Discretization: (Galerkin's Method)

Assume

$$w(x,t)=\sum_{i=1}^{n}\phi_i(x)q_i(t) \tag{22}$$

where $\phi_i(x)$ satisfies all B.C.s

Sub. eq. (22) into eq. (20),

$$\begin{aligned}
&\rho_b A_b \sum_{i=1}^{n}\phi_i(x)\ddot{q}_i(t)+E_b I_b \sum_{i=1}^{n}\phi_i^{(4)}(x)q_i(t) \\
&+\left[\rho_p A_p \sum_{i=1}^{n}\phi_i(x)\ddot{q}_i(t)+E_p I_p \sum_{i=1}^{n}\phi_i^{(4)}(x)q_i(t)\right]\left[H(x-x_1)-H(x-x_2)\right] \\
&+2E_p I_p \sum_{i=1}^{n}\phi_i^{(3)}(x)q_i(t)\left[H'(x-x_1)-H'(x-x_2)\right]+E_p I_p \sum_{i=1}^{n}\phi_i''(x)q_i(t)\left[H''(x-x_1)-H''(x-x_2)\right] \\
&+E_p d_{31}abv(t)\left[H''(x-x_1)-H''(x-x_2)\right]-f(x,t)=\varepsilon
\end{aligned} \tag{23}$$

Min ε by $\left\langle \varepsilon,\phi_j \right\rangle = 0$

$$\Rightarrow \left\langle \varepsilon,\phi_j \right\rangle = \int_0^L \varepsilon(x,t)\phi_j(x)dx = 0 \qquad j=1,2,\ldots,n \tag{24}$$

$$\begin{aligned}
&\Rightarrow\left[\rho_b A_b\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\phi_j(x)dx\right)+\rho_p A_p\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]\ddot{q}_i(t) \\
&\quad+\left[E_b I_b\left(\sum_{i=1}^{n}\int_0^L \phi_i^{(4)}(x)\phi_j(x)dx\right)+E_p I_p\left(\sum_{i=1}^{n}\int_0^L \phi_i^{(4)}(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]q_i(t) \\
&\quad+\left[2E_p I_p\left(\sum_{i=1}^{n}\int_0^L \phi_i^{(3)}(x)\phi_j(x)\left[H'(x-x_1)-H'(x-x_2)\right]dx\right)\right]q_i(t) \\
&\quad+\left[E_p I_p\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\phi_j(x)\left[H''(x-x_1)-H''(x-x_2)\right]dx\right)\right]q_i(t) \\
&\quad+E_p d_{31}abv(t)\left(\int_0^L \phi_j(x)\left[H''(x-x_1)-H''(x-x_2)\right]dx\right)-\int_0^L f(x,t)\phi_j(x)dx=0
\end{aligned} \tag{25}$$

- $\int_0^L \phi_i^{(4)}(x)\phi_j(x)dx = \int_0^L \phi_i''(x)\phi_j''(x)dx$ (26)

- $\int_0^L \phi_i^{(4)}(x)\phi_j(x)[H(x-x_1)-H(x-x_2)]dx$ (27)

- $2\int_0^L \phi_i^{(3)}(x)\phi_j(x)[H'(x-x_1)-H'(x-x_2)]dx$

$= 2\phi_i^{(3)}(x)\phi_j(x)[H(x-x_1)-H(x-x_2)]\Big|_0^L - 2\int_0^L \phi_i^{(4)}(x)\phi_j(x)[H(x-x_1)-H(x-x_2)]dx$

$-2\int_0^L \phi_i^{(3)}\phi_j'(x)[H(x-x_1)-H(x-x_2)]dx$ (28)

- $\int_0^L \phi_i''(x)\phi_j(x)[H''(x-x_1)-H''(x-x_2)]dx$

$= \phi_i''(x)\phi_j(x)[H'(x-x_1)-H'(x-x_2)]\Big|_0^L - \int_0^L \phi_i^{(3)}(x)\phi_j(x)[H'(x-x_1)-H'(x-x_2)]dx$

$-\int_0^L \phi_i''(x)\phi_j'(x)[H'(x-x_1)-H'(x-x_2)]dx$ (29)

$= \int_0^L \phi_i^{(4)}(x)\phi_j(x)[H(x-x_1)-H(x-x_2)]dx + \int_0^L \phi_i^{(3)}(x)\phi_j'(x)[H(x-x_1)-H(x-x_2)]dx$

$+\int_0^L \phi_i^{(3)}(x)\phi_j'(x)[H(x-x_1)-H(x-x_2)]dx + \int_0^L \phi_i''(x)\phi_j''(x)[H(x-x_1)-H(x-x_2)]dx$

- (27)+(28)+(29) = $\int_0^L \phi_i''(x)\phi_j''(x)[H(x-x_1)-H(x-x_2)]dx$ (30)

- $\int_0^L \phi_j(x)[H''(x-x_1)-H''(x-x_2)]dx$

$= \phi_j(x)[H'(x-x_1)-H'(x-x_2)]\Big|_0^L - \int_0^L \phi_j'(x)[H'(x-x_1)-H'(x-x_2)]dx$

$= -\phi_j'(x)[H(x-x_1)-H(x-x_2)]\Big|_0^L + \int_0^L \phi_j''(x)[H(x-x_1)-H(x-x_2)]dx$

$= \int_0^L \phi_j''(x)[H(x-x_1)-H(x-x_2)]dx = \phi_j'(x_2)-\phi_j'(x_1)$ (31)

Substituting (26), (30), (31) into (25),

$$\left[\rho_b A_b\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\phi_j(x)dx\right)+\rho_p A_p\left(\sum_{i=1}^{n}\int_0^L \phi_i(x)\phi_j(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]\ddot{q}_i(t)$$

$$+\left[E_b I_b\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\phi_j''(x)dx\right)+E_p I_p\left(\sum_{i=1}^{n}\int_0^L \phi_i''(x)\phi_j''(x)\left[H(x-x_1)-H(x-x_2)\right]dx\right)\right]q_i(t)$$

$$+E_p d_{31} abv(t)\left(\phi_j'(x_2)-\phi_j'(x_1)\right)=\int_0^L f(x,t)\phi_j(x)dx \qquad (32)$$

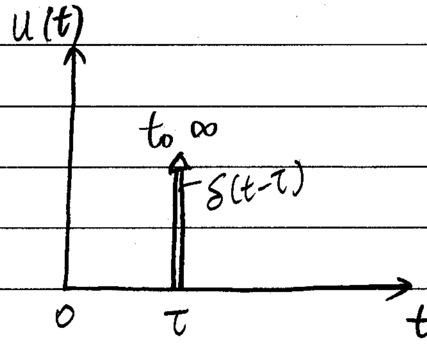
$$\sum_{i=1}^{n} m_{ij}\ddot{q}_i(t)+\sum_{i=1}^{n} k_{ij} q_i(t)=f_{c_j}(t)+f_{d_j}(t), \qquad j = 1,\ 2,\ \dots,\ n$$

where $m_{ij}=\rho_b A_b\int_0^L \phi_i(x)\phi_j(x)dx+\rho_p A_p\int_{x_1}^{x_2}\phi_i(x)\phi_j(x)dx$

$$k_{ij}=E_b I_b\int_0^L \phi_i''(x)\phi_j''(x)dx+E_p I_p\int_{x_1}^{x_2}\phi_i''(x)\phi_j''(x)dx$$

$$f_{c_j}=E_p d_{31} abv(t)\left[\left(\phi_j'(x_1)-\phi_j'(x_2)\right)\right]$$

$$f_{d_j}=\int_0^L f(x,t)\phi_j(x)dx$$



unit impulse at $t=\tau$

$$u(t) = \delta(t-\tau)$$

where $\delta(t-\tau)$ is the <u>Dirac delta function</u>

$$\begin{cases} \delta(t-\tau) \to \infty, & t=\tau \\ \delta(t-\tau) = 0, & t \neq \tau \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = 1$$

Consider discrete force at $x = X_d$,

$$f(x,t) = F_d(t)\,\delta(x-X_d)$$

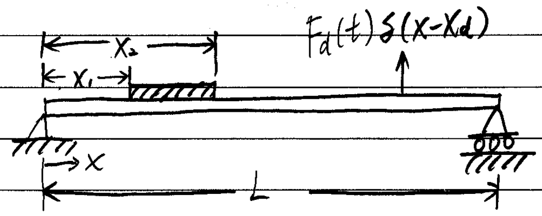
Sifting property: $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$

$$\Rightarrow f_{d_s} = \int_0^L f(x,t)\,\phi_s(x)\,dx$$

$$= \int_0^L F_d(t)\,\phi_s(x)\,\delta(x-X_d)\,dx$$

$$= F_d(t)\,\phi_s(X_d)$$

e.g. A Simply Supported Beam with a Piezoelectric Element



Choose $\phi_r(x) = \sin\left(\frac{r\pi x}{L}\right)$, $r = 1, 2, \cdots, N$

satisfy all B.C.s :

$x = 0$: $w = 0$, $w'' = 0$

$x = L$: $w = 0$, $w'' = 0$

Assume

$$w(x,t) \simeq \sum_{r=1}^{N} \phi_r(x)\, q_r(t)$$

$\Rightarrow$ Discretized eq. of motion :

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{f_c\} + \{f_d\}$$

where

$$m_{sr} = \int_0^L \rho_b A_b \phi_s(x)\phi_r(x)\,dx + \int_{x_1}^{x_2} \rho_p A_p \phi_s(x)\phi_r(x)\,dx$$

$$k_{sr} = \int_0^L E_b I_b \phi_s''(x)\phi_r''(x)\,dx + \int_{x_1}^{x_2} E_p I_p \phi_s''(x)\phi_r''(x)\,dx$$

$$f_{cs} = E_p d_{31} ab\, v(t) \left[\phi_s'(x_1) - \phi_s'(x_2) \right]$$

$$f_{ds} = F_d(t)\, \phi_s(x_d)$$

$$[C] = \alpha [M] + \beta [K]$$

for $s = r$:

$$m_{rr} = \frac{\rho_b A_b L}{2} + \rho_p A_p \left(\frac{x_2 - x_1}{2} \right) + \frac{\rho_p A_p L}{4\pi r} \left[\sin\left(\frac{2\pi r x_1}{L} \right) - \sin\left(\frac{2\pi r x_2}{L} \right) \right]$$

$$k_{rr} = \left(\frac{\pi r}{L} \right)^4 \left\{ \frac{E_b I_b L}{2} + E_p I_p \left(\frac{x_2 - x_1}{2} \right) + \frac{E_p I_p L}{4\pi r} \left[\sin\left(\frac{2\pi r x_1}{L} \right) - \sin\left(\frac{2\pi r x_2}{L} \right) \right] \right\}$$

for $s \neq r$:

$$m_{sr} = \frac{\rho_p A_p L}{\pi} \left[\frac{r \sin\left(\frac{s\pi x}{L} \right) \cos\left(\frac{r\pi x}{L} \right)}{(s^2 - r^2)} + \frac{s \cos\left(\frac{s\pi x}{L} \right) \sin\left(\frac{r\pi x}{L} \right)}{(r^2 - s^2)} \right]_{x_1}^{x_2}$$

$$k_{sr} = \frac{E_p I_p L}{\pi} \left(\frac{s r \pi^2}{L^2} \right)^2 \left[\frac{r \sin\left(\frac{s\pi x}{L} \right) \cos\left(\frac{r\pi x}{L} \right)}{(s^2 - r^2)} + \frac{s \cos\left(\frac{s\pi x}{L} \right) \sin\left(\frac{r\pi x}{L} \right)}{(r^2 - s^2)} \right]_{x_1}^{x_2}$$

$$M\ddot{q} + C\dot{q} + Kq = f_c + f_d \qquad (a)$$

Let state vector $x(t) = \begin{Bmatrix} q(t) \\ \dot{q}(t) \end{Bmatrix} \qquad (b)$

We can rewrite eq. (a) in state space form:

$$\dot{x} = Ax + Bu + \hat{B}u_d$$

$$y = C_0 x + Du$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \varepsilon_p d_{31} ab \begin{bmatrix} \phi_1'(x_1) - \phi_1'(x_2) \\ \vdots \\ \phi_N'(x_1) - \phi_N'(x_2) \end{bmatrix} \quad \text{for } u = v(t)$$

$$\hat{B} = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \begin{bmatrix} \phi_1(x_d) \\ \vdots \\ \phi_N(x_d) \end{bmatrix} \quad \text{for } u_d = F_d(t)$$

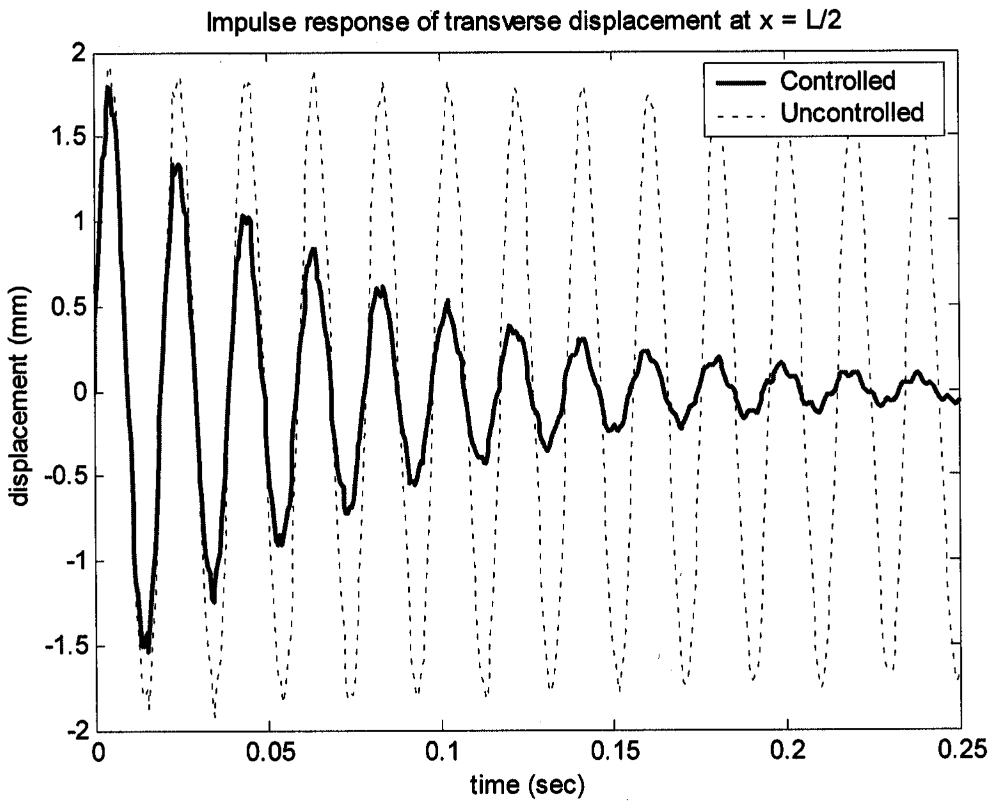
if $y = w(x_0, t) = \sum_{r=1}^{N} \phi_r(x_0)\, q_r(t)$

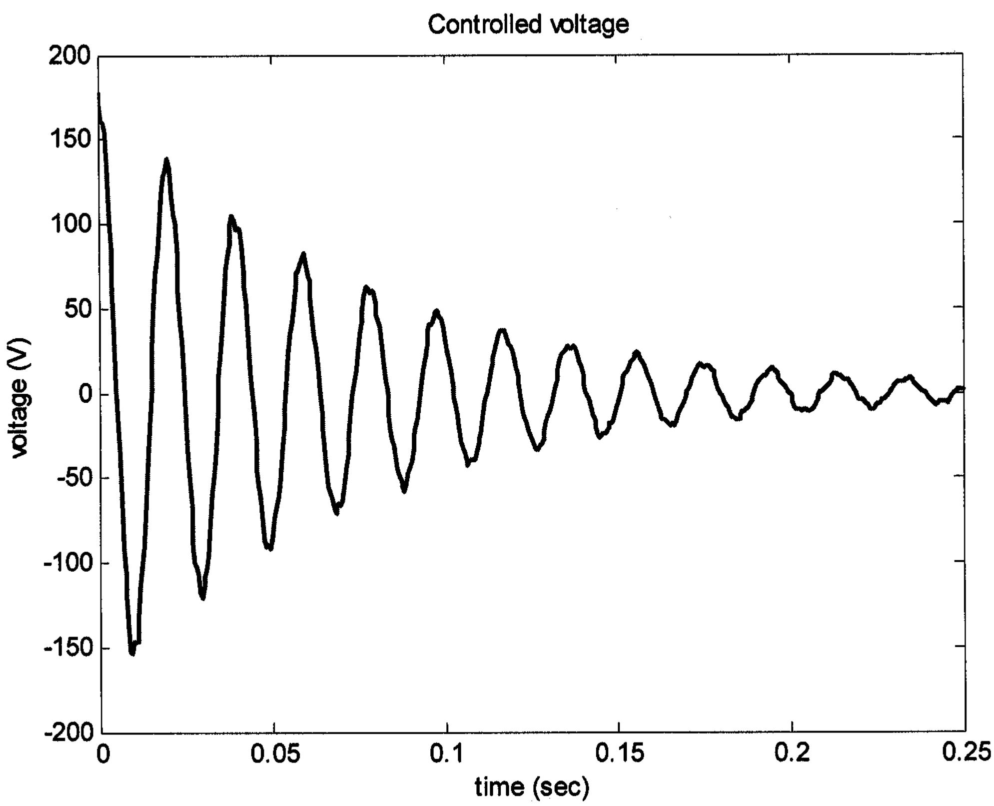
$$\Rightarrow C_0 = [\phi_1(x_0) \ \phi_2(x_0) \ \cdots \ \phi_N(x_0) \ 0 \ 0 \ \cdots \ 0]$$

$$D = 0$$

```
% Simply supported beam with a PZT element

clear

% initialize

b=1.27e-2;
L=0.3;
x1=0.1; x2=0.16;
xd=0.18;

Eb=7.1e10;
pb=2700;  tb=2.286e-3;

Ec=6.49e10;
pc=7600;  tc=0.762e-3;
d31=-175e-12;

Ac=b*tc;  Ab=b*tb;
Ib=b*tb^3/12;  Ic=b*tc^3/12;
a=(tb+tc)/2;

% stiffness and mass matrices

N=5; % no. of expansion terms

K=zeros(N);
M=zeros(N);
C=zeros(N);
Fc=zeros(N,1); Fd=zeros(N,1);

for r=1:N;
    for s=1:N;
        if r == s
    K(r,s)=(pi*r/L)^4*(Eb*Ib*L/2+Ec*Ic*(x2-x1)/2+...
        Ec*Ic*L/(4*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L)));
    M(r,s)=pb*Ab*L/2+pc*Ac*(x2-x1)/2+...
        pc*Ac*L/(4*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L));
        else
    K(r,s)=Ec*Ic*L/pi*(pi^2*r*s/L^2)^2*...
        ((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
        (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
        ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
        (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
    M(r,s)=pc*Ac*L/pi*((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
        (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
        ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
        (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
        end;
    end;
```

```
    % due to voltage input
    Fc(r)=-a*Ec*d31*b*(pi*r/L)*(cos(r*pi*x2/L)-cos(r*pi*x1/L));
    % due to discrete force with magnitude 1/100
    Fd(r)=1/100*sin(r*pi*xd/L);
end;

% add internal damping

C=0.64*M+1.2e-6*K;

% state-space model

AL=-inv(M)*K;
AR=-inv(M)*C;
A=[zeros(N) eye(N);...
   AL AR];
BL1=inv(M)*Fc; BL2=inv(M)*Fd;
B1=[zeros(N,1);BL1];
B2=[zeros(N,1);BL2];
for r=1:N;
    CCw(1,r)=sin(r*pi/2);  % displacement w at midpoint (x=L/2)
end;
CC=[CCw zeros(1,N)];
D=[0];

% control gain
Kc=1.0e+004*[-1.4035 -0.1594 0.5005 0.2457 -0.1712...
        -0.0289 -0.0033 0.0076 0.0045 -0.0030];
Ac=A-B1*Kc;

% impulse response

t=0:0.0005:0.25;
IU=1;
[y,x,t]=impulse(A,B2,CC,D,IU,t); % uncontrolled response
[yc,x,t]=impulse(Ac,B2,CC,D,IU,t); % controlled response
u=-Kc*x'; % controlled voltage

% plot results

figure(1),plot(t,yc*1000,t,y*1000,':') % unit (mm)
title('Impulse response of transverse displacement at x = L/2')
xlabel('time (sec)')
ylabel('displacement (mm)')

figure(2),plot(t,u)
title('Controlled voltage')
xlabel('time (sec)')
ylabel('voltage (V)')
```


Impulse response of transverse displacement at x = L/2
Controlled
Uncontrolled
displacement (mm)
time (sec)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.05
0.1
0.15
0.2
0.25


Controlled voltage
voltage (V)
time (sec)
200
150
100
50
0
-50
-100
-150
-200
0
0.05
0.1
0.15
0.2
0.25

Smart Materials and Structures

- **Overview**
  - Smart materials
  - Smart structures

- **Characteristics of Smart Materials**
  - Piezoelectric materials
    - Constitutive equations
    - Polarization field, coercive field, Young's modulus, Curie temperature
    - PZT vs. PVDF
  - Electrostrictive materials
  - Magnetostrictive materials
  - Shape memory alloys
    - Nitinol
    - Martensite, austenite
    - Pseudoelastic, shape memory effect
  - Electro-rheological (ER) fluids
    - Bingham plastic
  - Magneto-rheological (MR) fluids
    - Three basic modes of operation
  - Optic fibers

- **Vibration**
  - SDOF system
    - Damping
    - Natural frequency
    - Harmonic excitation
    - Resonance
    - Quality factor
    - Transient, steady-state responses
    - Energy dissipated
    - Impulse responses, convolution integral
    - Base excitation
    - Transfer function, FRF
  - MDOF system
    - Equations of motion
    - Natural frequencies
    - Mode shapes

- **Structural Control**
  - Passive, active, active-passive, semi-active
  - State space model

- **Transducers**
  - Sensors, actuators
  - Smart sensors & actuators
    - Solid-state
    - Smart fluids
  - Piezoelectric actuators/sensors
    - Motor, generator
    - Performance (blocked force, free deflection)
    - Design (stack, unimorph, bimorph, Moonie)
    - Piezoelectric accelerometer
  - MR damper
    - Accumulator, offset
    - Model
    - On-off control
    - Applications (suspension systems, buildings)

- **Structural Dynamics**
  - Hamilton's principle
  - Lagrange's equation
  - Generalized coordinates
  - Constraints
  - Equation of motion

- **Integrated Systems**
  - Suspension systems with MR dampers
  - Beam with piezoelectric elements
  - Galerkin's method
    - Discretization
    - Comparison functions
  - Discrete forces
  - Rayleigh damping

- **ACE 3220/5120 Final Exam:**
  - **December 9, 2006 (Saturday), 3:00 – 5:00 pm**
  - **ERB 407**
  - **Closed-books/closed-notes**
  - **One A4 hand-written sheet (double-sided)**
  - **Calculator**
  - **Cover concepts, theory, and applications**
  - **Review lecture & tutorial notes, handouts, examples, and homework problems**