Chapter 8 Adaptive Control

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Outline

1 Introduction

Many dynamic systems to be controlled have constant or slowly-varying uncertain parameters. For instance,

- Robot manipulators may carry large objects with unknown inertial parameters;
- Power systems may be subjected to large variations in loading conditions;
- Fire-fighting aircraft may experience considerable mass changes as they load and unload large quantities of water.

Adaptive control is an approach to the control of such systems.

1 Introduction

- The basic idea in adaptive control is to estimate the uncertain plant parameters (or, equivalently, the corresponding controller parameters) on-line based on the measured system signals, and use the estimated parameters in the control input computation.
- An adaptive control system can be regarded as a control system with on-line parameter estimation.
- An adaptive control law, whether developed for linear plants or for nonlinear plants, is nonlinear, the analysis and design of an adaptive control system is thus intimately connected with Lyapunov theory.

1 Introduction

Why adaptive control?

- Robot manipulation: Robots have to manipulate loads of various sizes, weights, and mass distributions (Figure 8.1). It is very restrictive to assume that the inertial parameters of the loads are well known before a robot picks them up and moves them away. If controllers with constant gains are used and the load parameters are not accurately known, robot motion can be either inaccurate or unstable. Adaptive control, on the other hand, allows robots to move loads of unknown parameters with high speed and high accuracy;

1 Introduction

Why adaptive control?

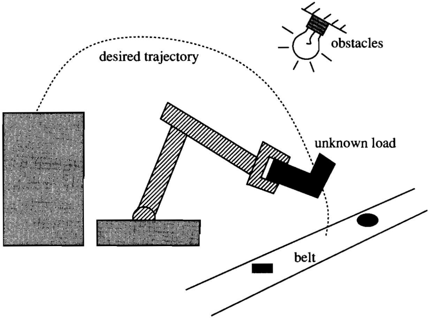

Figure 8.1 : A robot carrying a load of uncertain mass properties

- Ship steering;
- Aircraft control;
- Process control.

2 Basic Concepts in Adaptive Control

An adaptive controller differs from an ordinary controller in that the controller parameters are variable, and there is a mechanism for adjusting these parameters online based on signals in the system.

There are two main approaches for constructing adaptive controllers:

1. model-reference adaptive control method;
2. self-tuning method.

3 How To Design Adaptive Controllers?

- In conventional (non-adaptive) control design, a controller structure (e.g., pole placement) is chosen first, and parameters of the controller are then computed based on the known parameters of the plant.
- In adaptive control, the major difference is that the plant parameters are unknown, so that the controller parameters have to be provided by an adaptation law which can learn the unknown parameters online.
- As a result, the adaptive control design is more involved, with the additional needs of choosing an adaptation law and proving the stability of the system with adaptation.

3 How To Design Adaptive Controllers?

The design of an adaptive controller usually involves the following three steps:

1. choose a control law containing variable parameters;
2. choose an adaptation law for adjusting those parameters;
3. analyze the convergence properties of the resulting control system.

For uncertain linear systems, there are two main adaptive control methods, namely, model reference adaptive control (MRAC) and the self-tuning approach.

3 Adaptive Control of Nonlinear Systems

Recently, adaptive control has been successfully developed for some important classes of nonlinear control problems. Such problems usually satisfy the following conditions:

1. the nonlinear plant dynamics can be linearly parameterized
2. the full state is measurable
3. nonlinearities can be canceled stably (i.e., without unstable hidden modes or dynamics) by the control input if the parameters are known

3.1 Problem Statement

Consider $n^{th}-$ order nonlinear systems in companion form

$$y^{(n)} + \sum_{i=1}^{m} \alpha_i f_i(x,t) = bu \tag{8.1}$$

where $x = [y \ \dot{y} \ \cdots \ y^{(n-1)}]^T$ is the state vector, f_i are known nonlinear functions of the state and time, and the parameters α_i and b are unknown constants. We assume that the state is measured, and that the sign of b is known. One example of such dynamics is

$$m\ddot{x} + cf_1(\dot{x}) + kf_2(x) = u \tag{8.2}$$

which represents a mass-spring-damper system with nonlinear friction and nonlinear damping.

3.1 Problem Statement

The objective of the adaptive control design is to make the output asymptotically track a desired output $y_d(t)$ despite the parameter uncertainty. To facilitate the adaptive controller derivation, let us rewrite equation (8.1) as

$$a_0 y^{(n)} + \sum_{i=1}^{m} a_i f_i(x,t) = u \tag{8.3}$$

by dividing both sides by the unknown constant b, where $a_0 = 1/b$ and $a_i = \alpha_i / b$.

3.2 Design of Control Law

Similarly to the sliding control approach of chapter 7, let us define a combined error

$$s = e^{(n-1)} + \alpha_1 e^{(n-2)} + \cdots + \alpha_{n-1} e \tag{8.4}$$

where $e = y - y_d$ is the output tracking error and $\alpha_1, \ \cdots, \ \alpha_{n-1}$ are real numbers such that $\Delta(\lambda) = \lambda^{(n-1)} + \alpha_1 \lambda^{(n-2)} + \cdots + \alpha_{n-1}$ is a stable (Hurwitz) polynomial. Note that s can be rewritten as

$$s = y^{(n-1)} - y_r^{(n-1)} \tag{8.5}$$

where $y_r^{(n-1)}$ is defined as

$$y_r^{(n-1)} = y_d^{(n-1)} - \alpha_1 e^{(n-2)} - \cdots - \alpha_{n-1} e \tag{8.6}$$

3.2 Design of Control Law

Consider the control law

$$u = a_0 f_0(x,t) - ks + \sum_{i=1}^{m} a_i f_i(x,t) \tag{8.7}$$

where k is a constant of the same sign as a_0, and $f_0(x,t)$ is the derivative of $y_r^{(n-1)}$, i.e.,

$$f_0(x,t) = y_d^{(n)} - \alpha_1 e^{(n-1)} - \cdots - \alpha_{n-1}\dot{e} \tag{8.8}$$

Note that $f_0(x,t)$, the so-called "reference" value of $y^{(n)}$, is obtained by modifying $y_d^{(n)}$ according to the tracking errors.
If the parameters are all known, this choice leads to the tracking error dynamics

$$a_0\dot{s} + ks = 0 \tag{8.9}$$

and therefore gives exponential convergence of s, which, in turn, guarantees the convergence of e since $s = 0$ is a stable system.

3.3 Design of Adaptation Law

If the parameters are unknown, then the control law (8.7) is replaced by

$$u = \hat{a}_0 f_0(x,t) - ks + \sum_{i=1}^{m} \hat{a}_i f_i(x,t) \qquad (8.10)$$

where a_i have been replaced by their estimated value $\hat{a}_i$. The tracking error from this control law can be easily shown to be

$$a_0 \dot{s} + ks = -\sum_{i=0}^{m} \tilde{a}_i f_i(x,t) \qquad (8.11)$$

where $\tilde{a}_i = a_i - \hat{a}_i$, $i = 0, 1, \cdots, m$.

3.3 Design of Adaptation Law

Choose the following adaptation law

$$\dot{\hat{a}}_i = -\gamma_i sgn(a_0)\ s\ f_i,\ i = 0, \cdots, m \tag{8.12}$$

where $\gamma_i > 0$. The overall adaptive control law consists of (8.7) and (8.12).

Theorem 8.1: Suppose $f_i(x, t)$, $i = 1, \cdots, m$, and $y_d(t), \dot{y}_d(t), \cdots, y_d^{(n)}(t)$ are bounded. Then, for any initial condition, the solution of the closed-loop system composed of the plant (8.3) and the control law (8.7) and (8.12) is bounded, and $\lim_{t\to\infty} e(t) = 0$.

3.4 Stability Analysis

Proof: Consider the Lyapunov function candidate

$$V = \frac{1}{2} sgn(a_0) a_0 s^2 + \frac{1}{2} \sum_{i=0}^{m} \gamma_i^{-1} \tilde{a}_i^2 \tag{8.13}$$

whose derivative along the solution of the closed-loop system is

$$\begin{aligned} \dot{V} &= sgn(a_0) a_0 s \dot{s} + \sum_{i=0}^{m} \gamma_i^{-1} \tilde{a}_i \dot{\hat{a}}_i \\ &= sgn(a_0) s(-ks - \sum_{i=0}^{m} \tilde{a}_i f_i(x,t)) + \sum_{i=0}^{m} \gamma_i^{-1} \tilde{a}_i \dot{\hat{a}}_i \\ &= -|k| s^2 - \sum_{i=0}^{m} \tilde{a}_i (sgn(a_0) s f_i(x,t) - \gamma_i^{-1} \dot{\hat{a}}_i) = -|k| s^2 \leq 0 \end{aligned} \tag{8.14}$$

As a result, s and all $\tilde{a}_i$ are bounded. We will further show $\lim_{t \to \infty} s(t) = 0$. For this purpose, note that

$$\ddot{V} = -2|k| s \dot{s} \tag{8.15}$$

3.4 Stability Analysis

Since (8.4) can be viewed as a stable $(n-1)^{th}$ order linear differential equation with bounded input s, the solution $(e, \dot{e}, \cdots, e^{(n-1)})$ of (8.4) is bounded. By assumption, $y_d^{(n)}$ is bounded. Thus, from (8.8), $f_0(x,t)$ is bounded.
Since s and $\tilde{a}_i$ have been shown to be bounded, and, by assumption, $f_i(x,t)$, $i = 1, \cdots, m$, are bounded, from (8.11), $\dot{s}$ is bounded. Thus, $\dot{V}$ is uniformly continuous. By Babalat's Lemma, $\lim_{t\to\infty} s(t) = 0$.
Finally, note that (8.4) can be viewed as a stable $(n-1)^{th}$ order linear differential equation with the input s satisfying $\lim_{t\to\infty} s(t) = 0$. Thus, the solution $(e, \dot{e}, \cdots, e^{(n-1)})$ of (8.4) tends to zero asymptotically. Since $y_d(t), \dot{y}_d(t), \cdots, y_d^{(n-1)}(t)$ are bounded, $x(t)$ is bounded.

Exercise: Show in detail that (i) $f_0(x,t)$ is bounded, and (ii) the solution $(e, \dot{e}, \cdots, e^{(n-1)})$ of (8.4) tends to zero asymptotically.

Remark 8.1:

(i) γ_i is called the adaptation gain. A smaller gain leads to slower convergence, but a too large gain may incur poor transient response such as large oscillations.
(ii) $sgn(a_0)$ determines the direction of the search.
(iii) $\tilde{a}_i$ may not tend to zero. The convergence of $\tilde{a}_i$ is related to the concept of persistence of excitation.
(iv) If a_0 is known, then the control law can be simplified to

$$\begin{aligned} u &= a_0 f_0(x,t) - ks + \sum_{i=1}^{m} \hat{a}_i f_i(x,t) \\ \dot{\hat{a}}_i &= -\gamma_i sgn(a_0)\ s\ f_i,\ i = 1, \cdots, m \end{aligned}$$

(v) If f_i, $i = 0, 1, \cdots, m$, are independent of t, then $f_i(x)$ are bounded since x is bounded.

4 Adaptive Control of Euler-Lagrange Systems[1]

Euler-Lagrange Systems

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u \tag{8.16}$$

where $q, u \in \mathbb{R}^n$ are the generalized coordinate and force vector, respectively, $C(q,\dot{q})\dot{q} \in \mathbb{R}^n$ with $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ represents the Coriolis and centripetal force vector and $G(q) \in \mathbb{R}^n$ denotes the gravity force vector.

Some properties of system (8.16) are listed as follows.

Property 1: For all $q \in \mathbb{R}^n$, $M(q) \geq k_{\underline{m}} I_n$ for some $k_{\underline{m}} > 0$.

Remark 8.2: By Property 1, $k_{\underline{m}} M(q)^{-1} \leq I_n$, and hence $M(q)^{-1}$ is bounded for all $q \in \mathbb{R}^n$.

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[1]This Section is based on Chapter 5 of the book: He Cai, Youfeng Su, and Jie Huang "Cooperative Control of Multi-agent Systems: Distributed Observer and Distributed Internal Model Approaches," Springer, 2022.

The vector $C(q,\dot{q})\dot{q}$ is such that

$$C(q,\dot{q})\dot{q} = \dot{M}(q)\dot{q} - \frac{1}{2}\frac{\partial(\dot{q}^T M(q)\dot{q})}{\partial q}$$

Thus, the matrix $C(q,\dot{q})$ is not uniquely defined. If one adopts the following so-called the Christofflel symbols:

$$C_{ij} = \frac{1}{2}\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_k$$

where C_{ij} and M_{ij} denote the elements of $C(q,\dot{q})$ and $M(q)$ on the i^{th} row and j^{th} column, respectively, then the following property holds.

Property 2: The matrix $\dot{M}(q) - 2C(q,\dot{q})$ is anti-symmetric[2].

[2] A matrix $A \in \mathbb{R}^{n\times n}$ is anti-symmetric if $A + A^T = 0$. If A is anti-symmetric, then, for any $x \in \mathbb{R}^n$, $x^T A x = 0$.

Property 3:

$$M(q)x + C(q,\dot{q})y + G(q) = Y(q,\dot{q},x,y)\Theta,\ \forall x,y \in \mathbb{R}^n$$

where $Y(q,\dot{q},x,y) \in \mathbb{R}^{n\times l}$ is a known regression matrix and $\Theta \in \mathbb{R}^l$ is a nonzero constant vector consisting of the uncertain parameters. Moreover, if $q(t)$, $\dot{q}(t)$, $x(t)$, and $y(t)$ are bounded over $[0,\infty)$, then $C(q(t),\dot{q}(t))$ and $Y(q(t),\dot{q}(t),x(t),y(t))$ are also bounded over $[0,\infty)$.

The Desired Generalized Position Vector $q_0(t)$: It is assumed that q_0 is twice differentiable over $t \geq 0$, and $q_0(t)$, $\dot{q}_0(t)$ and $\ddot{q}_0(t)$ are bounded over $[0, \infty)$.

Trajectory Tracking Problem: Given system (8.16), design a control law of the following form:

$$u = f(q, \dot{q}, q_0, \dot{q}_0, \ddot{q}_0, \hat{\Theta})$$
$$\dot{\hat{\Theta}} = g(q, \dot{q}, q_0, \dot{q}_0, \ddot{q}_0)$$

where $\hat{\Theta}$ is the estimate of Θ, such that

$$\lim_{t\to\infty} (q(t) - q_0(t)) = 0, \quad \lim_{t\to\infty} (\dot{q}(t) - \dot{q}_0(t)) = 0.$$

To introduce our specific control law, let $e = q - q_0$, and

$$s = \dot{e} + \alpha e = \dot{q} - \zeta \tag{8.17}$$

where

$$\zeta = \dot{q}_0 - \alpha(q - q_0) \tag{8.18}$$

with $\alpha > 0$.
Let $Y = Y(q, \dot{q}, \dot{\zeta}, \zeta)$. Then our control law is given as follows:

$$u = -Ks + Y\hat{\Theta} \tag{8.19a}$$

$$\dot{\hat{\Theta}} = -\Lambda^{-1}Y^T s \tag{8.19b}$$

where $K \in \mathbb{R}^{n\times n}$, $\Lambda \in \mathbb{R}^{l\times l}$ are positive definite gain matrices.
Remark 8.3: The control law (8.19) is a type of adaptive control law in which (8.19b) provides the estimate $\hat{\Theta}$ of the unknown parameter vector Θ.
Theorem 8.2: Given system (8.16), the Trajectory Tracking Problem is solvable by the control law (8.19).

Proof: Substituting (8.19a) into (8.16) gives

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = -Ks + Y\hat{\Theta} \tag{8.20}$$

and subtracting $Y\Theta$ on both sides of (8.20) gives

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) - Y\Theta = -Ks + Y\hat{\Theta} - Y\Theta. \tag{8.21}$$

Substituting $Y(q,\dot{q},\dot{\zeta},\zeta)\Theta = M(q)\dot{\zeta} + C(q,\dot{q})\zeta + G(q)$, which is due to Property 3, into the left hand side of (8.21) gives

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) - M(q)\dot{\zeta} - C(q,\dot{q})\zeta - G(q) = -Ks + Y\tilde{\Theta}$$

where $\tilde{\Theta} = \hat{\Theta} - \Theta$. Then, we have

$$M(q)\dot{s} + C(q,\dot{q})s + Ks = Y\tilde{\Theta}. \tag{8.22}$$

Let

$$V = \frac{1}{2}(s^T M(q)s + \tilde{\Theta}^T \Lambda \tilde{\Theta}).$$

Noting $s^T(\dot{M}(q) - 2C(q,\dot{q}))s \equiv 0$ since $\dot{M}(q) - 2C(q,\dot{q})$ is anti-symmetric gives

$$\begin{aligned}
\dot{V} &= s^T M(q)\dot{s} + \frac{1}{2}s^T \dot{M}(q)s + \tilde{\Theta}^T \Lambda \dot{\tilde{\Theta}} \\
&= s^T(-C(q,\dot{q})s - Ks + Y\tilde{\Theta}) + \frac{1}{2}s^T \dot{M}(q)s + \tilde{\Theta}^T \Lambda \dot{\tilde{\Theta}} \\
&= -s^T Ks + \frac{1}{2}s^T(\dot{M}(q) - 2C(q,\dot{q}))s + s^T Y\tilde{\Theta} - \tilde{\Theta}^T \Lambda \Lambda^{-1} Y^T s \\
&= -s^T Ks + s^T Y\tilde{\Theta} - \tilde{\Theta}^T \Lambda \Lambda^{-1} Y^T s \\
&= -s^T Ks \leq 0.
\end{aligned} \tag{8.23}$$

Since $M(q)$ is positive definite by Property 1, $V(t)$ is lower bounded for all $t \geq 0$. By (8.23), $\dot{V} \leq 0$, which implies that $\lim_{t\to\infty} V(t)$ exists and is finite, and thus $s(t)$ and $\tilde{\Theta}(t)$ are bounded over $[0, \infty)$.

Next, we will show that $\dot{s}(t)$ is bounded over $[0, \infty)$. For this purpose, note that (8.17) can be viewed as a stable first order system in e with a bounded input s. Thus e and $\dot{e}$ are bounded. Since q_0, $\dot{q}_0$ are bounded, $q(t)$ and $\dot{q}(t)$ are also bounded over $[0, \infty)$.
From (8.17), we have

$$\dot{s} = \ddot{q} - \dot{\zeta}.$$

By (8.18),

$$\dot{\zeta} = \ddot{q}_0 - \alpha(\dot{q} - \dot{q}_0). \quad (8.24)$$

Therefore, $\zeta(t)$ and $\dot{\zeta}(t)$ are also bounded over $[0, \infty)$, which in turn implies that $Y(t)$ is bounded over $[0, \infty)$ by Property 3. Then, by (8.22), Remark 8.2 and Property 3 again, $\dot{s}(t)$ is bounded over $[0, \infty)$.

Since $\dot{s}(t)$ is bounded over $[0, \infty)$, so is $\ddot{V}(t)$. Then, by Barbalat's Lemma, $\lim_{t\to\infty} s(t) = 0$. By (8.17), e satisfies

$$\dot{e} + \alpha e = s \tag{8.25}$$

which is a stable first order system in e with the input s decaying to zero. Thus, $\lim_{t\to\infty} e(t) = 0$. The proof is thus complete.

Exercise: Show that the solution of (8.25) is bounded if $s(t)$ is bounded, and $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} \dot{e}(t) = 0$ if $\lim_{t\to\infty} s(t) = 0$.

Example: Shown in Fig. 1 is a three-link cylindrical robot arms whose motion equations[3] are described by (8.16) where $q = \text{col}(\theta, h, r)$ and

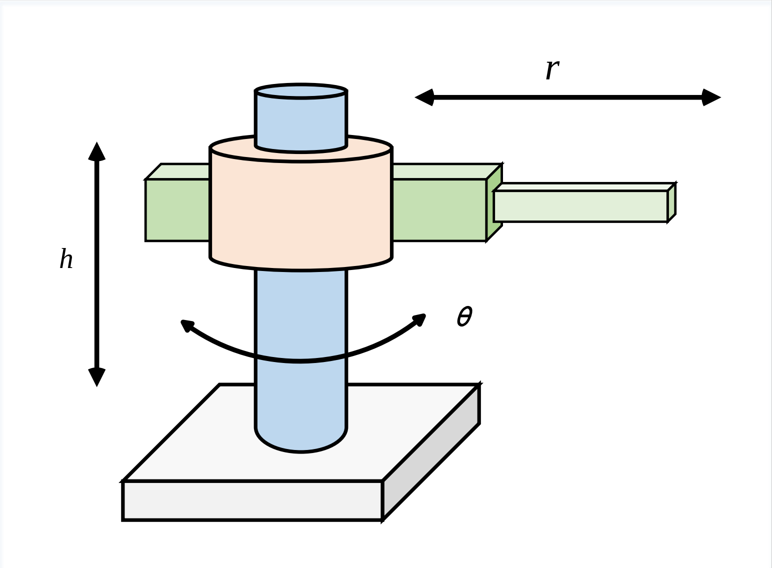
$$M(q) = \begin{bmatrix} J + m_2 r^2 & 0 & 0 \\ 0 & m_1 + m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix}$$

$$C(q, \dot{q}) = \begin{bmatrix} m_2 r \dot{r} & 0 & m_2 r \dot{\theta} \\ 0 & 0 & 0 \\ -m_2 r \dot{\theta} & 0 & 0 \end{bmatrix}, \quad G(q) = \begin{bmatrix} 0 \\ (m_1 + m_2) g \\ 0 \end{bmatrix}$$

where J is the moment of inertia of the base link, and m_1 and m_2 are the masses of the vertical link and the horizontal link, respectively. The values of J, m_1, m_2 are unknown. The actual values of the unknown parameters are $J = 1\ \text{kg}\cdot\text{m}^2$, $m_1 = 2\ \text{kg}$, $m_2 = 3\ \text{kg}$.

[3] The detailed derivation of the equations can be found in Lewis FL, Dawson MD, Abdallah TC (2004) Robot Manipulator Control: Theory and Practice. 2nd edition, Marcel Dekker, New York.

Fig. 1: Three-Link Cylindrical Robot Arm



Let $x = \text{col}(x_1, x_2, x_3)$ and $y = \text{col}(y_1, y_2, y_3)$. Then

$$\begin{aligned}
& M(q)x + C(q, \dot{q})y + G(q) \\
= & \begin{bmatrix} (J + m_2 r^2)x_1 + m_2 r\dot{r}y_1 + m_2 r\dot{\theta}y_3 \\ (m_1 + m_2)x_2 + (m_1 + m_2)g \\ m_2 x_3 - m_2 r\dot{\theta}y_1 \end{bmatrix} \\
= & \begin{bmatrix} x_1 & 0 & r^2 x_1 + r\dot{r}y_1 + r\dot{\theta}y_3 \\ 0 & x_2 + g & x_2 + g \\ 0 & 0 & x_3 - r\dot{\theta}y_1 \end{bmatrix} \cdot \begin{bmatrix} J \\ m_1 \\ m_2 \end{bmatrix} \\
\triangleq & Y(q, \dot{q}, x, y)\Theta.
\end{aligned}$$

where

$$Y(q,\dot{q},x,y) = \begin{bmatrix} x_1 & 0 & r^2x_1 + r\dot{r}y_1 + r\dot{\theta}y_3 \\ 0 & x_2 + g & x_2 + g \\ 0 & 0 & x_3 - r\dot{\theta}y_1 \end{bmatrix}$$

$$\Theta = \begin{bmatrix} J \\ m_1 \\ m_2 \end{bmatrix}.$$

By Theorem 8.2, we can synthesize an adaptive control law of the form (8.19) with the design parameters being $\alpha = 10$, $K = 10I_3$, $\Lambda = 0.2I_3$. To evaluate the performance of the control law (8.19), let the desired generalized position vector $q_0(t) = \text{col}(\pi/6, 0.1\sin 4t, 0.4\cos 4t)$. Then, q_0, $\dot{q}_0$ and $\ddot{q}_0$ are bounded.
Let us evaluate the control law (8.19) with the initial values being given by $q_i(0) = 0$, $\dot{q}_i(0) = 0$, $\hat{\Theta}_i(0) = 0$. Figs. 2 and 3 show the tracking performance of the position and velocity for each link, respectively. It is observed that all the tracking errors tend to zero asymptotically.

Fig. 2: Tracking Performance of the Position

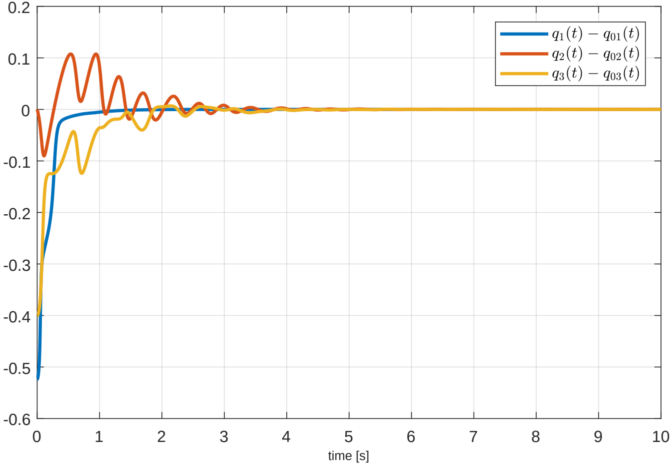
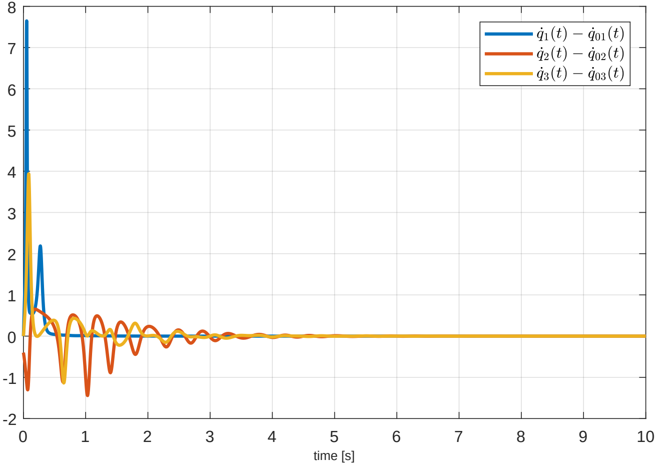


Fig. 3: Tracking Performance of the Velocity



Thank you!

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