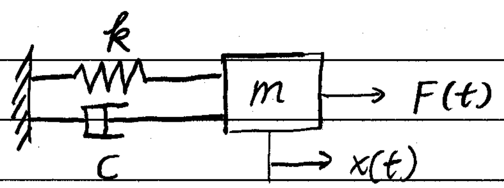
Vibration

- Any motion that repeats itself after an interval of time is called vibration or oscillation.

- The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the degree of freedom of the system.

Single-Degree-of-Freedom (SDOF) Systems



		Forces
k	spring	"restoring"
c	dashpot	"resisting"
m	mass	"inertial"

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad (1)$$

- Free Vibration of Undamped Systems:

Assume zero damping and external forces,

$$\ddot{x} + \omega_n^2 x = 0 \qquad (2)$$

where $\omega_n = \sqrt{\frac{k}{m}}$ (3)

: Harmonic Oscillator

The solution of eq. (2) is

$$x(t) = A\cos(\omega_n t - \phi) \qquad (4)$$

A : amplitude

ϕ : phase angle

ω_n : natural frequency (rad/sec)

To find A and ϕ, need <u>initial conditions</u>.

let $x(0) \triangleq x_0$, $\dot{x}(0) \triangleq v_0$

$$\Rightarrow \quad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \qquad (5)$$

$$\phi = \tan^{-1} \frac{v_0}{x_0 \omega_n} \qquad (6)$$

The time necessary to complete one cycle of motion defines the <u>period</u>

$$T = \frac{2\pi}{\omega_n} \quad \text{seconds} \qquad (7)$$

Natural frequency

$$f_n = \frac{1}{T} = \frac{\omega_n}{2\pi} \quad Hz \qquad (8)$$

where Hz denotes <u>hertz</u> [$1\,Hz$ = 1 cycle per second]

- Free Vibration of Damped Systems:

Let $F(t)=0$ and divide through by m, eq (1) becomes

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = 0 \qquad (9)$$

where
$$\zeta = \frac{c}{2m\omega_n} \qquad (10)$$

: damping ratio

I. $0<\zeta<1$ (underdamped)

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \qquad (11)$$

where
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \qquad (12)$$

: frequency of damped free vibration

Solve for the amplitude and phase angle from ICs:

$$A = \sqrt{x_0^2 + (\zeta\omega_n x_0 + v_0)^2/\omega_d^2} \qquad (13)$$

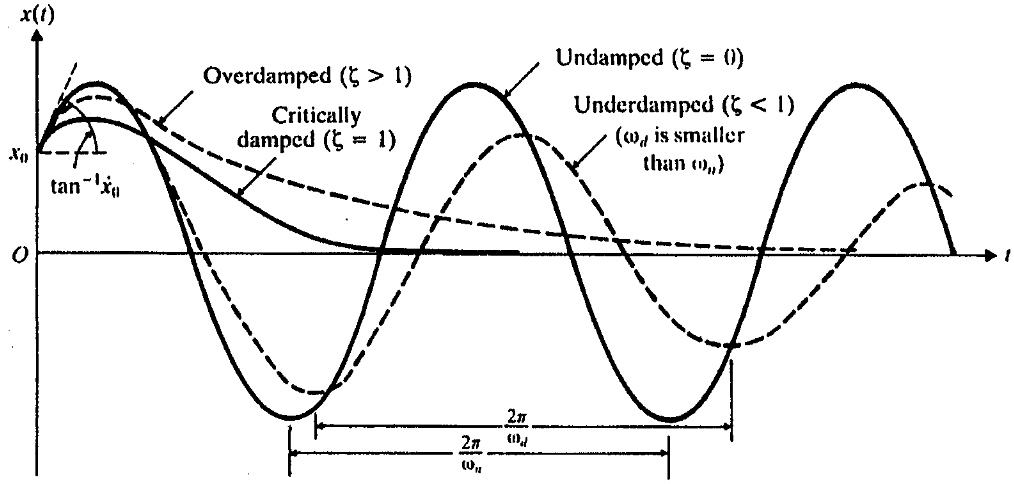
$$\phi = \tan^{-1}\left(\frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d}\right) \qquad (14)$$

II $\zeta=1$ (critical damping)

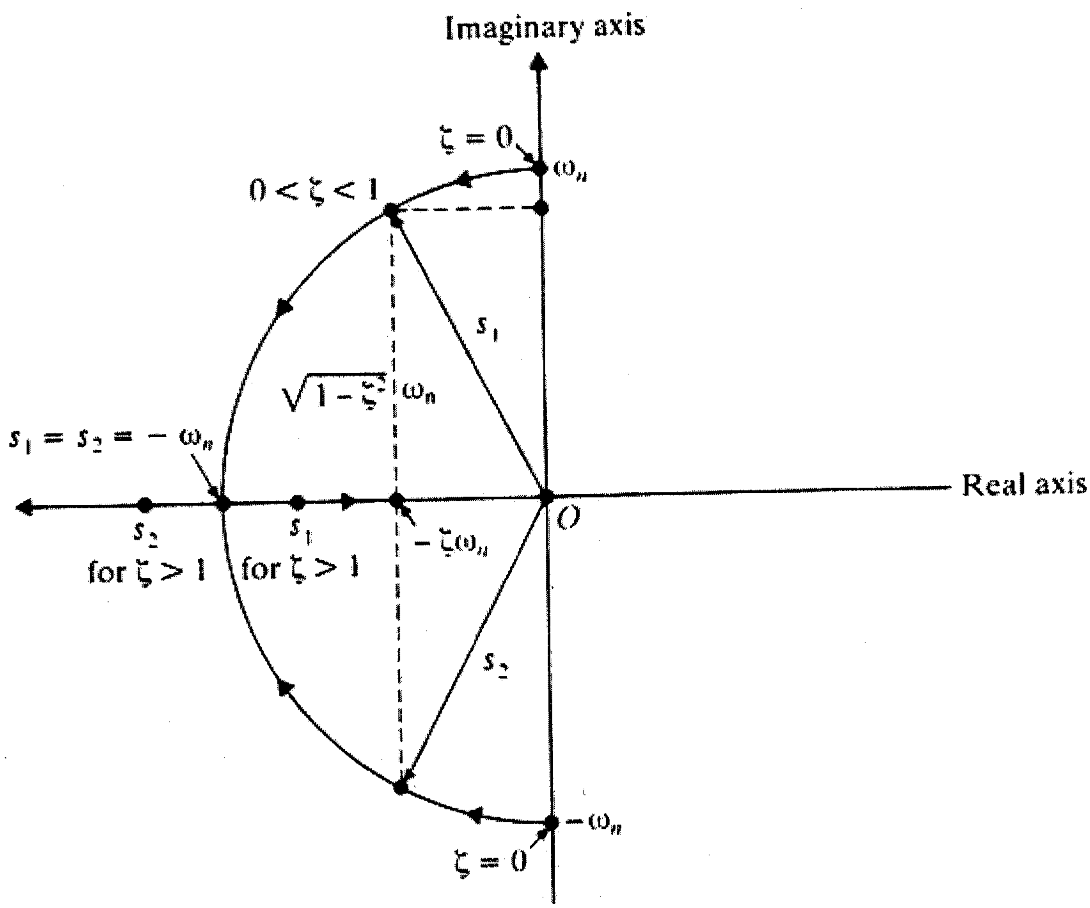
$$c_c = 2m\omega_n \qquad (15)$$

: critical damping coefficient

III. $\zeta>1$ (overdamped)



Comparison of motions with different types of damping



Locus of s_1 and s_2

- Response to Harmonic Excitations

Excitation force $F(t) = F_0 \cos\omega t$ (16)

where F_0 is the magnitude

ω is the excitation (or forcing) frequency

Equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t \quad (17)$$

$$\Rightarrow \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos\omega t \quad (18)$$

where $f_0 = \dfrac{F_0}{m}$

The particular solution of Eq. (17) can be expressed as

$$x_p(t) = X \cos(\omega t - \phi) \quad (19)$$

By substituting eq. (19) into eq. (17),

$$X\left[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)\right] = F_0 \cos\omega t \quad (20)$$

Using the trigonometric relations

$$\cos(\omega t - \phi) = \cos\omega t \cos\phi + \sin\omega t \sin\phi$$

$$\sin(\omega t - \phi) = \sin\omega t \cos\phi - \cos\omega t \sin\phi$$

Equating the coefficients of $\cos\omega t$ and $\sin\omega t$ on both sides of the resulting eq.

We obtain

$$X[(k-m\omega^2)\cos\phi + c\omega\sin\phi] = F_0$$

$$X[(k-m\omega^2)\sin\phi - c\omega\cos\phi] = 0 \qquad (21)$$

Solution of eqs (21) gives

$$X = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \qquad (22)$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right) \qquad (23)$$

Making the following substitutions:

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}\ ; \quad \frac{c}{m} = 2\zeta\omega_n$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0$$

$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

We obtain

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{\frac{1}{2}}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{24}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \tag{25}$$

where $M = \frac{X}{\delta_{st}}$ is called magnification factor

Characteristics:

- For $\zeta = 0$ (undamped system),

$$\frac{X}{\delta_{st}} = \frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|} = \frac{1}{|1-r^2|} \tag{26}$$

for $0 < r < 1 \Rightarrow \phi = 0°$: response is <u>in phase</u> with excitation

for $r > 1 \Rightarrow \phi = 180°$: response is <u>out of phase</u> with excitation

for $r \to 1 \Rightarrow M \to \infty$: when $\frac{\omega}{\omega_n} = 1$, the amplitude X becomes infinite : <u>resonance</u>

- For $\zeta > 0$,

for any r, $\zeta \uparrow$, $M \downarrow$

for $r = 0$, $M = 1$

for $r = 1$, $\phi = 90°$

for $r \to \infty$, $M \to 0$

• For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum M occurs

when $r = \sqrt{1-2\zeta^2}$ or $\omega = \omega_n \sqrt{1-2\zeta^2}$ (27)

$$M_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad (28)$$

ω at peak shifts to left as $\zeta \uparrow$

for $\zeta \ll 1$, $M_{max} \simeq \frac{1}{2\zeta} = M_{\omega=\omega_n} = Q$ (29)

: quality factor

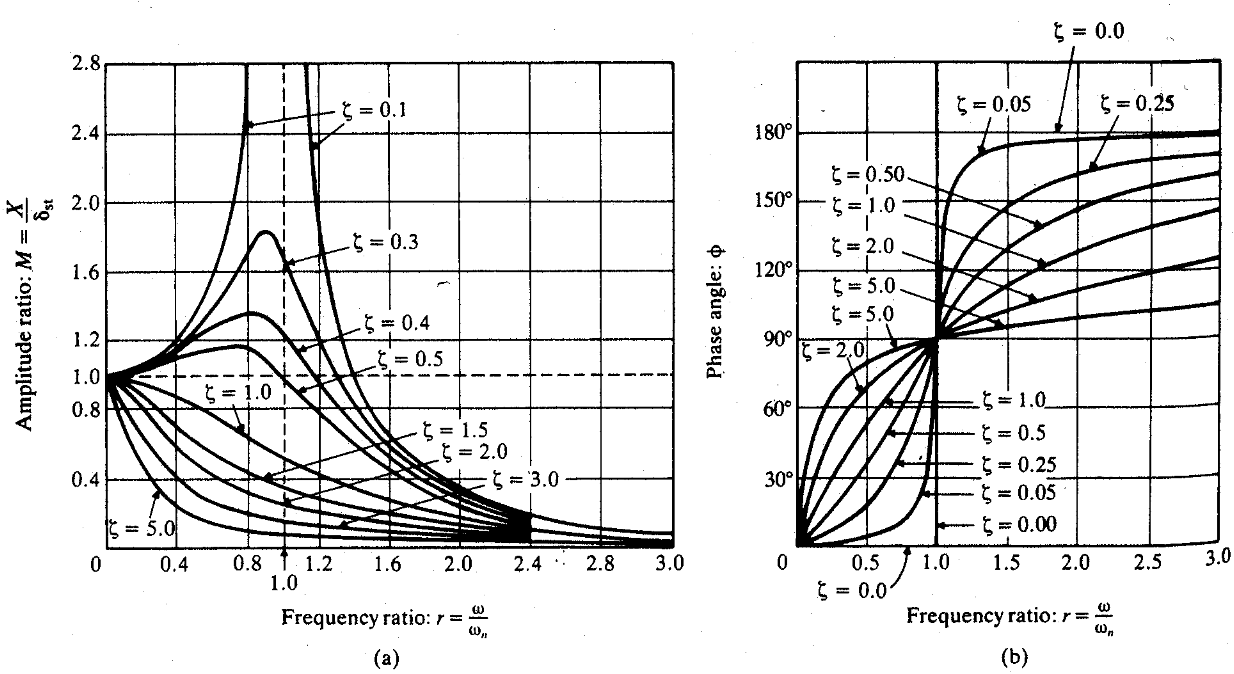


FIGURE 3.11 Variation of X and ϕ with frequency ratio r.