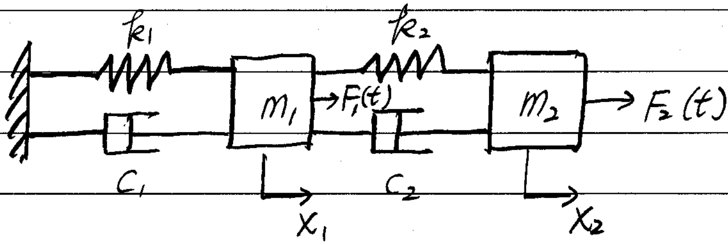
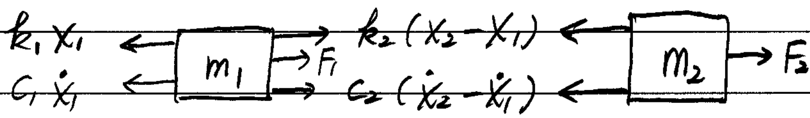
Multi-DOF Systems



Newtonian method $\Rightarrow$ Eqns of motion



$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2 x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + c_2\dot{x}_2 - k_2 x_1 + k_2 x_2 = F_2$$

Write equations in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\}$$

mass matrix	damping matrix	stiffness matrix	force vector

- Free Vibration of an Undamped System

Equations of motion:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or $M\ddot{X} + KX = 0$

Synchronous motion
(All coordinates execute the same motion in time)

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t}$$

$$\Rightarrow -\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\therefore \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$ must satisfy homogeneous linear equation:

$$\underbrace{\begin{bmatrix} (k_1+k_2)-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{bmatrix}}_{\det = 0 \text{ for nonzero } A_1,\ A_2} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\Rightarrow$ Characteristic equation:

$$[(k_1+k_2)-m_1\omega^2][k_2-m_2\omega^2] - k_2^2 = 0$$

Natural frequencies ω_1 and ω_2

Let $\omega_r = \sqrt{\lambda_r}$

$$k_1 k_2 + k_2^2 - \lambda m_1 k_2 - \lambda m_2 k_1 - \lambda m_2 k_2 + \lambda^2 m_1 m_2 - k_2^2 = 0$$

$$\lambda^2 (m_1 m_2) + \lambda\,(-m_1 k_2 - m_2 k_1 - m_2 k_2) + (k_1 k_2) = 0$$

roots

$$\lambda_r = \frac{m_1 k_2 + m_2 k_1 + m_2 k_2 \pm \sqrt{k_2^2 m_1^2 - 2k_1 k_2 m_1 m_2 + 2k_2^2 m_1 m_2 + k_1^2 m_2^2 + 2k_1 k_2 m_2^2 + k_2^2 m_2^2}}{2 m_1 m_2}$$ (cont.)

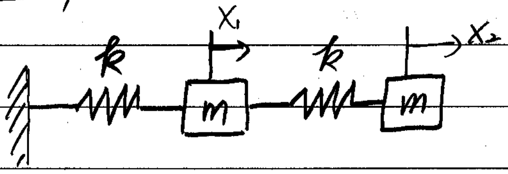
Special case : $m_1 = m_2 = m$

$$\lambda_r = \frac{(k_1 + 2k_2) \pm \sqrt{k_1^2 + 4k_2^2}}{2m}$$

Special case : $m_1 = m_2 = m$

$k_1 = k_2 = k$, $\lambda_r = \dfrac{3 \pm \sqrt{5}}{2} \dfrac{k}{m}$

Example



Equations of motion :

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Characteristic equation

$$\det \begin{vmatrix} 2k-\lambda m & -k \\ -k & k-\lambda m \end{vmatrix} = \lambda^2(m^2) + \lambda(-3mk) + k^2 = 0$$

Eigenvalues $\qquad \lambda_1 = \frac{3-\sqrt{5}}{2} \frac{k}{m}$

$$\lambda_2 = \frac{3+\sqrt{5}}{2} \frac{k}{m}$$

Natural frequencies $\qquad \omega_r = \sqrt{\lambda_r}$

$$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}} \sqrt{\frac{k}{m}} \simeq 0.618 \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}} \sqrt{\frac{k}{m}} \simeq 1.618 \sqrt{\frac{k}{m}}$$

Modal vectors

$$[K - \lambda_r M]\{A\}_r = \{0\}$$

Mode 1 : $\lambda_1 = \frac{3-\sqrt{5}}{2}\frac{k}{m}$

$$\begin{bmatrix} 2k-\left(\frac{3-\sqrt{5}}{2}\right)k & -k \\ -k & k-\left(\frac{3-\sqrt{5}}{2}\right)k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

absolute magnitudes of the elements of the modal vector are not unique

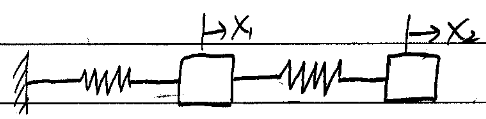
$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \alpha_1 \begin{Bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{Bmatrix} \simeq \alpha_1 \begin{Bmatrix} 1.000 \\ 1.618 \end{Bmatrix}$$

Mode 2 : $\lambda_2 = \frac{3+\sqrt{5}}{2}\frac{k}{m}$

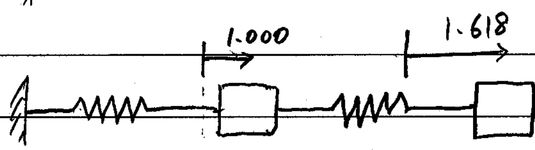
$$\begin{bmatrix} 2k-\left(\frac{3+\sqrt{5}}{2}\right)k & -k \\ -k & k-\left(\frac{3+\sqrt{5}}{2}\right)k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \alpha_2 \begin{Bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{Bmatrix} \simeq \alpha_2 \begin{Bmatrix} 1.000 \\ -0.618 \end{Bmatrix}$$

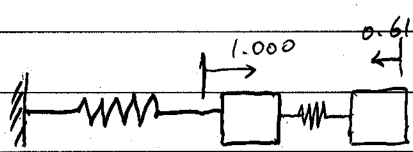
Summary - Natural Modes of Vibration :



Natural Frequencies	Modal Vectors
$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$	$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \alpha_1 \begin{Bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{Bmatrix}$
$\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$	$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \alpha_2 \begin{Bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{Bmatrix}$



move same direction



move opposite direction

Smart Materials and Structures

- **Overview**
  - Smart materials
  - Smart structures

- **Characteristics of Smart Materials**
  - Piezoelectric materials
    - Constitutive equations
    - PZT vs. PVDF
  - Electrostrictive materials
  - Magnetostrictive materials
  - Shape memory alloys
    - Nitinol
    - Martensite, austenite
    - Pseudoelastic, shape memory effect
  - Electro-rheological (ER) fluids
    - Bingham plastic
  - Magneto-rheological (MR) fluids
    - Three basic modes of operation
  - Optic fibers

- **Vibration**
  - SDOF system
    - Damping
    - Natural frequency
    - Harmonic excitation
    - Resonance
    - Quality factor
    - Transient, steady-state responses
    - Energy dissipated
    - Impulse responses, convolution integral
    - Base excitation
    - Transfer function, FRF
  - MDOF system
    - Equations of motion
    - Natural frequencies
    - Mode shapes
  - Piezoelectric accelerometer

- **ACE 3220/5120 Midterm Exam:**
  - **October 18, 2006 (Wednesday), 10:30 a.m. - 12:10 p.m.**
  - **ELB 403**
  - **Closed-books/closed-notes**
  - **One A4 hand-written sheet (single-sided)**
  - **Calculator**
  - **Cover concepts, theory, and applications**
  - **Review lecture & tutorial notes, handouts, examples, and homework problems**