Chapter 7 Sliding Control

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1 Introduction

Consider

$$y^{(n)} = \alpha(x) + \beta(x)u$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \alpha(x) + \beta(x)u \\ y &= x_1 \end{aligned} \tag{7.1}$$

1 Introduction (cont.)

Input-output linearization consists of two steps.

Step 1. Input transformation

$$u = \frac{-\alpha(x) + v}{\beta(x)}$$

which results in a linear system in chain integrator form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
&\cdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= v \\
y &= x_1.
\end{aligned}$$

Step 2. Using linear control

$$v = y_d^{(n)} - \alpha_1 e^{(n-1)} - \ldots - \alpha_{n-1}\dot{e} - \alpha_n e$$

to yield

$$e^{(n)} + \alpha_1 e^{(n-1)} + \ldots + \alpha_{n-1}\dot{e} + \alpha_n e = 0.$$

Limitation: $\alpha(x)$ & $\beta(x)$ must be known precisely.

Example 1. Consider a single link robot

$$J\ddot{\theta} - mgl\sin\theta = u$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{u + mgl\sin\theta}{J} \\ y &= x_1 \end{aligned}$$

where J is moment of inertia, m is total mass. Clearly,

$$u = Jv - mgl\sin\theta \tag{7.2}$$

makes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v. \end{aligned}$$

1 Introduction (cont.)

However, generally J (moment of inertia) and m (total mass) are not known exactly. Therefore instead of (7.2), one usually uses

$$u = \hat{J}v - \hat{m}gl\sin\theta \tag{7.3}$$

where $\hat{J}$ and $\hat{m}$ are estimates of J and m, respectively. Under (7.3), we have

$$\dot{x}_2 = \frac{\hat{J}v + (m - \hat{m})gl\sin\theta}{J}$$

or

$$\ddot{y} = \frac{\hat{J}v + (m - \hat{m})gl\sin\theta}{J}.$$

Further application of $v = \ddot{y}_d - \alpha_2 e + \alpha_1 \dot{e}$ will not give

$$\ddot{e} + \alpha_1 \dot{e} + \alpha_2 e = 0.$$

Therefore the control law cannot guarantee $\lim_{t \to \infty} e = 0$.

Conclusion: Input-output linearization cannot achieve asymptotic tracking when there is uncertainty in the plant.

2 Sliding Control

2.1 Problem description

Given (7.1) and $y_d(t)$, let $e = y - y_d$ and define

$$s = e^{(n-1)} + \alpha_1 e^{(n-2)} + ... + \alpha_{n-1} e \tag{7.4}$$

where α_1, ...,α_{n-1} are such that

$$\lambda^{n-1} + \alpha_1 \lambda^{n-2} + ... + \alpha_{n-2}\lambda + \alpha_{n-1} \tag{7.5}$$

is a stable polynomial.

2.1 Problem description (cont.)

We call a nonlinear controller

$$u = k(x, y_d, ..., y_d^{(n)}) \tag{7.6}$$

a sliding controller if u achieves

$$\frac{1}{2}\frac{ds^2}{dt} \leq -\eta|s| \qquad \text{for some } \eta > 0 \tag{7.7}$$

Geometrically, $s = 0$ can be viewed as a surface in $\Re^n$ space (Fig. 7.2) and is called a *sliding surface*. The condition described by (7.7) is called *sliding condition*.

2.1 Problem description (cont.)

Roughly speaking, (7.7) states that the squared "distance" to the surface decreases along all system trajectories.

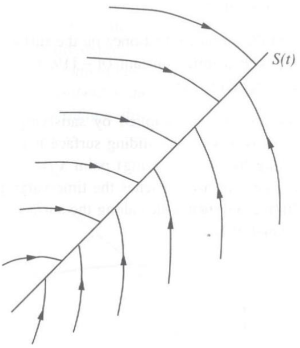


Figure 7.2 : The sliding condition

2.2 Interpretation of the sliding condition

Observation: If for some t_0, the trajectories of the closed-loop system $x(t)$ satisfies

$$s(t) = 0, \qquad t \geq t_0, \tag{7.8}$$

then (7.4) implies

$$\lim_{t \to \infty} e(t) = 0$$

since (7.5) is stable.

Next assume $s(0) \neq 0$. Without loss of generality, assume $s(0) > 0$, and note that (7.7) is equivalent to

$$\dot{s} \leq -\eta, \qquad s > 0 \tag{7.9}$$

$$\dot{s} \geq \eta, \qquad s < 0 \tag{7.10}$$

2.2 Interpretation of the sliding condition (cont.)

Assume $s(t) > 0$ for $0 \leq t < T$. Integrating (7.9) gives

$$\int_0^T \dot{s} d\tau \leq \int_0^T -\eta d\tau \qquad \text{whenever} \quad s > 0$$

$$\iff \quad s(T) - s(0) \leq -\eta T \quad \iff \quad s(T) \leq -\eta T + s(0).$$

Thus for some $T \leq \frac{s(0)}{\eta}$, $\lim_{t \to T} s(t) = 0$.

Similarly, if $s(0) < 0$, we have, for some $T \leq \frac{-s(0)}{\eta}$

$$\lim_{t \to T} s(t) = 0.$$

2.2 Interpretation of the sliding condition (cont.)

Therefore, for any $s(0)$, there exists some $T \leq \frac{|s(0)|}{\eta}$ such that

$$s(t) = 0, \quad t \geq T.$$

2.2 Interpretation of the sliding condition (cont.)

Conclusion: If a controller is such that the closed-loop system satisfies sliding condition (7.7) for some $\eta > 0$, then for any $x(0)$, the trajectories $x(t, x_0)$ will reach the sliding surface in $T \leq \frac{|s(0)|}{\eta}$, and then $x(t, x_0)$ will remain in $s(t) = 0$ for all $t \geq T$. Thus (7.4) and (7.5) guarantee

$$\lim_{t \to \infty} e(t) = 0.$$

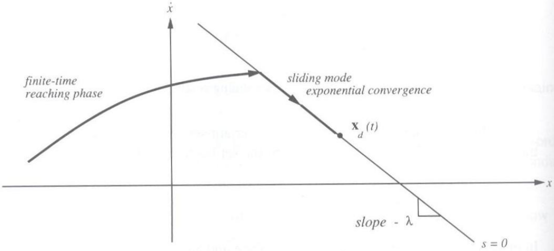


Figure 7.3 : Graphical interpretation of equations (7.3) and (7.5) ($n = 2$)

2.3 Achieving sliding condition

Step 1: Achieving

$$\dot{s} = v.$$

Since

$$\begin{aligned} \dot{s} &= e^{(n)} + \alpha_1 e^{(n-1)} + \ldots + \alpha_{n-1}\dot{e} \\ &= (y^{(n)} - y_d^{(n)}) + \alpha_1 e^{(n-1)} + \ldots + \alpha_{n-1}\dot{e} \\ &= (\alpha(x) + \beta(x)u - y_d^{(n)}) + \alpha_1 e^{n-1} + \ldots + \alpha_{n-1}\dot{e}, \end{aligned}$$

letting

$$u = \frac{v - \alpha(x) + y_d^{(n)} - \alpha_1 e^{(n-1)} - \ldots - \alpha_{n-1}\dot{e}}{\beta(x)}$$

gives $\dot{s} = v$.

2.3 Achieving sliding condition (cont.)

Step 2: Define a sign function $sgn(s)$ as follows

$$sgn(s) = \begin{cases} 1 & s > 0 \\ -1 & s < 0 \\ 0 & s = 0 \end{cases}$$

Let $v = -\phi(x) sgn(s)$ where $\phi(x)$ is a function satisfying

$$\phi(x) \geq \eta \qquad \forall x.$$

2.3 Achieving sliding condition (cont.)

Then

$$\dot{s} = -\phi(x) sgn(s)$$

which is equivalent to

$$\dot{s} = \begin{cases} -\phi(x) \leq -\eta & s > 0 \\ \phi(x) \geq \eta & s < 0 \end{cases}$$

or

$$\frac{1}{2}\frac{ds^2}{dt} \quad \leq \quad -\eta|s|.$$

Thus

$$u \quad = \quad \frac{-\phi(x)sgn(s) - \alpha(x) + y_d^{(n)} - \alpha_1 e^{(n-1)} - \ldots - \alpha_{n-1}\dot{e}}{\beta(x)} \quad (7.11)$$

achieves sliding condition.

2.4 Example

$$\ddot{y} = -1.5\dot{y}^2 \cos 3y + u$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -1.5x_2^2 \cos 3x_1 + u = \alpha(x) + \beta(x)u \\ y &= x_1 \end{aligned}$$

where $\alpha(x) = -1.5x_2^2 \cos x_1$, $\beta(x) = 1$. Given any $y_d(t)$, we want to achieve

$$\frac{1}{2}\frac{ds^2}{dt} \leq -\eta|s| \qquad \text{for} \quad \eta = 1$$

2.4 Example 2(cont.)

Solution: Since $n = 2$, define $s(t)$ as follows

$$s(t) = \dot{e}(t) + \alpha_1 e(t) \qquad \text{where} \quad \alpha_1 > 0.$$

Therefore by (7.11),

$$u = \frac{-\phi(x)sgn(s) - \alpha(x) + \ddot{y}_d - \alpha_1 \dot{e}}{\beta(x)} \tag{7.12}$$

where $\phi(x) \geq 1 \quad \forall x$.
Substituting $\phi(x) = 2$, $\alpha(x) = -1.5x_2^2 \cos 3x_1$, $\beta(x) = 1$ and $s = \dot{e} + \alpha_1 e = (\dot{y} - \dot{y}_d) + \alpha_1(y - y_d) = (x_2 - \dot{y}_d) + \alpha_1(x_1 - y_d)$ into (7.12) gives

$$\begin{aligned} u &= -2sgn(x_2 + \alpha_1 x_1 - (\dot{y}_d + \alpha_1 y_d)) + 1.5x_2^2 \cos x_1 + \ddot{y}_d - \alpha_1(x_2 - \dot{y}_d) \\ &= k(x, y_d, \dot{y}_d, \ddot{y}_d). \end{aligned}$$

2.5 Robustness property of sliding control

Consider

$$y^{(n)} = \alpha(x) + \beta(x)u.$$

Assume $\alpha(x)$ is not known exactly. Let

$$\alpha(x) = \hat{\alpha}(x) + \Delta\alpha(x)$$

where $\hat{\alpha}(x)$ is an estimation of $\alpha(x)$, and $\Delta\alpha(x)$ is the estimation error.

2.5 Robustness property of sliding control (cont.)

Assume the bound of $\Delta\alpha(x)$ is known, i.e. there exists a known function $F(x)$ such that

$$|\Delta\alpha(x)| \leq F(x), \qquad \forall x.$$

Let $\phi(x) = \eta + F(x)$. Then under the controller

$$u = \frac{-\phi(x)sgn(s) - \hat{\alpha}(x) + y_d^{(n)} - \alpha_1 e^{(n-1)} - ... - \alpha_{n-1}\dot{e}}{\beta(x)} \tag{7.13}$$

we have

$$y^{(n)} = -\phi(x)sgn(s) + \Delta\alpha(x) + y_d^{(n)} - \alpha_1 e^{(n-1)} - ... - \alpha_{n-1}\dot{e}$$

or

$$
\begin{aligned}
\dot{s} &= -\phi(x)sgn(s) + \Delta\alpha(x) \\
&= \begin{cases} -\phi(x) + \Delta\alpha(x) & s > 0 \\ \phi(x) + \Delta\alpha(x) & s < 0 \end{cases} \\
&= \begin{cases} -\eta - F(x) + \Delta\alpha(x) & s > 0 \\ \eta + F(x) + \Delta\alpha(x) & s < 0 \end{cases} \\
&= \begin{cases} -\eta - (F(x) - \Delta\alpha(x)) \leq -\eta & s > 0 \\ \eta + (F(x) + \Delta\alpha(x)) \geq \eta & s < 0 \end{cases}
\end{aligned}
$$

or

$$
\frac{1}{2}\frac{ds^2}{dt} \leq -\eta|s|.
$$

That is, even if $\alpha(x)$ is not known exactly, controller (7.13) can still achieve sliding condition by utilizing the estimation of $\alpha(x)$ in the controller.

2.5 Robustness property of sliding control (cont.)

Remark:

(i) Since the controller (7.13) can tolerate certain model uncertainty, we say that the controller is robust with respect to the model uncertainty.

(ii) Sliding controller can also handle model uncertainty associated with $\beta(x)$.

Example 3.

$$\ddot{y} = -a\dot{y}^2 \cos 3y + u \qquad \text{where} \quad 1 \le a \le 2$$

$$\Longleftrightarrow \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha(x) + \beta(x)u \\ y &= x_1 \end{aligned}$$

where $x_1 = y$, $x_2 = \dot{y}$, $\alpha(x) = -ax_2^2 \cos 3x_1$, $\beta(x) = 1$.

Objective: find $u = k(x, y_d, \dot{y}_d, \ddot{y}_d)$ so that

$$\frac{1}{2}\frac{ds^2}{dt} \le -\eta|s| \qquad \text{for some } \eta > 0$$

where $s = \dot{e} + \alpha_1 e$ with $e = y - y_d$ and $\alpha_1 > 0$.

2.5 Robustness property of sliding control (cont.)

Let $\hat{\alpha}(x) = -1.5x_2^2 \cos 3x_1$. Then $\alpha(x) = \hat{\alpha}(x) + \Delta\alpha(x)$
where $\Delta\alpha(x) = \alpha(x) - \hat{\alpha}(x) = (-a + 1.5)x_2^2 \cos 3x_1$.

Clearly $|\Delta\alpha(x)| \leq 0.5x_2^2 |\cos 3x_1| = F(x)$.

Let $\phi(x) = \eta + 0.5x_2^2 |\cos 3x_1|$. Then, by (7.13),

$$u = -\phi(x) sgn(s) - \hat{\alpha}(x) + \ddot{y}_d - \alpha_1 \dot{e} \tag{7.14}$$

achieves

$$\frac{1}{2}\frac{ds^2}{dt} \leq -\eta |s|.$$

2.6 Remarks

(i) $\alpha_1,\ldots,\alpha_{n-1}$ are such that $\lambda^{n-1} + \alpha_1\lambda^{n-2} + \ldots + \alpha_{n-2}\lambda + \alpha_{n-1}$ is stable.

For simplicity, we can always choose $\alpha_1,\ldots,\alpha_{n-1}$ such that

$$\lambda^{n-1} + \alpha_1\lambda^{n-2} + \ldots + \alpha_{n-2}\lambda + \alpha_{n-1} = (\lambda + \alpha)^{n-1} \qquad \text{for some } \alpha > 0.$$

For example, when $n = 3$, $\lambda^2 + \alpha_1\lambda + \alpha_2 = (\lambda + 2)^2 = \lambda^2 + 4\lambda + 4$

(ii) η is such that $T \leq \frac{|s(0)|}{\eta}$. The smaller η is, the larger T is.

(iii) Make $\phi(x)$ as small as possible but larger than $\eta + F(x)$ to reduce the control power and the chattering phenomenon.

2.6 Remarks (cont.)

(iv) Chattering phenomenon

Due to the presence of the function $sgn(s)$ in the control law, the control law is not continuous across $s(t) = 0$. Since the implementation of the control switching is imperfect in practice because, for instance, the switching is not instantaneous and the value of s is not known with infinite precision, the control law may lead to chattering phenomenon shown in Fig. 7.4. Figure 7.7 shows the control input and tracking performance under control law (7.14).

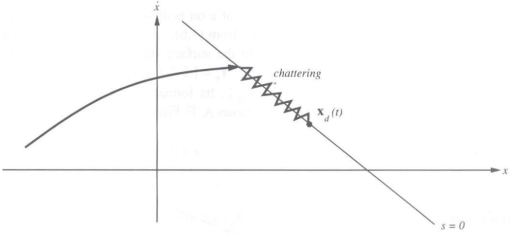


Figure 7.4 : Chattering as a result of imperfect control switchings

2.6 Remarks (cont.)

Consider again Example 3 given in the robustness property analysis, and assume that the desired trajectory is $y_d = \sin(\pi t/2)$. Figure 7.7 shows the tracking error and control law using the switched control law. We see that tracking performance is excellent, but is obtained at the price of high control chattering.

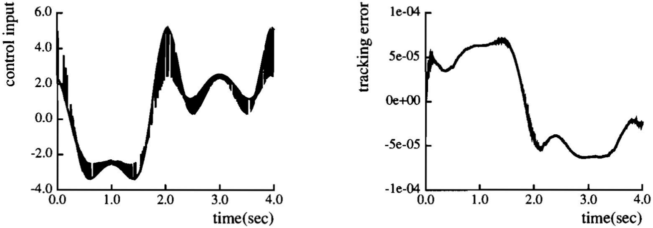
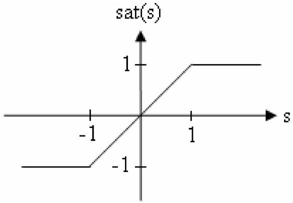


Figure 7.7 : Switched control input and resulting tracking performance

2.6 Remarks (cont.)

Chattering is undesirable in practice because it incites high frequency dynamics neglected in the course of modeling. To avoid chattering, we can introduce a saturation function $sat(s)$ as follows

$$sat(s) = \begin{cases} 1 & s > 1 \\ s & -1 \leq s \leq 1 \\ -1 & s < -1 \end{cases}$$



2.6 Remarks (cont.)

Replacing $sgn(s)$ by $sat(\frac{s}{\varepsilon})$ with $\varepsilon > 0$ in the control law gives an approximate control law as follows:

$$u = \frac{-\phi(x)sat(\frac{s}{\varepsilon}) - \hat{\alpha}(x) + y_d^{(n)} - \alpha_1 e^{(n-1)} - ... - \alpha_{n-1}\dot{e}}{\beta(x)}.$$

This control law cannot make $\lim_{t\to\infty} e(t) = 0$, but can make $e(t)$ sufficiently small by having ε sufficiently small.

It can be expected that the discontinuous control law u is suitably smoothed to achieve an optimal trade-off between control bandwidth and tracking precision, and chattering is avoided.

2.6 Remarks (cont.)

If we utilize the control law with saturation function, the control input and tracking performance are shown in Figure 7.8. It can be seen that the tracking performance, while not as "perfect" as above, is still very good, and is now achieved using a smooth control law.

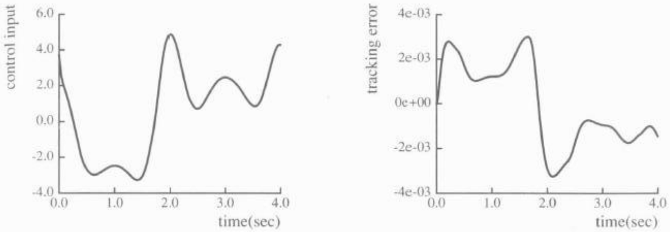


Figure 7.8 : Smooth control input and resulting tracking performance

Exercises

Consider the single-input, single output system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \vdots &= \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \alpha(x) + \beta(x)u \\ y &= x_1 \end{aligned}$$

where $x = (x_1, \cdots, x_{n-1}, x_n)^T$, $\alpha(x)$ and $\beta(x)$ are continuous for all $x \in R^n$, and are not known exactly. Assume there exist known functions $\hat{\alpha}(x)$ and $F(x)$ such that, for all x, $|\alpha(x) - \hat{\alpha}(x)| \leq F(x)$. Also, assume there exist two known positive real numbers b_{min} and b_{max} such that

$$b_{min} \leq \beta(x) \leq b_{max} \quad \text{for all } x \tag{7.15}$$

Exercises (cont.)

(a) Define the estimate $\hat{\beta}$ of $\beta(x)$ by $\hat{\beta} = (b_{min} b_{max})^{1/2}$ and let $b = (b_{max}/b_{min})^{1/2}$. Show that

$$b^{-1} \leq \frac{\hat{\beta}}{\beta(x)} \leq b \quad \text{for all } x$$

(b) Show that

$$\left| \frac{\hat{\beta}}{\beta(x)} - 1 \right| \leq b - 1 \tag{7.16}$$

(c) Given any smooth time function $y_d(t)$, $t \geq 0$, let $e = y - y_d$ and $s = e^{(n-1)} + \alpha_1 e^{(n-2)} + \cdots + \alpha_{n-1} e$ for some real number $\alpha_1, \ \cdots, \ \alpha_{n-1}$. Show that, for any $\eta > 0$, $u = \hat{\beta}^{-1}[\hat{u} - \phi(x) sgn(s)]$, where $\hat{u} = -\hat{\alpha}(x) + y_d^{(n)} - \alpha_1 e^{(n-1)} - \cdots - \alpha_{n-1}\dot{e}$ and $\phi(x) \geq b(F(x) + \eta) + (b-1)|\hat{u}|$ for all x, is such that

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|$$

Exercises (cont.)

Solution:

(a) Since

$$b_{min} \le \beta(x) \le b_{max},$$

we have

$$\frac{1}{b_{max}} \le \frac{1}{\beta(x)} \le \frac{1}{b_{min}}.$$

Then

$$\frac{(b_{min}b_{max})^{\frac{1}{2}}}{b_{max}} \le \frac{\hat{\beta}}{\beta(x)} \le \frac{(b_{min}b_{max})^{\frac{1}{2}}}{b_{min}},$$

which is equivalent to

$$b^{-1} \le \frac{\hat{\beta}}{\beta(x)} \le b. \tag{7.17}$$

Exercises (cont.)

(b) By definition, $b \geq 1$. Thus,

$$\left(\frac{1}{b} - 1\right) - (1 - b) = \frac{b^2 - b - b + 1}{b} = \frac{(b-1)^2}{b} \geq 0,$$

which gives (7.16) since

$$1 - b \leq \frac{1}{b} - 1 \leq \frac{\hat{\beta}}{\beta(x)} - 1 \leq b - 1. \tag{7.18}$$

Exercises (cont.)

(c) By (7.16), we have

$$b(F(x)+\eta)+(b-1)|\hat{u}| \geq \frac{\hat{\beta}}{\beta(x)}(F(x)+\eta)+\left|\frac{\hat{\beta}}{\beta(x)}-1\right||\hat{u}|. \tag{7.19}$$

Now consider

$$\begin{aligned}
\dot{s} &= e^{(n)}+\alpha_1 e^{(n-1)}+\cdots+\alpha_{n-1}\dot{e} \\
&= \hat{\alpha}(x)+\Delta(x)+\beta(x)u-y_d^{(n)}+\alpha_1 e^{(n-1)}+\cdots+\alpha_{n-1}\dot{e} \\
&= \Delta(x)+\frac{\beta(x)}{\hat{\beta}}(\hat{u}-\phi(x)sgn(s))-\hat{u} \\
&= \Delta(x)+\left(\frac{\beta(x)}{\hat{\beta}}-1\right)\hat{u}-\frac{\beta(x)}{\hat{\beta}}\phi(x)sgn(s).
\end{aligned}$$

Exercises (cont.)

For $s > 0$,

$$\begin{aligned}
\dot{s} &= \Delta(x) + \left(\frac{\beta(x)}{\hat{\beta}} - 1\right)\hat{u} - \frac{\beta(x)}{\hat{\beta}}\phi(x) \\
&\leq F(x) + \left|\frac{\beta(x)}{\hat{\beta}} - 1\right||\hat{u}| - \frac{\beta(x)}{\hat{\beta}}\left(\frac{\hat{\beta}}{\beta(x)}(\eta + F(x)) + \left|\frac{\hat{\beta}}{\beta(x)} - 1\right||\hat{u}|\right) \\
&= -\eta, \quad s > 0. \qquad (7.20)
\end{aligned}$$

For $s < 0$,

$$\begin{aligned}
\dot{s} &= \Delta(x) + \left(\frac{\beta(x)}{\hat{\beta}} - 1\right)\hat{u} + \frac{\beta(x)}{\hat{\beta}}\phi(x) \\
&\geq \Delta(x) - \left|\frac{\beta(x)}{\hat{\beta}} - 1\right||\hat{u}| + \frac{\beta(x)}{\hat{\beta}}\left(\frac{\hat{\beta}}{\beta(x)}(\eta + F(x)) + \left|\frac{\hat{\beta}}{\beta(x)} - 1\right||\hat{u}|\right) \\
&\geq \eta, \quad s < 0. \qquad (7.21)
\end{aligned}$$

Exercises (cont.)

(7.20) and (7.21) imply

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|,$$

which completes the proof.