rho - 1$$

for x in an open neighborhood of the origin. By the Frobenius Theorem, it is possible to choose $\phi_i(x),\ i = \rho + 1, \cdots, n$, such that

$$L_g \phi_i(x) = 0, \quad i = \rho + 1, \cdots, n \tag{6.54}$$

for x in an open neighborhood of the origin, and $\frac{\partial \phi(x)}{\partial x}|_{x=0}$ is nonsingular, i.e., $\phi(x)$ is locally invertible. It is clear from (6.53) that this set of choices will render the equations (6.53) a more special expression as follows:

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= \left(L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u \right)\Big|_{x=\phi^{-1}(z)} = \alpha(z) + \beta(z) u \\
\dot{z}_{\rho+1} &= L_f \phi_{\rho+1}(x)|_{x=\phi^{-1}(z)} = \gamma_{\rho+1}(z) \\
&\vdots \\
\dot{z}_n &= L_f \phi_n(x)|_{x=\phi^{-1}(z)} = \gamma_n(z) \\
y &= z_1.
\end{aligned} \tag{6.55}
$$

4.3 Normal Form and Zero Dynamics (cont.)

Under the control law

$$u = \frac{-\alpha(z) - \alpha_1 z_\rho - \cdots - \alpha_\rho z_1}{\beta(z)}, \tag{6.56}$$

the closed-loop system is

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= -\alpha_1 z_\rho - \cdots - \alpha_\rho z_1 \\
\dot{z}_{\rho+1} &= \gamma_{\rho+1}(z) \\
&\vdots \\
\dot{z}_n &= \gamma_n(z) \\
y &= z_1.
\end{aligned} \tag{6.57}$$

4.3 Normal Form and Zero Dynamics (Cont.)

If $\alpha_1, \cdots, \alpha_\rho$ are such that $s^\rho + \alpha_1^{(\rho-1)} + \cdots + \alpha_{\rho-1}s + \alpha_\rho$ is stable, then $\lim_{t\to\infty}(z_1, z_2, \cdots, z_\rho) = 0$ and $(z_{\rho+1}, \cdots, z_n)$ will be governed by

$$\begin{aligned} \dot{z}_{\rho+1} &= \gamma_{\rho+1}(0, \cdots, 0, z_{\rho+1}, \cdots, z_n) \\ &\cdots \\ \dot{z}_n &= \gamma_n(0, \cdots, 0, z_{\rho+1}, \cdots, z_n). \end{aligned} \tag{6.58}$$

(6.58) is called the zero dynamics of (6.46).

Thus under the input-output feedback control law (6.56), the origin of the closed-loop system (6.57) is A.S. only if the origin of (6.58) is A.S.

4.3 Normal Form and Zero Dynamics (Cont.)

Remark: (i) A more general definition of the minimum phase system can be given in terms of the zero dynamics (6.58). System (6.46) is said to be (global) minimum phase if the origin of (6.58) is (globally) A.S, and is said to be non-minimum phase if the origin of (6.58) is unstable.

(ii) The above definition also applies to the critical case.

(iii) If $\rho = n$, system (6.46) has a trivial zero dynamics since the dimension of (6.58) is zero and it can be fully linearized to a chain integrator form by an input transform $\alpha(z) + \beta(z)u = v$.

4.3 Normal Form and Zero Dynamics (Cont.)

(iv) Let $\xi = (z_{\rho+1}, \cdots, z_n)^T$ and

$$\gamma(\xi) = \begin{bmatrix} \gamma_{\rho+1}(0, \cdots, 0, z_{\rho+1}, \cdots, z_n) \\ \vdots \\ \gamma_n(0, \cdots, 0, z_{\rho+1}, \cdots, z_n) \end{bmatrix}$$

Then the zero dynamics (6.58) can be put in the following compact form:

$$\dot{\xi} = \gamma(\xi)$$

It can be shown that the roots of $b(s)$, i.e., the zeros of the Jacobian linearization of (6.46) at the origin coincide with the eigenvalues of the matrix $\frac{\partial \gamma}{\partial \xi}(0)$.

Exercise: The origin of (6.29) is asymptotically stable if and only if the origin of (6.57) is asymptotically stable.

4.3 Normal Form and Zero Dynamics (Cont.)

Example:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} ax_1 \\ x_1x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} u \\
y &= h(x) = x_3
\end{aligned}
$$

where a is some constant.
It is known that this system has a relative degree 2 at the origin with $h(x) = x_3$ and $L_f h(x) = x_2$. Let $\phi_1(x) = x_3$, $\phi_2(x) = x_2$ and $\phi_3(x) = x_1 - \exp(x_2) + 1$. Then $z = \phi(x)$ is a global diffeomorphism since we have $x = \phi^{-1}(z) = (z_3 + \exp(z_2) - 1, z_2, z_1)$ and

$$L_g\phi_3 = [1, -\exp(x_2), 0]g(x) = 0$$

.

4.3 Normal Form and Zero Dynamics (Cont.)

Thus the (global) normal form of the system is

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \dot{x}_2 = x_1 x_2 + u = (z_3 + \exp(z_2) - 1) z_2 + u \\
\dot{z}_3 &= \dot{x}_1 - \exp(x_2)\dot{x}_2 = a x_1 + \exp(x_2) u - \exp(x_2)(x_1 x_2 + u) \\
&= a x_1 - \exp(x_2) x_1 x_2 = (z_3 + \exp(z_2) - 1)(a - \exp(z_2) z_2)
\end{aligned}
$$

and the zero dynamics is

$$\dot{z}_3 = a z_3$$

The eigenvalue of the zero dynamics is equal to a which coincides with the zero of the Jacobian linearization of the given system at the origin.

5.1 Lie Brackets

Definition 6.2 Let f and g be two vector fields on $\Re^n$. The Lie bracket of f and g is a third vector field defined by

$$[f, g] = \nabla g \ f - \nabla f \ g.$$

The Lie bracket $[f, g]$ is commonly written as $ad_f g$ (where ad stands for "adjoint").

Repeated Lie brackets can then be defined recursively by

$$\begin{aligned} ad_f^0 g &= g \\ ad_f^i g &= [f, ad_f^{i-1} g], \quad \text{for } i = 1, 2, \ldots. \end{aligned}$$

5.1 Lie Brackets (cont.)

Example 6.7: The following system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1\\ \dot{x}_2 &= -x_2\cos x_1 + \cos(2x_1)u\end{aligned}$$

can be written in the form $\dot{x} = f(x) + g(x)u$ with the two vector fields defined by

$$f(x) = \begin{bmatrix} -2x_1 + ax_2 + \sin x_1 \\ -x_2 \cos x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \cos(2x_1) \end{bmatrix}.$$

Their Lie bracket can be computed as

$$\begin{aligned}[f, g] &= \begin{bmatrix} 0 & 0 \\ -2\sin(2x_1) & 0 \end{bmatrix} f - \begin{bmatrix} -2 + \cos x_1 & a \\ x_2 \sin x_1 & -\cos x_1 \end{bmatrix} g\\ &= \begin{bmatrix} a\cos(2x_1) \\ \cos x_1 \cos(2x_1) - 2\sin(2x_1)(-2x_1 + ax_2 + \sin x_1) \end{bmatrix}.\end{aligned}$$

5.1 Lie Brackets (cont.)

Lemma 6.1 Lie brackets have the following properties

(i) bilinearity:

$$[\alpha_1 f_1 + \alpha_2 f_2, g] = \alpha_1 [f_1, g] + \alpha_2 [f_2, g]$$
$$[f, \alpha_1 g_1 + \alpha_2 g_2] = \alpha_1 [f, g_1] + \alpha_2 [f, g_2]$$

where f, f_1, f_2 and g, g_1, g_2 are smooth vector fields, and α_1, α_2 are constant scalars.

(ii) skew-commutativity:

$$[f, g] = -[g, f].$$

(iii) Jacobi identity:

$$L_{ad_f g} h = L_f L_g h - L_g L_f h$$

where $h(x)$ is a smooth scalar function.

Proof

(1) Bilinearity:

$$\begin{aligned}
[\alpha_1 f_1 + \alpha_2 f_2, g] &= \nabla g(\alpha_1 f_1 + \alpha_2 f_2) - \nabla(\alpha_1 f_1 + \alpha_2 f_2) g \\
&= \alpha_1(\nabla g\, f_1 - \nabla f_1\, g) + \alpha_2(\nabla g\, f_2 - \nabla f_2\, g) \\
&= \alpha_1 [f_1, g] + \alpha_2 [f_2, g] \\
[f, \alpha_1 g_1 + \alpha_2 g_2] &= \nabla(\alpha_1\, g_1 + \alpha_2\, g_2) f - \nabla f(\alpha_1\, g_1 + \alpha_2\, g_2) \\
&= \alpha_1(\nabla g_1 f - \nabla f g_1) + \alpha_2(\nabla g_2 f - \nabla f g_2) \\
&= \alpha_1 [f, g_1] + \alpha_2 [f, g_2]
\end{aligned}$$

(2) Skew-commutativity:

$$[f, g] = \nabla g\, f - \nabla f\, g = -(\nabla f\, g - \nabla g\, f) = -[g, f]$$

5.1 Lie Brackets (cont.)

(3) Jacobi identity: The identity can be rewritten as

$$\nabla h[f,g] = \nabla(L_g h)f - \nabla(L_f h)g$$

The left-hand side of the above equation can be expanded as

$$\nabla h[f,g] = \nabla h(\nabla g\, f - \nabla f\, g) = \frac{\partial h}{\partial x}(\frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g)$$

while the right-hand side can be expanded as

$$\begin{aligned}\nabla(L_g h)f - \nabla(L_f h)g &= \nabla(\frac{\partial h}{\partial x}g)f - \nabla(\frac{\partial h}{\partial x}f)g \\ &= (\frac{\partial h}{\partial x}\frac{\partial g}{\partial x} + g^T\frac{\partial^2 h}{\partial x^2})f - (\frac{\partial h}{\partial x}\frac{\partial f}{\partial x} + f^T\frac{\partial^2 h}{\partial x^2})g \\ &= \frac{\partial h}{\partial x}(\frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g)\end{aligned}$$

where $\frac{\partial^2 h}{\partial x^2} = [\frac{\partial^2 h}{\partial x_i \partial x_j}]_{i,j=1,\cdots n}$ is the Hessian of h which is a symmetric matrix.

5.1 Lie Brackets (cont.)

Remark: The Jacobi identity can be used recursively to obtain useful technical identities. Using it twice yields

$$\begin{aligned} L_{ad_f^2 g}h &= L_{ad_f(ad_f g)}h = L_f L_{ad_f g}h - L_{ad_f g}L_f h \\ &= L_f\left(L_f L_g h - L_g L_f h\right) - \left(L_f L_g - L_g L_f\right)L_f h \\ &= L_f^2 L_g h - 2L_f L_g L_f h + L_g L_f^2 h. \end{aligned} \tag{6.59}$$

Similar identities can be obtained for higher-order Lie brackets.

5.2 the Frobenius Theorem

Consider the set of first-order partial differential equations

$$
\begin{aligned}
\frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \frac{\partial h}{\partial x_3} f_3 = 0 \\
\frac{\partial h}{\partial x_1} g_1 + \frac{\partial h}{\partial x_2} g_2 + \frac{\partial h}{\partial x_3} g_3 = 0
\end{aligned} \tag{6.60}
$$

where $f_i, g_i, i = 1, 2, 3$, are known scalar functions, and h is an unknown function. Clearly, this set of partial differential equations is uniquely defined by the two vectors $f = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^T, g = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}^T$.

If a solution $h(x_1, x_2, x_3)$ exists for the above partial differential equations, we shall say that the set of vector fields $\{f, g\}$ is *completely integrable*.

5.2 The Frobenius Theorem (cont.)

Formal definition of complete integrability of a set of vector fields:

Definition 6.4 A linearly independent set of vector fields $\{f_1, f_2, \ldots, f_m\}$ on $\Re^n$ is said to be completely integrable iff, there exist $(n-m)$ scalar functions $h_1(x), h_2(x), \ldots, h_{n-m}(x)$ satisfying the system of partial differential equations

$$(\nabla h_i) f_j = 0$$

where $1 \leq i \leq n-m, 1 \leq j \leq m$, and the gradients ∇h_i are linearly independent.

Note that the number of unknown scalar functions h_i involved is $(n-m)$ and the number of partial differential equations is $m(n-m)$.

5.2 The Frobenius Theorem (cont.)

Definition 6.5 A linearly independent set of vector fields $\{f_1, f_2, \ldots, f_m\}$ is said to be involutive iff $\exists$ scalar functions $\alpha_{ijk} : \Re^n \mapsto \Re$ such that

$$[f_i, f_j](x) = \sum_{k=1}^{m} \alpha_{ijk}(x) f_k(x), \quad \forall\ i, j.$$

Involutivity means that if one forms the Lie bracket of any pairs of vector fields from the set $\{f_1, f_2, \ldots, f_m\}$, then the resulting vector field can be expressed as a linear combination of the original set of vector fields.

5.2 The Frobenius Theorem (cont.)

i.e., involutivity condition guarantees the integrability of a set of vector fields.

Theorem 6.1 (Frobenius) Let $\{f_1, f_2, \ldots, f_m\}$ be a set of linearly independent vector fields. The set is completely integrable iff it is involutive.

For example, equation (6.60) has a solution $h(x_1, x_2, x_3)$ iff $\exists$ scalar functions $\alpha_1(x_1, x_2, x_3)$ and $\alpha_2(x_1, x_2, x_3)$ such that

$$[f, g] = \alpha_1 f + \alpha_2 g$$

i.e., if the Lie bracket of f and g can be expressed as a linear combination of f and g. Geometrically it means that the vector $[f, g]$ is in the plane formed by f and g.

5.2 The Frobenius Theorem (cont.)

Remarks:

- Constant vector fields are always involutive. Indeed, the Lie bracket of two constant vectors is simply the zero vector, which can be trivially expressed as linear combination of the vector fields.
- A set composed of a single vector f is involutive. Indeed,

  $$[f, f] = (\nabla f)f - (\nabla f)f = 0.$$

  Thus, the following set of equations is always completely integrable

  $$(\nabla h_i)f = 0$$

  where $1 \leq i \leq n-1$, and the gradients ∇h_i are linearly independent.
- From Definition 6.5, checking whether a set of vector fields $\{f_i, f_2, \ldots, f_m\}$ is involutive amounts to checking whether

  $$\text{rank}\,[f_1(x) \ldots f_m(x)] = \text{rank}\big[f_1(x) \ldots f_m(x)\ [f_i, f_j](x)\big]$$

  for all x and all i, j.

Example 6.9: Consider the set of partial differential equations

$$4x_3 \frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} = 0$$
$$-x_1 \frac{\partial h}{\partial x_1} + (x_3^2 - 3x_2) \frac{\partial h}{\partial x_2} + 2x_3 \frac{\partial h}{\partial x_3} = 0.$$

The associated vector fields are $\{f_1, f_2\}$ with

$$f_1 = \begin{bmatrix} 4x_3 & -1 & 0 \end{bmatrix}^T \qquad f_2 = \begin{bmatrix} -x_1 & (x_3^2 - 3x_2) & 2x_3 \end{bmatrix}^T.$$

One can easily finds that

$$[f_1, f_2] = \begin{bmatrix} -12x_3 & 3 & 0 \end{bmatrix}^T.$$

Since $[f_1, f_2] = -3f_1 + 0f_2$, this set of vector fields is involutive. Therefore, the two partial differential equations are solvable.

5.3 Input-State Linearization of SISO Systems

In this subsection, we discuss input-state linearization for single-input nonlinear systems, linear in control or affine

$$\dot{x} = f(x) + g(x)u. \tag{6.61}$$

with f and g being smooth vector fields.

We study

- when such systems can be linearized by state and input transformations,
- how to find such transformations,
- how to design controllers based on such feedback linearizations.

Definition 6.6 A single-input nonlinear system in the form (6.61), with $f(x)$ and $g(x)$ being smooth vector fields on $\Re^n$, is said to be *input-state linearizable* if there exists a region Ω in $\Re^n$, a diffeomorphism $\phi : \Omega \mapsto \Re^n$, and a nonlinear feedback control law

$$u = \alpha(x) + \beta(x)v \tag{6.62}$$

such that the new state variables $z = \phi(x)$ and the new input v satisfy a linear time-invariant relation

$$\dot{z} = Az + bv \tag{6.63}$$

where

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, \quad b = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}.$$

5.3 Input-State Linearization of SISO Systems

Remarks:

- The new state z is called the linearizing state, and the control law (6.62) is called the linearizing control law. To simplify the notation, we often use z to denote not only the transformed state, but the diffeomorphism ϕ itself, *i.e.*, $z = z(x)$.

- Note that the transformed linear dynamics has its A matrix and b vector of a special form, corresponding to a linear companion form. However, this loses no generality, because any representation of a linear controllable system is equivalent to the companion form (6.63) through a linear state transformation and pole placement. Therefore, if (6.61) can be transformed into a linear system, it can be transformed into the form prescribed by (6.63) by using additional linear transformations in state and input.

Remarks:

- From the canonical form (6.63), input-output linearization is a special case of input-state linearization, where the output function leads to a relative degree n. This means that if a system is input-output linearizable with relative degree n, it must be input-state linearizable.

- On the other hand, if a system is input-state linearizable, with the first new state z_1 representing the output, then $z_1^{(n)} = u$, i.e., the system is input-output linearizable with relative degree n.

We summarize the relationship between input-output linearization and input-state linearization into the following lemma.

Lemma 6.3 An nth-order nonlinear system is input-state linearizable iff there exists a scalar function $z_1(x)$ such that the system's input-output linearization with $z_1(x)$ as output function has relative degree n.

Note, however, that the above lemma provides no guidance about how to find the desirable output function $z_1(x)$.

5.3 Input-State Linearization of SISO Systems

At this point, a natural question is: can all nonlinear state equations in the form of (6.61) be input-state linearized? If not, when do such linearizations exist?

The following theorem provides a definitive answer to that question, and constitutes one of the most fundamental results of feedback linearization theory.

Theorem 6.2 The nonlinear system (6.61), with $f(x)$ and $g(x)$ being smooth vector fields, is input-state linearizable iff there exists a region Ω such that the following two conditions hold:

- the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ are linearly independent in Ω;
- the set $\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in Ω.

Remarks:

- The first condition can be interpreted as simply representing a *controllability condition* for the nonlinear system (6.61). For linear systems, since $f(x) = Ax$, $g(x) = b$, we have $ad_f^j g = (-1)^j A^j b$, $j = 1, 2, \cdots, (n-1)$. Thus, linear independence of the vector fields

$$\{g, ad_f g, \ldots, ad_f^{n-1} g\}$$

is equivalent to the invertibility of the familiar linear controllability matrix.

$$(b, Ab, \ldots, A^{n-1} b).$$

- The involutivity condition is trivially satisfied for linear systems (which have constant vector fields), but not generically satisfied in the nonlinear case.

Before proving Theorem 6.2, we state a technical lemma.

Lemma 6.4 Let $z(x)$ be a smooth function in a region Ω. Then, in Ω, the set of equations

$$L_g z = L_g L_f z = \ldots = L_g L_f^k z = 0 \tag{6.64}$$

is equivalent to

$$L_g z = L_{ad_f g} z = \ldots = L_{ad_f^k g} z = 0 \tag{6.65}$$

for any positive integer k.

5.3 Input-State Linearization of SISO Systems

Proof:
Let us show that (6.64) implies (6.65) .
When $k = 0$, the result is obvious. When $k = 1$, we have from Jacobi's identity (Lemma 6.1)

$$L_{ad_f\, g} z = L_f L_g z - L_g L_f z = 0 - 0 = 0$$

When $k = 2$, we further have, using Jacobi's identity twice (see (6.59))

$$L_{ad_f^2\, g} z = L_f^2 L_g\, z - 2 L_f L_g L_f\, z + L_g\, L_f^2 z = 0 - 0 + 0 = 0$$

Repeating this procedure, we can show by induction that (6.64) implies (6.65) for any k. One proceeds similarly to show that (6.65) implies (6.64) (by using Jacobi's identity the other way around).

Proof of Theorem 6.2:

(*Necessity*) Assume that there exist a state transformation $z = z(x)$ and an input transformation $u = \alpha(x) + \beta(x)v$ such that z and v satisfy (6.63). Expanding each line of (6.63), we obtain a set of differential equations

$$\begin{gathered}
\frac{\partial z_1}{\partial x} f + \frac{\partial z_1}{\partial x} gu = z_2 \\
\frac{\partial z_2}{\partial x} f + \frac{\partial z_2}{\partial x} gu = z_3 \\
\cdots \\
\frac{\partial z_n}{\partial x} f + \frac{\partial z_n}{\partial x} gu = v.
\end{gathered}$$

Since $z_1, \ldots, z_n$ are independent of u, while v is not, we conclude from the above equations that

$$L_g z_1 = L_g z_2 = \ldots = L_g z_{n-1} = 0, \quad L_g z_n \neq 0 \tag{6.66}$$

$$L_f z_i = z_{i+1}, \quad i = 1, \ldots, n-1. \tag{6.67}$$

By Lemma 6.4, equation (6.66) implies that

$$\nabla z_1 \; ad_f^k g = 0, \quad k = 0, 1, 2, \ldots, n-2. \tag{6.68}$$

Furthermore, we can show

$$\begin{aligned} \nabla z_1 \; ad_f^{n-1} g &= L_{ad_f(ad_f^{n-2} g)} z_1 \\ &= L_f L_{ad_f^{n-2} g} z_1 - L_{ad_f^{n-2} g} L_f z_1 \\ &= 0 - L_{ad_f^{n-2} g} z_2 \\ &= \cdots = (-1)^{n-1} L_g z_n. \end{aligned}$$

This implies that

$$\nabla z_1 \; ad_f^{n-1} g \neq 0. \tag{6.69}$$

(6.68) and (6.69) means the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ must be linearly independent for each $x \in \Omega$. In fact, if for some number $i \leq n-1$, there existed scalar functions $\alpha_1(x)$, $\cdots$, $\alpha_{i-1}(x)$ such that

$$ad_f^i g = \sum_{k=0}^{i-1} \alpha_k ad_f^k g,$$

then

$$[f, ad_f^i g] = ad_f^{i+1} g = \sum_{k=1}^{i} \alpha_k ad_f^k g, \cdots, [f, ad_f^{n-2} g] = ad_f^{n-1} g = \sum_{k=n-i-1}^{n-2} \alpha_k ad_f^k g.$$

This, together with (6.68), would imply that

$$\nabla z_1 ad_f^{n-1} g = \sum_{k=n-i-1}^{n-2} \alpha_k \nabla z_1 ad_f^k g = 0$$

a contradiction to (6.69).

The set $\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive follows from the existence of a scalar function z_1 satisfying the $(n-1)$ partial differential equations in (6.68), and from the necessity part of the Frobenius theorem.

5.3 Input-State Linearization of SISO Systems

(*Sufficiency*) Since the involutivity condition is satisfied, from Frobenius theorem, there exists a non-zero scalar function $z_1(x)$ satisfying

$$L_g z_1 = L_{ad_f g} z_1 = \ldots = L_{ad_f^{n-2} g} z_1 = 0. \qquad (6.70)$$

By Lemma 6.4, the above equations can be written as

$$L_g z_1 = L_g L_f z_1 = \ldots = L_g L_f^{n-2} z_1 = 0.$$

This means that if we use $z = [z_1 \ L_f z_1 \ \ldots \ L_f^{n-1} z_1]^T$ as a new set of state variables, the first $(n-1)$ state equations verify

$$\dot{z}_k = z_{k+1}, \quad k = 1, \ldots, n-1$$

while the last equation is

$$\dot{z}_n = L_f^n z_1 + L_g L_f^{n-1} z_1 u. \qquad (6.71)$$

5.3 Input-State Linearization of SISO Systems

We now show that $L_g L_f^{n-1} z_1$ cannot be equal to zero. In fact, by Jacobi identity,

$$\begin{aligned} L_{ad_f^{n-1} g} z_1 &= L_{ad_f (ad_f^{n-2} g)} z_1 \\ &= L_f L_{ad_f^{n-2} g} z_1 - L_{ad_f^{n-2} g} L_f z_1 \\ &= 0 - L_{ad_f^{n-2} g} L_f z_1 \\ &= \cdots = (-1)^{n-1} L_g L_f^{n-1} z_1, \end{aligned}$$

we must have

$$L_g L_f^{n-1} z_1 \neq 0, \quad \forall\, x \in \Omega.$$

Otherwise, the non-zero vector ∇z_1 would satisfy

$$\nabla z_1 \, [g \;\; ad_f g \;\; \ldots \;\; ad_f^{n-1} g] = 0$$

and thus would be orthogonal to n linearly independent vectors, a contradiction to the fact that the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ are linearly independent in Ω.

Therefore, by taking the control law to be

$$u = \frac{-L_f^n z_1 + v}{L_g L_f^{n-1} z_1}$$

equation (6.71) simply becomes

$$\dot{z}_n = v$$

which shows that the input-state linearization of the nonlinear system has been achieved. The proof is thus completed.

5.3 Input-State Linearization of SISO Systems

Based on the previous discussion, the input-state linearization of a nonlinear system can be performed through the following steps:

(1) Construct the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ for the given system.

(2) Check whether the controllability and involutivity conditions are satisfied.

(3) If both are satisfied, find the first state z_1 from equations (6.70), *i.e.*,

$$\nabla z_1 \ ad_f^i g = 0, \quad i = 0, \ldots, n-2, \quad \nabla z_1 \ ad_f^{n-1} g \neq 0.$$

(4) Compute the state transformation $z(x) = [z_1 \ L_f z_1 \ \ldots \ L_f^{n-1} z_1]^T$ and the input transformation (6.62), with

$$\alpha(x) = -\frac{L_f^n z_1}{L_g L_f^{n-1} z_1}, \qquad \beta(x) = \frac{1}{L_g L_f^{n-1} z_1}.$$

The Robot Example: Recall that with the state vector as

$$x = [q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2]^T$$

the corresponding vector fields f and g can be written

$$\begin{aligned} f &= [x_2 \quad -\frac{MgL}{I} sin x_1 - \frac{k}{I}(x_1 - x_3) \quad x_4 \quad \frac{k}{J}(x_1 - x_3)]^T \\ g &= [0 \quad 0 \quad 0 \quad \frac{1}{J}]^T \end{aligned}$$

5.3 Input-State Linearization of SISO Systems

The controllability matrix is obtained by simple computation

$$\begin{bmatrix} g & ad_f\, g & ad_f^2\, g & ad_f^3\, g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{k}{IJ} \\ 0 & 0 & \frac{k}{IJ} & 0 \\ 0 & -\frac{1}{J} & 0 & \frac{k}{J^2} \\ \frac{1}{J} & 0 & -\frac{k}{J^2} & 0 \end{bmatrix}$$

It has rank 4 for $k > 0, IJ < \infty$. Furthermore, since the vector fields $\begin{bmatrix} g & ad_f\, g & ad_f^2\, g \end{bmatrix}$ are constant, they form an involutive set. Therefore, the system is input-state linearizable.

5.3 Input-State Linearization of SISO Systems

Let us find out the state transformation $z = z(x)$ and the input transformation $u = \alpha(x) + \beta(x)v$ so that input-state linearization is achieved. From (6.68), $z_1(x)$ must satisfy the following equations:

$$\frac{\partial z_1}{\partial x} g = 0, \quad \frac{\partial z_1}{\partial x} ad_f g = 0, \quad \frac{\partial z_1}{\partial x} ad_f^2 g = 0$$

Given the above expression of the controllability matrix, the first component z_1 of the new state vector z should satisfy

$$\frac{\partial z_1}{\partial x_4} = 0, \quad \frac{\partial z_1}{\partial x_3} = 0, \quad \frac{\partial z_1}{\partial x_2} = 0$$

Thus, z_1 must be a function of x_1 only. In fact, any smooth function of x_1 satisfies the above equations, but the simplest non-zero solution to the above equations is

$$z_1 = x_1 \tag{6.72}$$

5.3 Input-State Linearization of SISO Systems

The other states can be obtained from z_1

$$
\begin{aligned}
z_2 &= \nabla z_1 \, f = x_2 && (6.73)\\
z_3 &= \nabla z_2 \, f = -\frac{MgL}{I} sin x_1 - \frac{k}{I}(x_1 - x_3) && (6.74)\\
z_4 &= \nabla z_3 \, f = -\frac{MgL}{I} x_2 cos x_1 - \frac{k}{I}(x_2 - x_4) && (6.75)
\end{aligned}
$$

Accordingly, the input transformation is

$$
u = \frac{v - \nabla z_4 \, f}{\nabla z_4 \, g} \tag{6.76}
$$

which can be written explicitly as

$$
u = \frac{IJ}{k}(v - \alpha(x)) \tag{6.77}
$$

where

$$\alpha(x) = \frac{MgL}{I} sin x_1 (x_2^2 + \frac{MgL}{I} cos x_1 + \frac{k}{I}) + \frac{k}{I}(x_1 - x_3)(\frac{k}{I} + \frac{k}{J} + \frac{MgL}{I} cos x_1)$$

As a result, we have

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v \end{aligned}$$

thus completing the input-state linearization.

5.3 Input-State Linearization of SISO Systems

Finally, note that

• The above input-state linearization is actually global, because the diffeomorphism $z(x)$ and the input transformation are well defined everywhere. Specifically, the inverse of the state transformation is

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= z_1 + \frac{I}{k}(z_3 + \frac{MgL}{I} sin z_1) \\ x_4 &= z_2 + \frac{I}{k}(z_4 + \frac{MgL}{I} z_2 cos z_1) \end{aligned}$$

which is well defined and differentiable everywhere. The input transformation (6.77) is also well defined everywhere.

5.3 Input-State Linearization of SISO Systems

• In this particular example, the transformed variables have physical meanings. We see that z_1 is the link position, z_2 the link velocity, z_3 the link acceleration, and z_4 the link jerk. This further illustrates our earlier remark that the complexity of a nonlinear physical model is strongly dependent on the choice of state variables.

5.3 Input-State Linearization of SISO Systems

• In hindsight, of course, we also see that the same result could have been derived simply by differentiating the first motion equation of the robot twice, i.e., from the input-output linearization perspective of Lemma 6.3. Note that inequality $\nabla z_1 ad_f^{n-1} g \neq 0$ can be replaced by the normalization equation

$$\nabla z_1 ad_f^{n-1} g = 1$$

without affecting the input-state linearization. This equation and $\nabla z_1 ad_f^i g = 0, i = 0, 1, ..., n-2$ constitute a total of n linear equations

5.3 Input-State Linearization of SISO Systems

$$\begin{bmatrix} ad_f^0 g & ad_f^1 g & \cdots & ad_f^{n-2} g & ad_f^{n-1} g \end{bmatrix} \begin{bmatrix} \frac{\partial z_1}{\partial x_1} \\ \frac{\partial z_1}{\partial x_2} \\ \cdots \\ \frac{\partial z_1}{\partial x_{n-1}} \\ \frac{\partial z_1}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$$

5.3 Input-State Linearization of SISO Systems

Given the independence condition on the vector fields, the partial derivatives $\frac{\partial z_1}{\partial x_1}, ..., \frac{\partial z_1}{\partial x_n}$ can be computed uniquely from the above equations. The state variable z_1 can then be found, in principle, by sequentially integrating these partial derivatives. Note that analytically solving this set of partial differential equations for z_1 may be a nontrivial step (although numerical solutions may be relatively easy due to the recursive nature of the equations).