Structural Dynamics

Newtonian

- Vector Calculus
- Newton's Law
- Space coordinates
- FBD – need to compute interacting forces

Lagrangian

- Calculus of Variations
- Hamilton's principle & Lagrange's eq.
- Generalized coordinates
- Treat the system as a whole
  - NOT individual bodies

Hamilton's Principle

Consider a system with N particles,

From d'Alembert's principle,

$$\sum_{n=1}^{N} (m_n \ddot{\underline{r}}_n - \underline{F}_n) \cdot \delta \underline{r}_n = 0 \qquad (1)$$

where

m_n is the mass of the nth particle

$\underline{r}_n$ is the position vector of the nth particle

$\underline{F}_n$ is the vector force applied to the nth particle

$\delta \underline{r}_n$ is the virtual displacement of the nth particle

Virtual displacement :

- infinitesimal, hypothetical change in the coordinate system from the particles actual path
- does NOT involve time

The <u>virtual work</u> is defined as

$$\delta W = \sum_{n=1}^{N} \underline{F}_n \cdot \delta \underline{r}_n \qquad (2)$$

Consider the 1st term in eq (1),

$$\ddot{\underline{r}}_n \cdot \delta \underline{r}_n = \frac{d}{dt}\left(\dot{\underline{r}}_n \cdot \delta \underline{r}_n\right) - \delta\left(\frac{1}{2}\dot{\underline{r}}_n \cdot \dot{\underline{r}}_n\right) \qquad (3)$$

Substitute eqs (2) and (3) into eq (1),

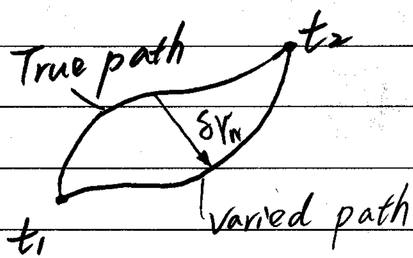
$$\sum_{n=1}^{N}\left[m_n\left(\frac{d}{dt}\left(\dot{\underline{r}}_n \cdot \delta \underline{r}_n\right) - \delta\left(\frac{1}{2}\dot{\underline{r}}_n \cdot \dot{\underline{r}}_n\right)\right)\right] - \delta W = 0 \qquad (4)$$

Define the total kinetic energy of the N particles as

$$T = \sum_{n=1}^{N} \frac{1}{2} m_n \dot{\underline{r}}_n \cdot \dot{\underline{r}}_n \qquad (5)$$

(4) $\Rightarrow$

$$\delta T + \delta W = \sum_{n=1}^{N} m_n \frac{d}{dt}\left(\dot{\underline{r}}_n \cdot \delta \underline{r}_n\right) \qquad (6)$$



Choose

$$\delta \underline{r}_n(t_1) = \delta \underline{r}_n(t_2) = 0$$

Integrate eq. (6) from $t = t_1$ to t_2

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} \sum_{n=1}^{N} m_n \frac{d}{dt} (\dot{\underline{r}}_n \cdot \delta \underline{r}_n)\, dt$$

$$= \sum_{n=1}^{N} m_n (\dot{\underline{r}}_n \cdot \delta \underline{r}_n) \Big|_{t_1}^{t_2}$$

$$= 0 \qquad (7)$$

$\boxed{\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0}$: Extended Hamilton's Principle (Generalized) $(\ast)$

$$\delta W = \delta W_c + \delta W_{NC} \qquad (8)$$

For conservative force,

$$\delta W_c = -\delta V \qquad (9)$$

V is potential energy (P.E.)

δV is variation of P.E.

$$(*) \Rightarrow \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{NC})\,dt = 0 \qquad (10)$$

Introduce Lagrangian

$$L = T - V \qquad (11)$$

$$(10) \Rightarrow \int_{t_1}^{t_2} (\delta L + \delta W_{NC})\,dt = 0 \qquad (12)$$

Lagrange's Equations

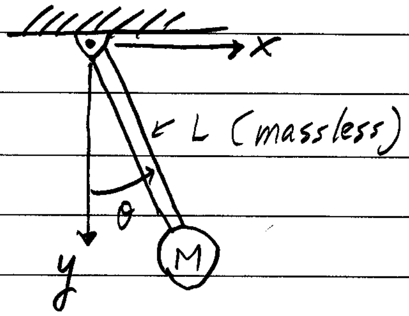
For a system of p particles, each with displacement vector $\underline{r}_i = \underline{r}_i(x_i, y_i, z_i)$, $i = 1, 2, \cdots, p$

$\Rightarrow$ the number n of independent (generalized) coordinates

$$n = 3p - c \qquad (1)$$

where c is the number of constraint eqs.

e.g.



the planar motion of a pendulum

constraints:

$$x^2 + y^2 = L^2$$

$$z = 0$$

$\Rightarrow n = 3 \times 1 - 2 = 1$,

only one generalized coordinate θ

n = DOF

= Minimum no. of independent coordinates needed

= No. of generalized coordinates

The position vector of each particle can be expressed as a function of the generalized coordinates and time,

$$\underline{r}_i = \underline{r}_i(q_1, q_2, \cdots, q_n, t), \quad i = 1, 2, \cdots, p \qquad (2)$$

The kinetic energy of a system of p particles:

$$T = \frac{1}{2}\sum_{i=1}^{p} m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i \qquad (3)$$

where

$$\dot{\underline{r}}_i = \sum_{r=1}^{n} \frac{\partial \underline{r}_i}{\partial q_r}\dot{q}_r + \frac{\partial \underline{r}_i}{\partial t} \qquad (4)$$

Introducing eq. (4) into eq. (3), one obtains

$$T = T(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t) \qquad (5)$$

$$\Rightarrow \quad \delta T = \sum_{r=1}^{n} \frac{\partial T}{\partial q_r}\delta q_r + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r}\delta \dot{q}_r \qquad (6)$$

where

$$\delta \dot{q}_r = \frac{d}{dt}\delta q_r \qquad (7)$$

$$\delta W = \sum_{k=1}^{p} \underline{F}_k \cdot \delta \underline{r}_k \qquad (8)$$

where

$$\delta \underline{r}_k = \sum_{r=1}^{n} \frac{\partial \underline{r}_k}{\partial q_r}\delta q_r \qquad (9)$$

Substituting eq (9) into eq. (8), one obtains

$$\delta W = \sum_{k=1}^{p} \sum_{r=1}^{n} \underline{F}_k \cdot \frac{\partial \underline{r}_k}{\partial q_r} \delta q_r \qquad (10)$$

$$= \sum_{r=1}^{n} Q_r \delta q_r \qquad (11)$$

where

$$Q_r = \sum_{k=1}^{p} \underline{F}_k \cdot \frac{\partial \underline{r}_k}{\partial q_r} \qquad (12)$$

: Generalized Force

Sub. eq. (6) and eq. (11) into the extended Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} \left(\sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \delta q_r + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \delta \dot{q}_r \right) dt$$

$$+ \int_{t_1}^{t_2} \left(\sum_{r=1}^{n} Q_r \delta q_r \right) dt$$

Integration by parts,

$$= \int_{t_1}^{t_2} \left[\sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \delta q_r - \sum_{r=1}^{n} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) \delta q_r \right] dt + \int_{t_1}^{t_2} \left(\sum_{r=1}^{n} Q_r \delta q_r \right) dt$$

$$= -\sum_{r=1}^{n} \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} - Q_r \right] \delta q_r \, dt = 0$$

Since δq_r is arbitrary

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r}, \quad r = 1, 2, \cdots, n \qquad (*)$$

Lagrange's equation

• If seperate

$$\delta W = \delta W_C + \delta W_{NC}$$

$$= -\delta V + \sum_{r=1}^{n} Q_{r_{NC}} \delta q_r$$

Extended Hamilton's principle,

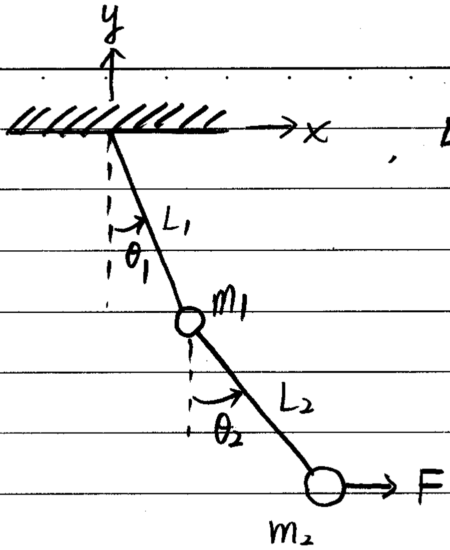
$$\Rightarrow \int_{t_1}^{t_2} \left(\delta T - \delta V + \sum_{r=1}^{n} Q_{r_{NC}} \delta q_r \right) dt = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = Q_{r_{NC}}}, \quad r = 1, 2, \cdots, n \qquad (**)$$

Lagrange's eq. in alter form

$$(*) \equiv (**)$$

Ex.



$$\underline{r}_1 = L_1 \sin\theta_1 \hat{i} - L_1 \cos\theta_1 \hat{j}$$

$$\underline{V}_1 = \dot{\underline{r}}_1 = \dot{\theta}_1 L_1 \cos\theta_1 \hat{i} + \dot{\theta}_1 L_1 \sin\theta_1 \hat{j}$$

$$V_1^2 = \underline{V}_1 \cdot \underline{V}_1 = \dot{\theta}_1^2 L_1^2 \cos^2\theta_1 + \dot{\theta}_1^2 L_1^2 \sin^2\theta_1 = \dot{\theta}_1^2 L_1^2$$

$$\underline{r}_2 = (L_1 \sin\theta_1 + L_2 \sin\theta_2)\hat{i} - (L_1 \cos\theta_1 + L_2 \cos\theta_2)\hat{j}$$

$$\underline{V}_2 = (L_1\dot{\theta}_1 \cos\theta_1 + L_2\dot{\theta}_2 \cos\theta_2)\hat{i} + (L_1\dot{\theta}_1 \sin\theta_1 + L_2\dot{\theta}_2 \sin\theta_2)\hat{j}$$

$$V_2^2 = \underline{V}_2 \cdot \underline{V}_2 = L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1\cos\theta_2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1\sin\theta_2$$

$$= L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$T = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2$$

$$V = m_1 g h_1 + m_2 g h_2$$

$$= -m_1 g L_1 \cos\theta_1 - m_2 g (L_1\cos\theta_1 + L_2\cos\theta_2)$$

$$L = T - V = \frac{1}{2}m_2\left[L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$$

$$+ \frac{1}{2}m_1\dot{\theta}_1^2L_1^2 + m_1 g L_1\cos\theta_1 + m_2 g(L_1\cos\theta_1 + L_2\cos\theta_2)$$

Let $q_1 = \theta_1$, $q_2 = \theta_2$

$$\delta W_{NC} = F\,\delta(L_1 \sin\theta_1 + L_2 \sin\theta_2)$$

$$= F(L_1 \cos\theta_1\,\delta\theta_1 + L_2 \cos\theta_2\,\delta\theta_2)$$

$$\Rightarrow Q_{1NC} = F L_1 \cos\theta_1 \quad , \quad Q_{2NC} = F L_2 \cos\theta_2$$

$$\frac{\partial L}{\partial \dot\theta_1} = m_2 L_1^2 \dot\theta_1 + m_2 L_1 L_2 \dot\theta_2 \cos(\theta_1 - \theta_2) + m_1 L_1^2 \dot\theta_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = (m_1+m_2) L_1^2 \ddot\theta_1 + m_2 L_1 L_2 \ddot\theta_2 \cos(\theta_1-\theta_2) - m_2 L_1 L_2 (\dot\theta_1 - \dot\theta_2)\dot\theta_2 \sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - g(m_1+m_2) L_1 \sin\theta_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = Q_{1NC}$$

$$\Rightarrow (m_1+m_2) L_1^2 \ddot\theta_1 + m_2 L_1 L_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 L_1 L_2 \dot\theta_2^2 \sin(\theta_1-\theta_2)$$
$$+ (m_1+m_2) g L_1 \sin\theta_1 = F L_1 \cos\theta_1$$

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 L_2^2 \dot\theta_2 + m_2 L_1 L_2 \dot\theta_1 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2 L_2^2 \ddot\theta_2 + m_2 L_1 L_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 L_1 L_2 \dot\theta_1 (\dot\theta_1 - \dot\theta_2) \sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - m_2 g L_2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = Q_{2NC}$$

$$\Rightarrow m_2 L_2^2 \ddot\theta_2 + m_2 L_1 L_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 L_1 L_2 \dot\theta_1^2 \sin(\theta_1-\theta_2)$$
$$+ m_2 g L_2 \sin\theta_2 = F L_2 \cos\theta_2$$

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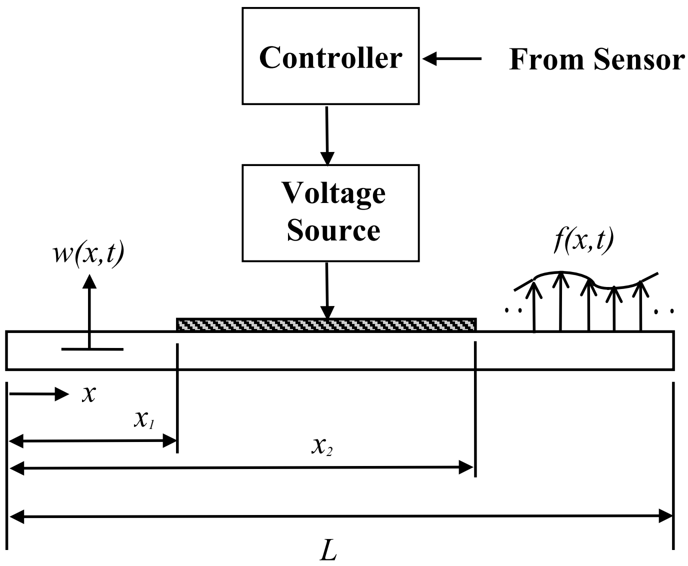
A Beam with a Surface Mounted Piezoelectric Element



Assumptions:

- The piezoelectric element is perfectly bonded
- The applied voltage is uniform along the beam, i.e., $v(x,t) = v(t)$

Potential energies:

$$V_b = \frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{1}$$

$$V_p = \frac{1}{2}\int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \left[H(x-x_1)-H(x-x_2)\right] dx \tag{2}$$

where H is the Heaviside's function.

Kinetic energies:

$$T_b = \frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx \tag{3}$$

$$T_p = \frac{1}{2}\int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 [H(x-x_1)-H(x-x_2)] dx \tag{4}$$

Virtual work:

$$\delta W_d = \int_0^L f(x,t)\delta w(x,t)dx \tag{5}$$

From the constitutive equation of the piezoelectric materials,

$$S_1 = s_{11}^E T_1 + d_{31}E_3 \tag{6}$$

$$T_1 = E_p(S_1 - d_{31}E_3) \tag{7}$$

$$\text{where } E_p = \frac{1}{s_{11}^E} \quad , \qquad\qquad E_3 = \frac{v(t)}{t_p} \tag{8}$$

The virtual work done by the induced strain (force) is:

$$\delta W_p = \int_0^L E_p d_{31} b v(t)\delta\left(\frac{\partial u_p}{\partial x}\right)\left[H(x-x_1) - H(x-x_2)\right]dx \tag{9}$$

where b is the width of beam and piezo layer

$$u_p = -\left(\frac{t_b + t_p}{2}\right)\frac{\partial w}{\partial x} \tag{10}$$

$$\text{Let } a = \frac{t_b + t_p}{2} \tag{11}$$

$$\delta W_p = -\int_0^L E_p d_{31} a b v(t)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H(x-x_1) - H(x-x_2)\right]dx \tag{12}$$

Apply extended Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{NC})dt = 0$$

$$\begin{aligned}
&\int_{t_1}^{t_2} (\delta\left\{\frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx\right\} + \delta\left\{\frac{1}{2}\int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 [H(x-x_1) - H(x-x_2)]dx\right\} \\
&- \delta\left\{\frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\} - \delta\left\{\frac{1}{2}\int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 [H(x-x_1) - H(x-x_2)]dx\right\} \\
&+ \int_0^L f(x,t)\delta w(x,t)dx - \int_0^L E_p d_{31} a b v(t)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1) - H(x-x_2)]dx)dt \\
&= 0
\end{aligned} \tag{13}$$

- $$\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \rho_b A_b\left(\frac{\partial w}{\partial t}\right)^2 dx\right\}dt = -\int_{t_1}^{t_2}\int_0^L \rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)\delta w dx dt \tag{14}$$

- $$-\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\}dt$$

$$= -\int_{t_1}^{t_2} E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt + \int_{t_1}^{t_2} E_b I_b\left(\frac{\partial^3 w}{\partial x^3}\right)\delta w\Bigg|_0^L dt - \int_{t_1}^{t_2}\int_0^L E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)\delta w dx dt \tag{15}$$

- $$\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \rho_p A_p\left(\frac{\partial w}{\partial t}\right)^2 [H(x-x_1)-H(x-x_2)]dx\right\}dt$$

$$= -\int_{t_1}^{t_2}\int_0^L \rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)[H(x-x_1)-H(x-x_2)]\delta w dx dt \tag{16}$$

- $$-\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)^2 [H(x-x_1)-H(x-x_2)]dx\right\}dt$$

$$= -\int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1)-H(x-x_2)]\delta\left(\frac{\partial^2 w}{\partial x^2}\right)dx dt$$

$$= -\int_{t_1}^{t_2} E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1)-H(x-x_2)]\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L dt$$

$$+\int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H(x-x_1)-H(x-x_2)]\delta\left(\frac{\partial w}{\partial x}\right)dx dt \tag{17}$$

$$+\int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H'(x-x_1)-H'(x-x_2)]\delta\left(\frac{\partial w}{\partial x}\right)dx dt$$

$$= \int_{t_1}^{t_2} E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H(x-x_1)-H(x-x_2)]\delta w\Bigg|_0^L dt - \int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)[H(x-x_1)-H(x-x_2)]\delta w dx dt$$

$$-\int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]\delta w dx dt + \int_{t_1}^{t_2} E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H'(x-x_1)-H'(x-x_2)]\delta w\Bigg|_0^L dt$$

$$-\int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]\delta w dx dt - \int_{t_1}^{t_2}\int_0^L E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H''(x-x_1)-H''(x-x_2)]\delta w dx dt$$

- $$-\int_{t_1}^{t_2}\int_0^L E_p d_{31} abv(t)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)[H(x-x_1)-H(x-x_2)]dxdt$$

$$= -\int_{t_1}^{t_2}\int_0^L E_p d_{31} abv(t)[H''(x-x_1)-H''(x-x_2)]\delta w dxdt \tag{18}$$

Substituting eqs. (14) - (18) into eq. (13),

$$\begin{aligned}&\int_{t_1}^{t_2}\{\int_0^L \ (-\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)-\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)[H(x-x_1)-H(x-x_2)]-E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)\\ &-E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)[H(x-x_1)-H(x-x_2)]-2E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]\\ &-E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H''(x-x_1)-H''(x-x_2)]+f(x,t)-E_p d_{31} abv(t)[H''(x-x_1)-H''(x-x_2)])\delta w dx\\ &-E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L+E_b I_b\left(\frac{\partial^3 w}{\partial x^3}\right)\delta w\Bigg|_0^L\}dt=0\end{aligned} \tag{19}$$

For arbitrary δw in $0 < x < L$,

Equation of motion:

$$\begin{aligned}&\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)+E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)+\left\{\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)+E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)\right\}[H(x-x_1)-H(x-x_2)]\\ &+2E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)[H'(x-x_1)-H'(x-x_2)]+E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)[H''(x-x_1)-H''(x-x_2)]\\ &+E_p d_{31} abv(t)[H''(x-x_1)-H''(x-x_2)]=f(x,t)\end{aligned} \tag{20}$$

with boundary conditions:

$$\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L=0 \text{ and } \left(\frac{\partial^3 w}{\partial x^3}\right)\delta w\Bigg|_0^L=0 \tag{21}$$