香港中文大學
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The Chinese University of Hong Kong

Department of Mechanical & Automation Engineering

MAEG5080 Smart Materials & Structures

Assignment #1

by

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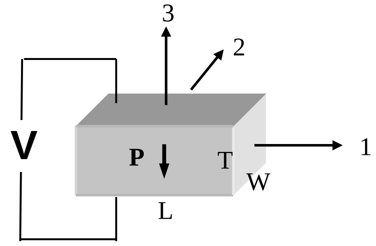
2022-23 Term 1

Problem 1

The piezoelectric constant matrix $\mathbf{d}$ of PZT is described as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

Consider a PZT element above used as a micro positioning device, in which in $L = 30$ mm, $T = 5$ mm, and $W = 12$ mm. With 110 volts applied, compute the changes in L, T, and W for a PSI-5A-S4 piezoceramic ($d_{31} = -190 \times 10^{-12}$ Meters/Volt; $d_{33} = 390 \times 10^{-12}$ Meters/Volt; $d_{15} = 550 \times 10^{-12}$ Meters/Volt).



Solution:

The electric fields after 110 volts are applied are given by:

$$E = \begin{bmatrix} 0 \\ 0 \\ \frac{110 \text{ V}}{T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{110 \text{ V}}{5 \text{ mm}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2.2 \times 10^4 \end{bmatrix} \text{ V/m} \tag{2}$$

The mechanical strains are calculated as:

$$S = d^t E = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix}^t \cdot \begin{bmatrix} 0 \\ 0 \\ 2.2 \times 10^4 \end{bmatrix} = \begin{bmatrix} -4.18 \times 10^{-6} \\ -4.18 \times 10^{-6} \\ 8.58 \times 10^{-6} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Therefore, the changes in L, T, and W for a PSI-5A-S4 piezoceramic can be computed as

$$\Delta L = S_1 L = -4.18 \times 10^{-6} \times 30 \text{ mm} = -1.254 \times 10^{-7} \text{ m} \tag{4}$$

$$\Delta W = S_2 W = -4.18 \times 10^{-6} \times 12 \text{ mm} = -5.02 \times 10^{-8} \text{ m} \tag{5}$$

$$\Delta T = S_3 T = 8.58 \times 10^{-6} \times 5 \text{ mm} = 4.29 \times 10^{-8} \text{ m} \tag{6}$$

Problem 2

Given the following differential equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{7}$$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{8}$$

where $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega_n}$. For initial conditions: $x(0) = x_0$, $\dot{x}(0) = v_0$,

(a) Show the solutions for the following cases in details:

(i) $\zeta = 0$ (undamped):

$$x(t) = A\cos(\omega_n t - \phi) \tag{9}$$

where $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)}$ and $\phi = \tan^{-1}\frac{v_0}{x_0\omega_n}$.

(ii) $0 < \zeta < 1$ (underdamped):

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{10}$$

where $\omega_d = \sqrt{1-\zeta^2}\omega_n$, $A = \sqrt{x_0^2 + \left(\frac{\zeta\omega_n x_0 + v_0}{\omega_d}\right)^2}$, and $\phi = \tan^{-1}\frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d}$.

(iii) $\zeta = 1$ (critically damped):

$$x(t) = [x_0(v_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{11}$$

(iv) $\zeta > 1$ (overdamped):

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{12}$$

where

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{13}$$

and

$$C_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right) - v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{14}$$

(b) Consider the following values of damping ratio:

(1) $\zeta = 0$; (2) $\zeta = 0.1$; (3) $\zeta = 1$; (4) $\zeta = 5$

where $\omega_n = 1.2\pi$ rad/sec, $x_0 = 1.5$ mm, and $v_0 = 2$ mm/sec.

Plot the following three figures (MATLAB is recommended):

(i) $x(t)$ versus t (0~8 sec)

(ii) $\dot{x}(t)$ versus t (0~8 sec)

(iii) $\dot{x}(t)$ versus $x(t)$ (called phase plane)

(c) Discuss the results in part (b)

Solution:

(a)

(i)

Assume that the solution $x(t)$ is of the form (Inman & Singh, 1994)

$$x(t) = ae^{\lambda t} \tag{15}$$

where a and λ are nonzero constants to be determined. Upon successive differentiation, Equation (15) becomes $\dot{x}(t) = \lambda a e^{\lambda t}$ and $\ddot{x}(t) = \lambda^2 a e^{\lambda t}$. Substitution of the assumed exponential form into Equation (8) yields

$$m\lambda^2 a e^{\lambda t} + k a e^{\lambda t} = 0 \tag{16}$$

Since the term $ae^{\lambda t}$ is never zero, Equation (16) can be divided by $ae^{\lambda t}$ to yield

$$m\lambda^2 + k = 0 \tag{17}$$

Solving this algebraically results in

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \tag{18}$$

where $j = \sqrt{-1}$ is the imaginary number and $\omega_n = \sqrt{k/m}$ is the natural frequency as before. Note that there are two values for λ, $\lambda = +\omega_n j$ and $\lambda = -\omega_n j$, because the equation for λ is of second order. This implies that there must be two solutions of Equation (8) as well. Substitution of Equation (18) into Equation (15) yields that the two solutions for $x(t)$ are

$$x(t) = a_1 e^{+j\omega_n t} \tag{19}$$

and

$$x(t) = a_2 e^{-j\omega_n t} \tag{20}$$

where a_1 and a_2 are complex-valued constants of integration. The Euler relations for trigonometric functions state that $2\sin\theta = \left(e^{\theta j} - e^{-\theta j}\right)$ and $2\cos\theta = \left(e^{\theta j} + e^{-\theta j}\right)$, where $j = \sqrt{-1}$. Using the Euler relations, Equation (20) can be written as

$$x(t) = A\cos(\omega_n t - \phi) \tag{21}$$

where A and ϕ are real-valued constants of integration. Each set of two constants is determined by the initial conditions, x_0 and v_0:

$$x_0 = x(0) = A\cos(\omega_n 0 - \phi) = A\cos\phi \tag{22}$$

and

$$v_0 = \dot{x}(0) = -\omega_n A\sin(\omega_n 0 - \phi) = \omega_n A\sin\phi \tag{23}$$

Solving these two simultaneous equations for the two unknowns A and ϕ yields

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)} \tag{24}$$

and

$$\phi = \tan^{-1} \frac{v_0}{x_0 \omega_n} \tag{25}$$

(ii)

Let $x(t)$ have the form given in Equation (15), $x(t) = ae^{\lambda t}$. Substitution of this form into Equation (8) yields

$$\left(m\lambda^2 + c\lambda + k\right) ae^{\lambda t} = 0 \tag{26}$$

Again, $ae^{\lambda t} \neq 0$, so that this reduces to a quadratic equation in λ of the form

$$m\lambda^2 + c\lambda + k = 0 \tag{27}$$

called the characteristic equation. This is solved using the quadratic formula to yield the two solutions

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{28}$$

In this case, the damping ratio ζ is less than 1 ($0 < \zeta < 1$) and the discriminant of Equation (28) is negative, resulting in a complex conjugate pair of roots. Factoring out (-1) from the discriminant in order to clearly distinguish that the second term is imaginary yields

$$\sqrt{\zeta^2 - 1} = \sqrt{(1 - \zeta^2)(-1)} = \sqrt{1 - \zeta^2}\, j \tag{29}$$

Thus the two roots become

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}\, j \tag{30}$$

Following the same argument as that made for the undamped response of Equation (20), the solution is then of the form

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{+j\sqrt{1-\zeta^2}\omega_n t} + a_2 e^{-j\sqrt{1-\zeta^2}\omega_n t}\right) \tag{31}$$

where a_1 and a_2 are arbitrary complex-valued constants of integration to be determined by the initial conditions. Using the Euler relations, this can be written as

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{32}$$

where A and ϕ are constants of integration and ω_d, called the damped natural frequency, is given by

$$\omega_d = \sqrt{1 - \zeta^2}\omega_n \tag{33}$$

in units of $\mathrm{rad/s}$. Each set of A and ϕ is determined by the initial conditions, x_0 and v_0:

$$x_0 = x(0) = Ae^{-\zeta\omega_n 0}\cos(\omega_d 0 - \phi) = A\cos\phi \tag{34}$$

Differentiating Equation (32) yields

$$\dot{x}(t) = -\zeta\omega_n A e^{-\zeta\omega_n t}\cos(\omega_d t - \phi) - \omega_d A e^{-\zeta\omega_n t}\sin(\omega_d t - \phi) \tag{35}$$

Let $t = 0$ and $A = x_0/\cos\phi$ in this last expression to get

$$v_0 = \dot{x}(0) = -\zeta\omega_n x_0 + x_0\omega_d \tan\phi \tag{36}$$

Solving this last expression for ϕ yields

$$\tan\phi = \frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d} \tag{37}$$

With this value of ϕ, the cosine becomes

$$\cos\phi = \frac{x_0\omega_d}{\sqrt{(\zeta\omega_n x_0 + v_0)^2 + (x_0\omega_d)^2}} \tag{38}$$

Thus the value of A and ϕ are determined to be

$$A = \sqrt{x_0^2 + \left(\frac{\zeta\omega_n x_0 + v_0}{\omega_d}\right)^2} \tag{39}$$

and

$$\phi = \tan^{-1}\frac{\zeta\omega_n x_0 + v_0}{x_0\omega_d} \tag{40}$$

(iii)

In this last case, the damping ratio is exactly one ($\zeta = 1$) and the discriminant of Equation (28) is equal to zero. This corresponds to the value of ζ that separates oscillatory motion from nonoscillatory motion. Since the roots are repeated, they have the value

$$\lambda_1 = \lambda_2 = -\omega_n \tag{41}$$

The solution takes the form

$$x(t) = (a_1 + a_2 t)\, e^{-\omega_n t} \tag{42}$$

where, again, the constants a_1 and a_2 are determined by the initial conditions. Substituting the initial displacement into Equation (42) and the initial velocity into the derivative of Equation (42) yields

$$a_1 = x_0, \quad a_2 = v_0 + \omega_n x_0 \tag{43}$$

(iv)

In this case, the damping ratio is greater than 1 ($\zeta > 1$). The discriminant of Equation (28) is positive, resulting in a pair of distinct real roots. These are

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{44}$$

The solution of Equation (8) then becomes

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{45}$$

which represents a nonoscillatory response. Again, the constants of integration C_1 and C_2 are determined by the initial conditions:

$$x_0 = x(0) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n 0} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n 0} = C_1 + C_2 \tag{46}$$

Differentiating Equation (32) yields

$$\dot{x}(t) = \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + \left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{47}$$

Let $t = 0$ in this last expression to get

$$v_0 = \dot{x}(0) = \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n C_1 + \left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n C_2 \tag{48}$$

Solving Equation (46) and (48) for C_1 and C_2 yields

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{49}$$

and

$$C_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right) - v_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{50}$$

(b) The MATLAB code in the main file is shown below:

```
clc; clf; clear all;
hold on;
t0 = 0; tf = 8;
tspan = [t0 tf];
x0v0 = [2; 1];
[t,x1] = ode45('Q2MotionFunction1', tspan , x0v0);
figure(1);
hold on;
plot(t, x1(:,1),'color',[238, 64, 53]/256,'LineWidth',2.5);
figure(2);
hold on;
plot(t, x1(:,2),'color',[238, 64, 53]/256,'LineWidth',2.5);
[t,x2] = ode45('Q2MotionFunction2', tspan , x0v0);
figure(1);
plot(t, x2(:,1),'color',[128, 0, 128]/256,'LineWidth',2.5);
figure(2);
plot(t, x2(:,2),'color',[128, 0, 128]/256,'LineWidth',2.5);
[t,x3] = ode45('Q2MotionFunction3', tspan , x0v0);
figure(1);
```

```
plot(t, x3(:,1),'color',[123, 192, 67]/256,'LineWidth',2.5);
figure(2);
plot(t, x3(:,2),'color',[123, 192, 67]/256,'LineWidth',2.5);
[t,x4] = ode45('Q2MotionFunction4', tspan , x0v0);
figure(1);
plot(t, x4(:,1),'color',[3, 146, 207]/256,'LineWidth',2.5);
figure(2);
plot(t, x4(:,2),'color',[3, 146, 207]/256,'LineWidth',2.5);
figure(3);
hold on;
plot(x1(:,1), x1(:,2),'color',[238, 64, 53]/256,'LineWidth',2.5);
plot(x2(:,1), x2(:,2),'color',[128, 0, 128]/256,'LineWidth',2.5);
plot(x3(:,1), x3(:,2),'color',[123, 192, 67]/256,'LineWidth',2.5);
plot(x4(:,1), x4(:,2),'color',[3, 146, 207]/256,'LineWidth',2.5);
figure(1);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x\left(t\right), \mathrm{\ \left(mm\right)}$', ...
    'interpreter','latex');
legend('$\zeta=0$', '$\zeta=0.1$', '$\zeta=1$', '$\zeta=5$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "x_vs_t"+".pdf";
saveas(gcf,filename);
figure(2);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\dot{x}\left(t\right), \mathrm{\ \left(mm/s\right)}$', ...
    'interpreter','latex');
legend('$\zeta=0$', '$\zeta=0.1$', '$\zeta=1$', '$\zeta=5$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "xdot_vs_t"+".pdf";
saveas(gcf,filename);
figure(3);
grid on;
xlabel('$x\left(t\right), \mathrm{\ \left(mm\right)}$', ...
    'interpreter','latex');
ylabel('$\dot{x}\left(t\right), \mathrm{\ \left(mm/s\right)}$', ...
    'interpreter','latex');
```

```
legend('$\zeta=0$', '$\zeta=0.1$', '$\zeta=1$', '$\zeta=5$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
hold off;
filename = "xdot_vs_x"+".pdf";
saveas(gcf,filename);
```

The MATLAB code in the function that defines the motion dynamics of the spring-damper system is shown below:

Case (1)

```
function xdot = Q2MotionFunction1(t,x)
omega_n = 1.5*pi;
zeta = 0;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

Case (2)

```
function xdot = Q2MotionFunction2(t,x)
omega_n = 1.5*pi;
zeta = 0.1;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

Case (3)

```
function xdot = Q2MotionFunction3(t,x)
omega_n = 1.5*pi;
zeta = 1;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

Case (4)

```
function xdot = Q2MotionFunction4(t,x)
omega_n = 1.5*pi;
zeta = 5;
xdot(1) = x(2);
xdot(2) = -omega_n^2*x(1)-2*zeta*omega_n*x(2);
xdot = xdot(:);
```

(i) The results for $x(t)$ are plotted in Figure 1.

(ii) The results for $\dot{x}(t)$ are plotted in Figure 2.

(iii) The results for phase portraits are plotted in Figure 3.

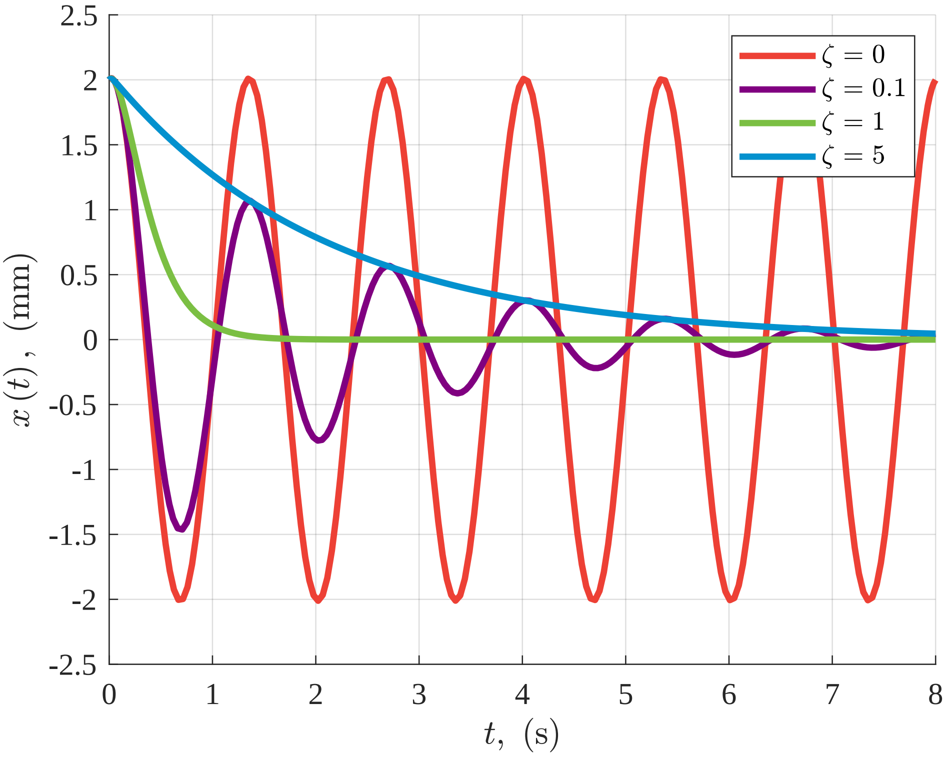


Figure 1: Results for $x(t)$ versus t.

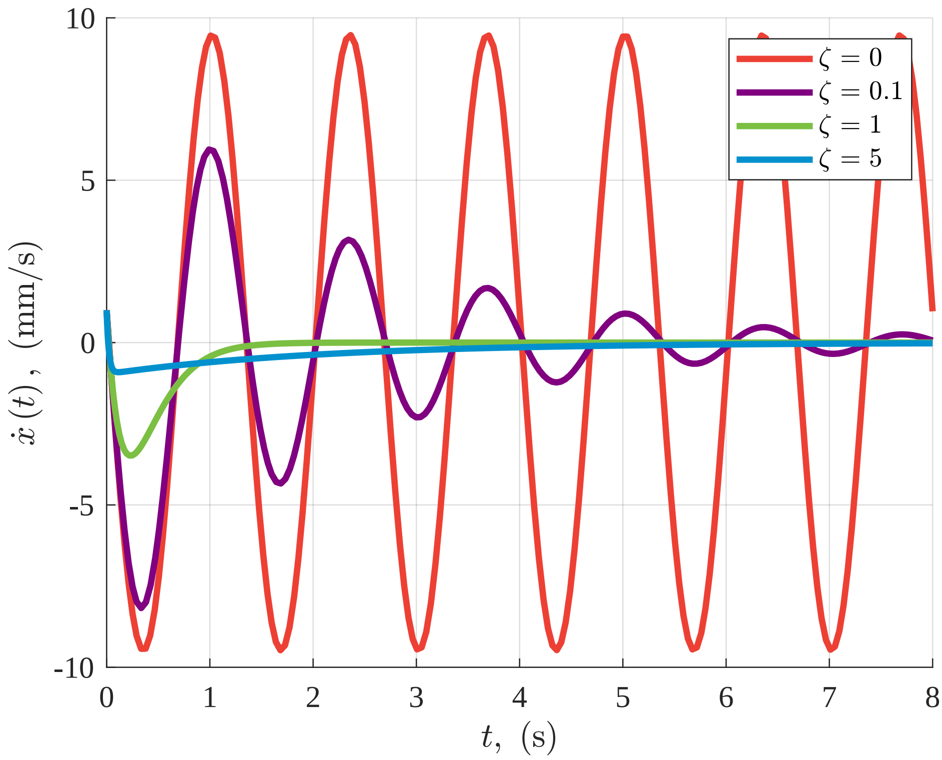


Figure 2: Results for $\dot{x}(t)$ versus t.

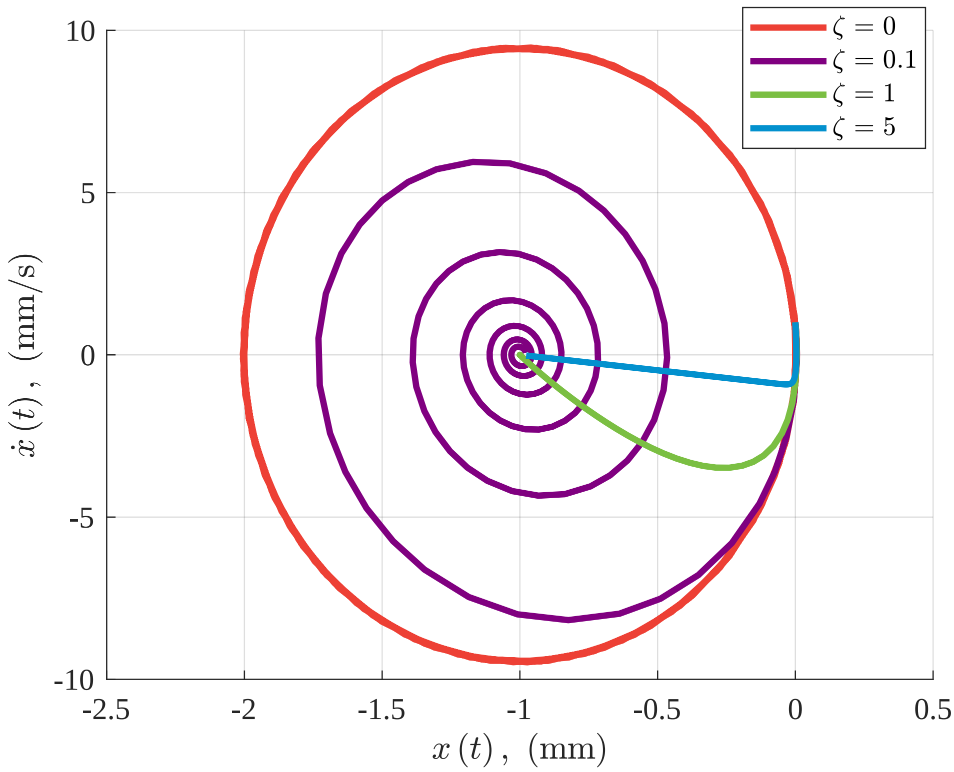


Figure 3: Results for $\dot{x}(t)$ versus $x(t)$ (called phase plane).

(c) The discussion is listed below:

- For undamped case, the system is in the harmonic motion and will never end, which is marginal stable.
- Critical damping returns the system to equilibrium as fast as possible without overshooting.
- An underdamped system will oscillate through the equilibrium position.
- An overdamped system moves more slowly toward equilibrium than one that is critically damped.

Problem 3

For a single degree of freedom damped system under harmonic force, the magnification factor M is found as

$$M = \frac{1}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}} \tag{51}$$

where $r = \frac{\omega}{\omega_n}$.
Show the maximum value of M for $0 < \zeta < \frac{1}{\sqrt{2}}$,

$$M_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{52}$$

where $r = \sqrt{1-2\zeta^2}$.
Solution:

To get the maximum value of M, we need to minimize $\left(1-r^2\right)^2 + (2\zeta r)^2$. We can regard $\left(1-r^2\right)^2 + (2\zeta r)^2$ as a function with respect to r:

$$f(r) = \left(1-r^2\right)^2 + (2\zeta r)^2 = r^4 + \left(4\zeta^2 - 2\right) r^2 + 1 \tag{53}$$

Because $0 < \zeta < \frac{1}{\sqrt{2}}$, $f(r)$ is a quadratic function with respect to r^2. This function reaches the minimum point at

$$r^2 = -\frac{4\zeta^2 - 2}{2} \tag{54}$$

that is

$$r = \sqrt{1-2\zeta^2} \tag{55}$$

In this case, the magnification factor M reaches maximum, which equals

$$M_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{56}$$

References

Inman, D. J. & Singh, R. C. (1994). *Engineering vibration*, volume 3. Prentice Hall Englewood Cliffs, NJ.

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DEPARTMENT OF MECHANICAL & AUTOMATION ENGINEERING

MAEG5080 Smart Materials & Structures

Assignment #2

by

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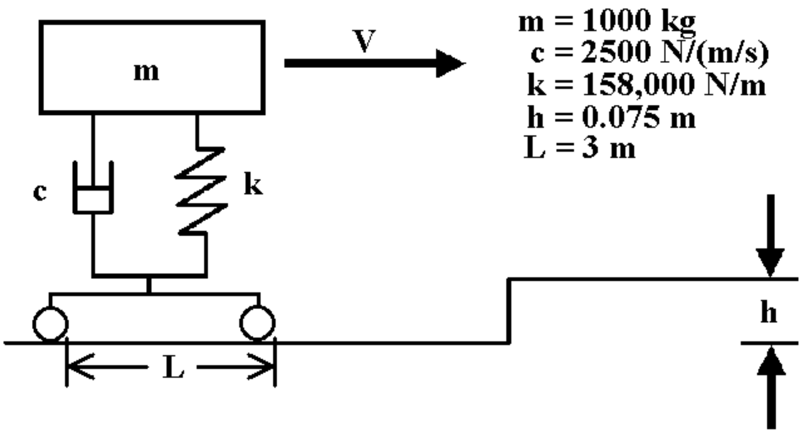
Liuchao Jin

2022-23 Term 1

Problem 1

The vertical dynamics of a two-axle vehicle with a suspension system are modeled as a SDOF system, as shown in the following figure. The vehicle travels to the right at a speed V and encounters a step change in the road height.



(a) Develop the equation of motion for this "base-driven" system.

(b) Develop an expression for the vertical motion of the vehicle following its encounter with the road height change. Use nominal speed V = 24 m/s, plot this motion as a function of time. Discuss your findings.

(c) Possibly computer-assisted: What is the maximum height attainable by the vehicle at any speed? At what speed(s) is this achieved? At what speed(s) will the residual suspension motion (when both wheels are on the higher part of the road) be minimized?

Solution:

(a) The free-body diagram of m is shown on the right side of Figure 1. The force exerted by the spring k on the mass m is downward as it tends to restore to the undeformed position. Note that the gravitational force, mg, is not included in the free-body diagrams.

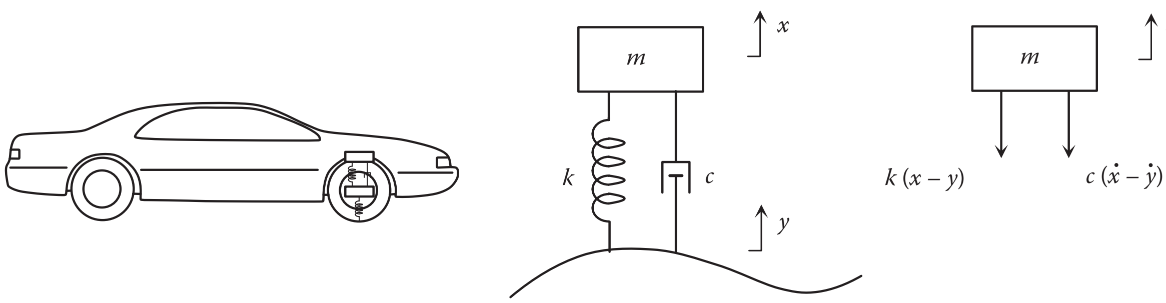


Figure 1: Simplified Suspension System of Car Model.

Applying Newton's second law to the mass m gives

$$+\uparrow x: \sum F_x = ma_x \tag{1}$$

$$k(x-y)+c(\dot{x}-\dot{y}) = m_2\ddot{x}_2 \tag{2}$$

Rearranging the equations into the standard input–output form,

$$m\ddot{x}+c\dot{x}+kx = c\dot{y}+ky \tag{3}$$

(b) When the vehicle encounters a step change in the road height, the equation of motion for this "base-driven" system can be expressed as

$$m\ddot{x}+c\dot{x}+kx = \frac{ch}{2}\delta\left(t\right)+\frac{ch}{2}\delta\left(t-\frac{L}{V}\right)+\frac{kh}{2}u\left(t\right)+\frac{kh}{2}u\left(t-\frac{L}{V}\right) \tag{4}$$

where $\delta\left(t-a\right)$ is the Dirac Delta Function, and $u\left(t-a\right)$ is the Heaviside Function. Taking the Laplace transform to Equation (4) with the initial condition $x\left(0\right)=0$ and $\dot{x}\left(0\right)=0$ yields that

$$ms^2X\left(s\right)+csX\left(s\right)+kX\left(s\right) = \frac{ch}{2}+\frac{ch}{2}e^{-\frac{L}{V}s}+\frac{kh}{2}\frac{1}{s}+\frac{kh}{2}\frac{e^{-\frac{L}{V}s}}{s} \tag{5}$$

Solving the Laplace transform in Equation (5) gets that

$$\begin{aligned} x\left(t\right) = & \frac{3u\left(t-\frac{1}{8}\right)}{80} \\ & + \frac{3\sqrt{2503}u\left(t-\frac{1}{8}\right)e^{-\frac{5}{4}t+\frac{5}{32}}\sin\left(\frac{\sqrt{2503}t}{4}-\frac{\sqrt{2503}}{32}\right)}{40048} \\ & - \frac{3u\left(t-\frac{1}{8}\right)e^{-\frac{5}{4}t+\frac{5}{32}}\cos\left(\frac{\sqrt{2503}t}{4}-\frac{\sqrt{2503}}{32}\right)}{40048} \\ & + \frac{3\sqrt{2503}e^{-\frac{5}{4}t}\sin\frac{\sqrt{2503}t}{4}}{40048} - \frac{3e^{-\frac{5}{4}t}\cos\frac{\sqrt{2503}t}{4}}{40048} + \frac{3}{80} \end{aligned} \tag{6}$$

The plot of this motion as a function of time is shown in Figure 2. From the system response shown in Figure 2, we can see that when the vehicle encounters a step change in the road height, the system is underdamped and the steady time is about 2 seconds.

(c) I use the following MATLAB codes to find the answer to this question:

```
clc; clf; clear all;
syms s t V
m = 1000; c = 2500; k = 158000; h = 0.075; L = 3;
f1 = ilaplace(c*h/2*1/(m*s^2+c*s+k),t);
f2 = ilaplace(c*h/2*exp(-L/V*s)/(m*s^2+c*s+k),t);
f3 = ilaplace(k*h/2*1/s*1/(m*s^2+c*s+k),t);
f4 = ilaplace(k*h/2*1/s*exp(-L/V*s)/(m*s^2+c*s+k),t);
```

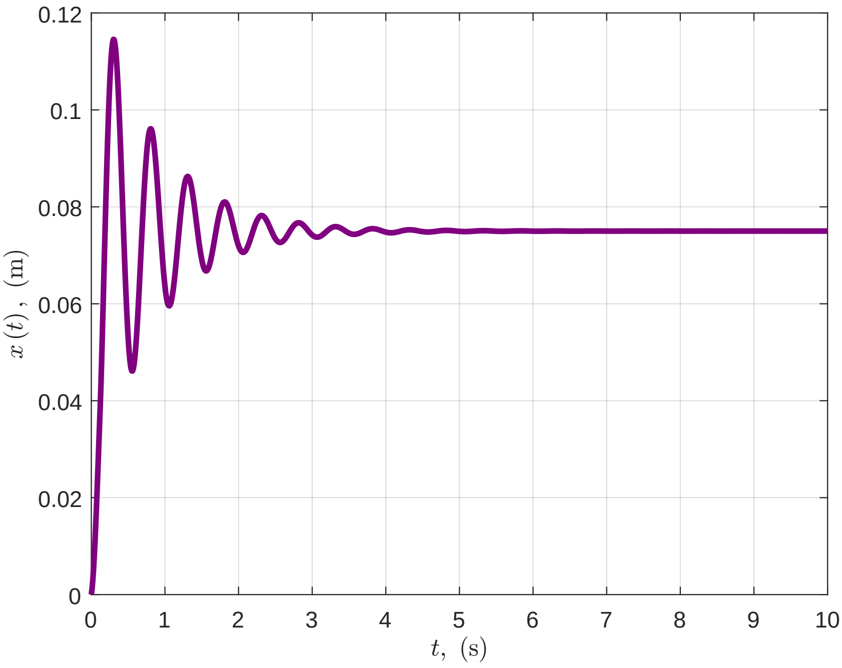


Figure 2: System motion diagram.

```
f(t,V) = f1+f2+f3+f4;
figure;
fplot(f,'color',[128, 0, 128]/256,'LineWidth',2.5)
xlim([0 10]);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x\left(t\right), \mathrm{\ \left(m\right)}$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2-1-Motion"+".pdf";
saveas(gcf,filename);
figure;
V = 0.01:0.01:3000;
sv = size(V);
Mx = zeros(1,sv(2));
for i = 1:sv(2)
    t = [0:0.01:2 3/V(i):0.01:(3/V(i)+2)];
    Mx(i) = max(f(t,V(i)));
end
plot(V,Mx,'color',[128, 0, 128]/256,'LineWidth',2.5);
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x\left(t\right), \mathrm{\ \left(m\right)}$', ...
    'interpreter','latex');
a = get(gca,'XTickLabel');
```

```
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q2-1-Motion"+".pdf";
saveas(gcf,filename);
```

The results gotten from MATLAB are shown in Figure 3.

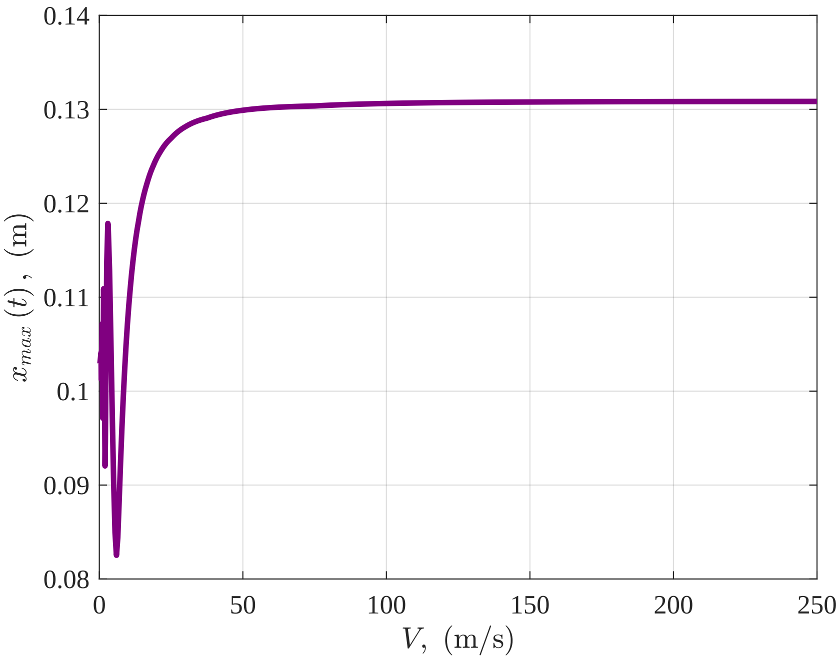


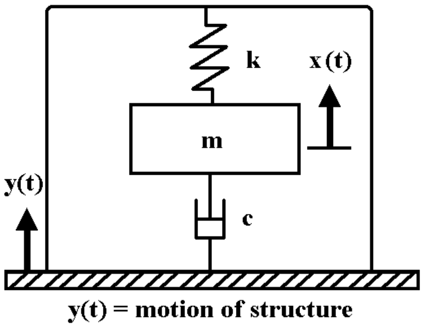
Figure 3: The maximum displacement versus the velocity.

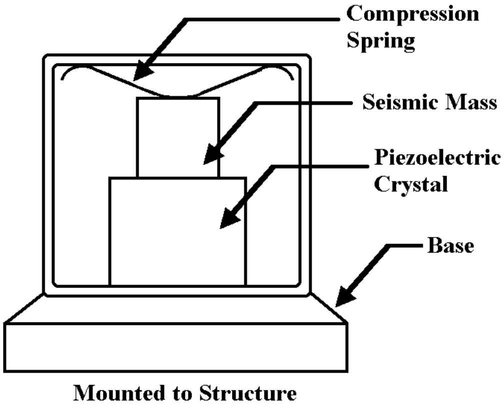
From Figure 3, it can be concluded that

- The maximum height attainable by the vehicle at any speed is 0.1309 m, with the velocity equal to 2560 m/s.
- The residual suspension motion (when both wheels are on the higher part of the road) is minimized when the velocity is equal to 11.9450 m/s. At this speed, the minimum displacement is equal to 0.0825 m

Problem 2

An accelerometer has a suspended mass of 0.01 kg with a damped natural frequency of vibration of 150 Hz. When mounted on an engine undergoing an acceleration of 1 g at an operating speed of 6000 rpm, the acceleration is recorded as 9.5 m/s2 by the instrument. Find the damping constant and the spring stiffness of the accelerometer (Choose damping ratio close to 0.7 if possible).





Solution:

The amplitude ratio of the accelerometer is equal to (Inman & Singh, 1994)

$$M = \frac{g_m}{g_t} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{9.5}{9.8} \tag{7}$$

In addition, the forcing function frequency and the damped natural frequency of the system are equal to

$$\omega = \frac{2\pi \times 6000}{60} = 200\pi \text{ rad/s} \tag{8}$$

and

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 2\pi f = 300\pi \text{ rad/s} \tag{9}$$

Solving Equation (7), (8), and (9) to obtain ζ and ω_n, we can know that $\zeta = 0.72$ and $\omega_n = 1363.9$ rad/s. Therefore, the damping constant and the spring stiffness of the accelerometer are equal to

$$c = 2m\zeta\omega_n = 19.72 \text{ N/(m/s)} \tag{10}$$

and

$$k = m\omega_n^2 = 1.86 \times 10^4 \text{ N/m} \tag{11}$$

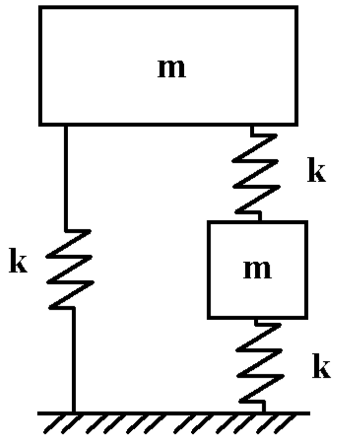
Problem 3

A spring-mass system (see figure), which is constrained to move in the vertical direction only.

(a) Derive the equations of motion.

(b) Find the natural frequencies.

(c) Find and sketch the mode shapes.



Solution:

(a) We set the coordinate for the upper block as x_1 and the lower block as x_2. Then, the equations of motion is derived as follows:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

(b) We can use determinate to calculate the natural frequencies.

$$\begin{aligned} \det\left([k] - [m]\,\omega^2\right) = 0 &\Rightarrow \left(2k - m\omega^2\right)\left(2k - m\omega^2\right) - k^2 = 0 \\ &\Rightarrow m^2\omega^4 - 4km\omega^2 + 3k^2 = 0 \\ &\Rightarrow \omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{3k}{m}} \end{aligned} \tag{13}$$

(c) For Mode 1 ($\omega_1 = \sqrt{\frac{k}{m}}$),

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{14}$$

For Mode 2 ($\omega_1 = \sqrt{\frac{3k}{m}}$),

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{15}$$

Based on this, the mode shapes are drawn as shown in Figure 4.

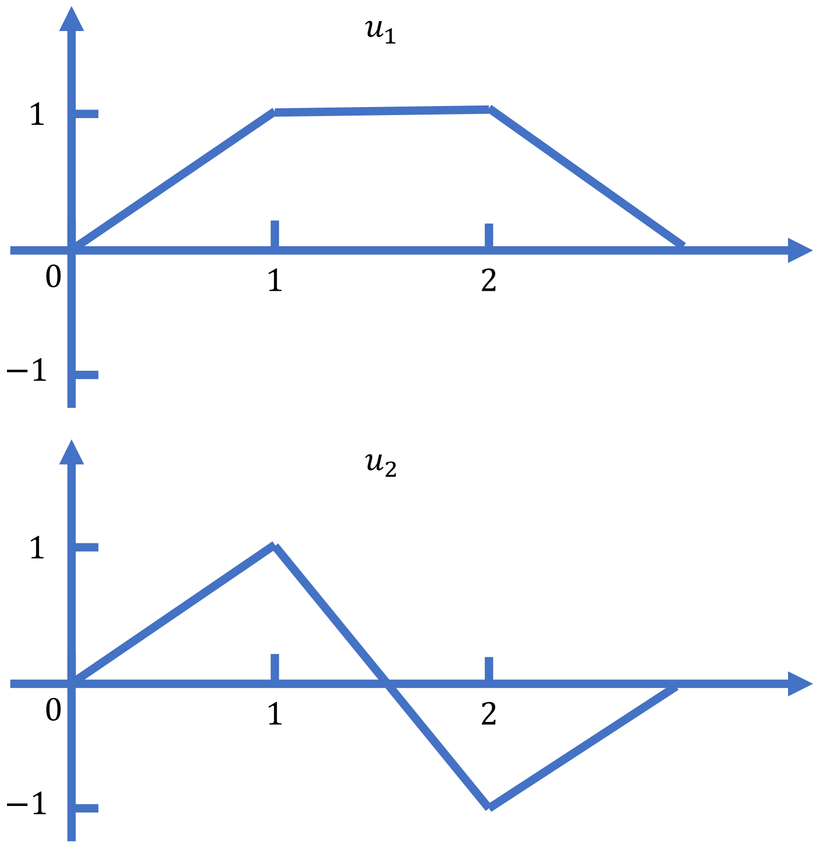


Figure 4: Mode shapes for the system.

References

Inman, D. J. & Singh, R. C. (1994). *Engineering vibration*, volume 3. Prentice Hall Englewood Cliffs, NJ.

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MAEG5080 Smart Materials & Structures

Assignment #3

by

Liuchao JIN (Student ID: 1155184008)

Liuchao Jin

2022-23 Term 1

Problem 1

Under sinusoidal input with amplitude 0.03 m, $90°$ phase angle and frequency 2.5 Hz, plot the following figures (SIMULINK is highly recommended) and hand in the results with your own program codes:

(i) Damping force $F_d(t)$ versus displacement $x(t)$ (for v = 0 V, 1 V, 2 V cases).

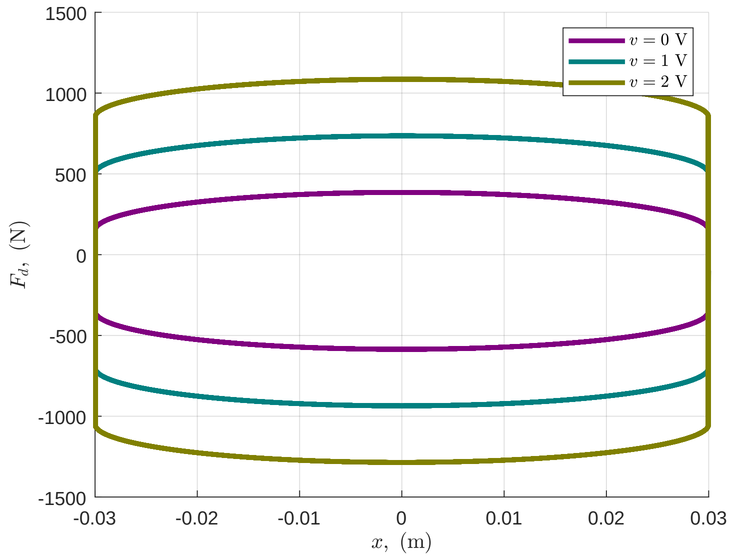

Figure 1: Damping force $F_d(t)$ versus displacement $x(t)$ (for v = 0 V, 1 V, 2 V cases).

(ii) Damping force $F_d(t)$ versus velocity $\dot{x}(t)$ (for v = 0 V, 1 V, 2 V cases).

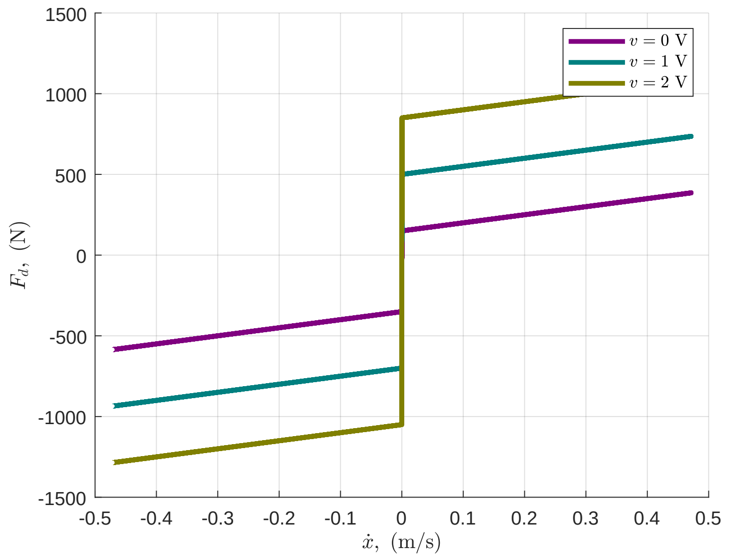

Figure 2: Damping force $F_d(t)$ versus velocity $\dot{x}(t)$ (for v = 0 V, 1 V, 2 V cases).

Problem 2

(a) The equation of motion is shown as follows:

$$\begin{cases} m_1\ddot{x}_1 + k_1x_1 - k_1x_2 + F_d = 0 \\ m_2\ddot{x}_2 + c_2\dot{x} + (k_1 + k_2)\,x_2 - c_2\dot{x}_b - k_2x_b - k_1x_1 - F_d = 0 \end{cases} \tag{1}$$

(b) The state space representation of the car suspension system with the outputs $x_1, \dot{x}_1, \dot{x}_2$ is

$$\begin{aligned} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & 0 & 0 \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & 0 & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{m_1} & 0 & 0 \\ \frac{1}{m_2} & \frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} F_d \\ x_b \\ \dot{x}_b \end{bmatrix} \\ &\triangleq AX + Bu \end{aligned} \tag{2}$$

$$\begin{aligned} Y = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &\triangleq CX + Du \end{aligned} \tag{3}$$

Problem 3

The on-off controller is set up as follows:

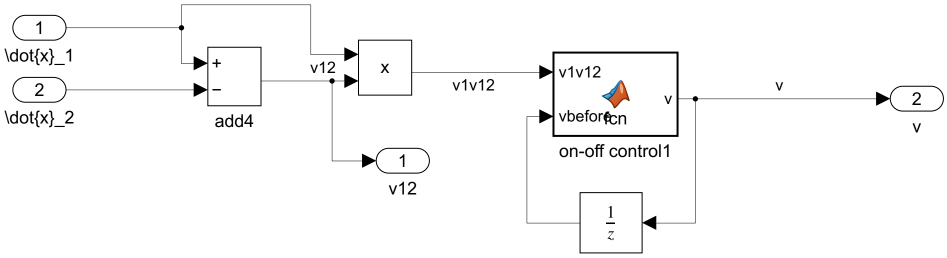

Figure 3: On-off controller.

Problem 4

The response for the system is list below:

(a) Bump excitation: displacement versus time for $t = 0 : 0.001 : 8$.

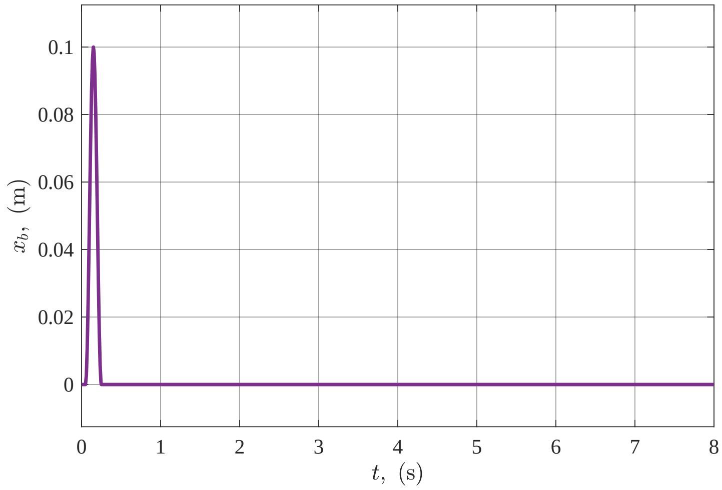

Figure 4: Bump excitation: displacement versus time for $t = 0 : 0.001 : 8$.

(b) $x_1(t)$ versus t for $t = 0 : 0.001 : 8$ (for $v = 0$ V, 2 V and controlled cases).

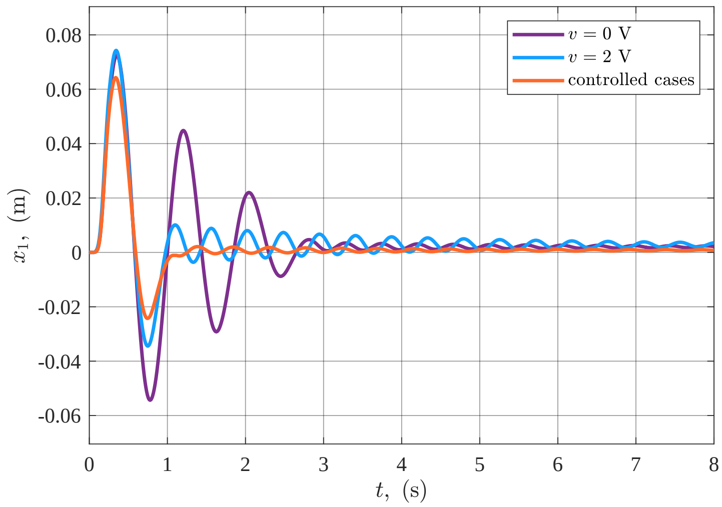

Figure 5: $x_1(t)$ versus t for $t = 0 : 0.001 : 8$ (for $v = 0$ V, 2 V and controlled cases).

(c) $\ddot{x}_1(t)$ versus t for $t = 0 : 0.001 : 8$ (for $v = 0$ V, 2 V and controlled cases).

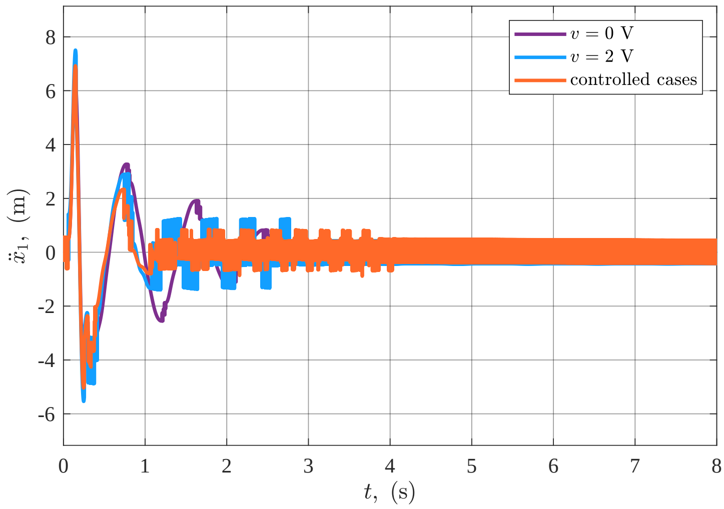

Figure 6: $\ddot{x}_1(t)$ versus t for $t = 0 : 0.001 : 8$ (for $v = 0$ V, 2 V and controlled cases).

(d) Voltage $v(t)$ versus t for $t = 0 : 0.001 : 8$ (only for the controlled case).

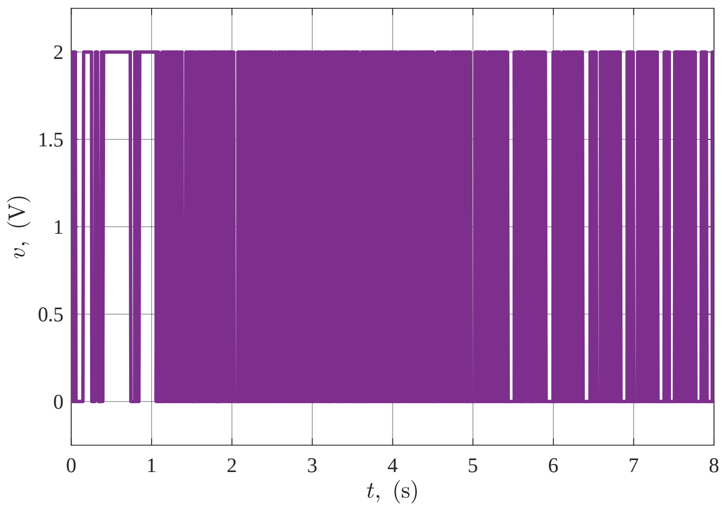

Figure 7: Voltage $v(t)$ versus t for $t = 0 : 0.001 : 8$ (only for the controlled case).

Discussion: From Figure 5, we can know that the controlled case converges to stability more quickly than the other two cases and has a lower overshoot and oscillation. The amplitude of oscillation for the acceleration when the system is controlled by on-off controller is also lower than that of other cases. Therefore, we can conclude that the MR damper has the better performance and can make the system stable to equilibrium point.

Appendix

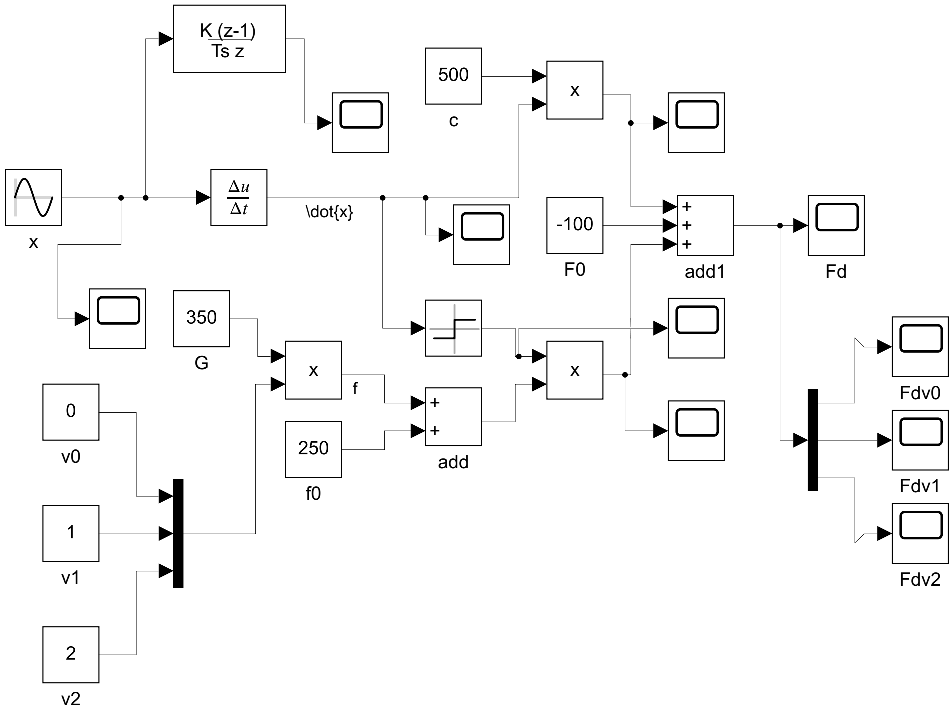

Figure 8: Block diagram for MR damper.

Input Matlab source for plot MR damper response diagram:

```
clf;
x = out.xin.signals.values;
dx = out.dxin.signals.values;
Fdv0 = out.Fdv0.signals.values;
Fdv1 = out.Fdv1.signals.values;
Fdv2 = out.Fdv2.signals.values;

figure(1);
hold on;
plot(x,Fdv0,"Color",[128, 0, 128]/256,'LineWidth',2.5);
plot(x,Fdv1,"Color",[0, 128, 128]/256,'LineWidth',2.5);
plot(x,Fdv2,"Color",[128, 128, 0]/256,'LineWidth',2.5);
hold off;
grid on;
xlabel('$x, \mathrm{\ \left(m\right)}$','interpreter','latex');
ylabel('$F_d, \mathrm{\ \left(N\right)}$', 'interpreter','latex');
legend('$v=0 \mathrm{\ V}$','$v=1 \mathrm{\ V}$','$v=2 \mathrm{\ V}$',...
    'interpreter','latex');
% a = get(gca,'XTickLabel');
% set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q3-1-Fd-x"+".pdf";
```

```
saveas(gcf,filename);
figure(2);
hold on;
plot(dx,Fdv0,"Color",[128, 0, 128]/256,'LineWidth',2.5);
plot(dx,Fdv1,"Color",[0, 128, 128]/256,'LineWidth',2.5);
plot(dx,Fdv2,"Color",[128, 128, 0]/256,'LineWidth',2.5);
hold off;
grid on;
xlabel('$\dot{x}, \mathrm{\ \left(m/s\right)}$','interpreter','latex');
ylabel('$F_d, \mathrm{\ \left(N\right)}$', 'interpreter','latex');
legend('$v=0 \mathrm{\ V}$','$v=1 \mathrm{\ V}$','$v=2 \mathrm{\ V}$',...
    'interpreter','latex');
% a = get(gca,'XTickLabel');
% set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q3-1-Fd-dx"+".pdf";
saveas(gcf,filename);
```

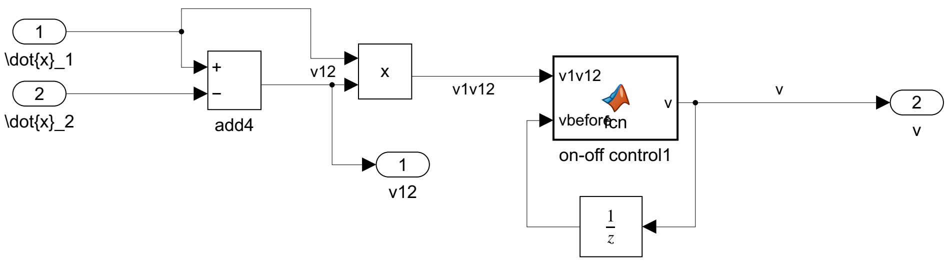

Figure 9: Block diagram for on-off controller.

Input Matlab source for on-off controller:

```
function v = fcn(v1v12,vbefore)

if v1v12 > 0
    v = 2;
elseif v1v12 < 0
    v = 0;
else
    v = vbefore;
end
```

Input Matlab source for plotting x_b, v, x_1, and $\ddot{x}_1$:

```
clear all; clc;
figg1 = openfig('v.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$v, \mathrm{\ \left(V\right)}$', 'interpreter','latex');
% legend('$v=0 \mathrm{\ V}$','$v=1 \mathrm{\ V}$','$v=2 \mathrm{\ V}$',...
```

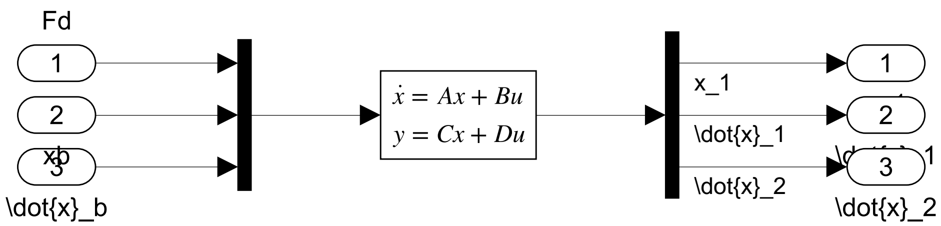

Figure 10: Block diagram for state space function.

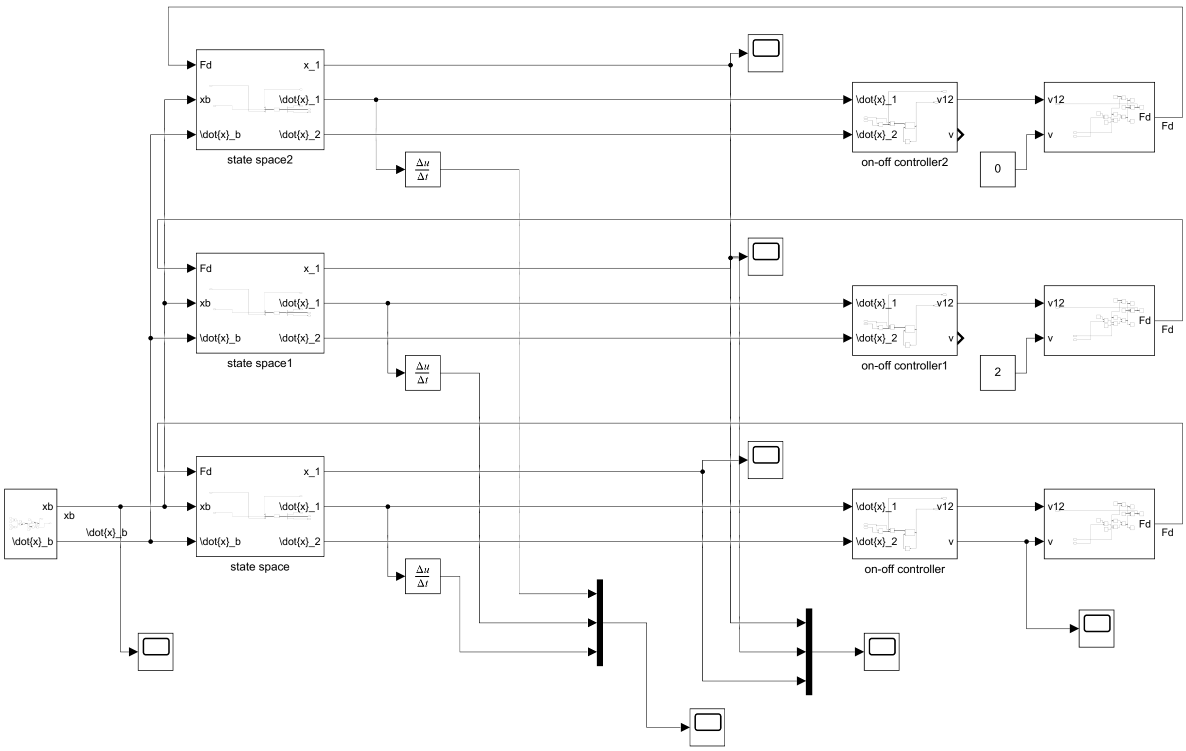

Figure 11: Block diagram for the whole system.

```
%      'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "v"+".pdf";
saveas(gcf,filename);
close(figg1);
figg2 = openfig('xb.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_b, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
title('');
% legend('$v=0 \mathrm{\ V}$','$v=2 \mathrm{\ V}$','$\text{controlled cases}$',...
%      'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
```

```
set(gcf,'renderer','painters');
filename = "xb"+".pdf";
saveas(gcf,filename);
close(figg2);
figg3 = openfig('x.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$x_1, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
title('');
legend('$v=0 \mathrm{\ V}$','$v=2 \mathrm{\ V}$','controlled cases',...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "x"+".pdf";
saveas(gcf,filename);
close(figg3);
figg4 = openfig('ddotx.fig','reuse');
grid on;
xlabel('$t, \mathrm{\ \left(s\right)}$','interpreter','latex');
ylabel('$\ddot{x}_1, \mathrm{\ \left(m\right)}$', 'interpreter','latex');
title('');
legend('$v=0 \mathrm{\ V}$','$v=2 \mathrm{\ V}$','controlled cases',...
    'interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "ddotx"+".pdf";
saveas(gcf,filename);
close(figg4);
```

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MAEG5080 Smart Materials & Structures

Assignment #4

by

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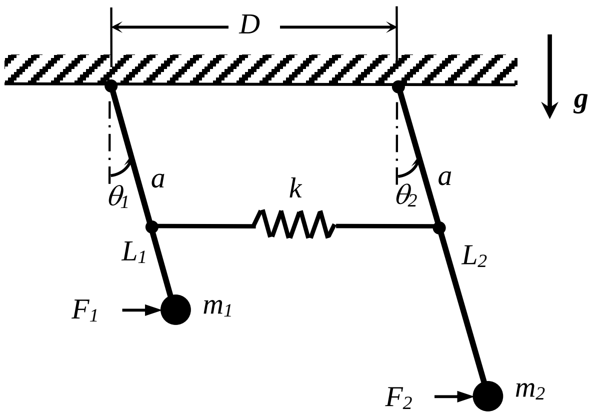
Liuchao Jin

2022-23 Term 1

Problem 1

(40 points) Two plane pendulums, masses m_1 and m_2, are connected by respective massless rigid links L_1 and L_2 shown in the figure. Those two links are coupled via a spring of stiffness k at a distance a from the supports. The spring is unstretched when the links are vertical. The pendulums are respectively excited by external forces F_1(t) and F_2(t), which remain horizontal at all times. Derive the equations of motion for the system. Assume small motions on the plane.



Solution:

In terms of generalized coordinate q, the Lagrange's equation subject to a generalized force has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} + \frac{\partial U}{\partial q} = Q \tag{1}$$

Kinetic energy:

$$T = \frac{1}{2}m_1\left(L_1\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(L_2\dot{\theta}_2\right)^2 \tag{2}$$

Potential energy:

$$U = m_1 g L_1\left(1-\cos\theta_1\right) + m_2 g L_2\left(1-\cos\theta_2\right) + \frac{1}{2}k\left(2a\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)^2 \tag{3}$$

Rayleigh's damping (or dissipation) function:

$$D = 0 \tag{4}$$

Generalized force:

$$Q = \sum_l F_l \cdot \frac{\partial r_l}{\partial q} = F_i L_i \cos\theta_i, \quad i = 1, 2 \tag{5}$$

For $q_1 = \theta_1$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} + \frac{\partial U}{\partial q_1} = Q \tag{6}$$

$$\frac{d}{dt}\left(m_1 L_1^2 \dot{\theta}_1\right) - 0 + 0 + m_1 g L_1 \sin\theta_1 + \frac{1}{2} \times 2k \left(2a \sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right) \times 2a \times \frac{1}{2}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = F_1 L_1 \cos\theta_1 \tag{7}$$

$$m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 \sin\theta_1 + ka^2 \sin(\theta_1 - \theta_2) = F_1 L_1 \cos\theta_1 \tag{8}$$

For $q_2 = \theta_2$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} + \frac{\partial U}{\partial q_2} = Q \tag{9}$$

$$\frac{d}{dt}\left(m_2 L_2^2 \dot{\theta}_2\right) - 0 + 0 + m_2 g L_2 \sin\theta_2 + \frac{1}{2} \times 2k \left(2a \sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right) \times 2a \times \left(-\frac{1}{2}\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\right) = F_2 L_2 \cos\theta_2 \tag{10}$$

$$m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \sin\theta_2 - ka^2 \sin(\theta_1 - \theta_2) = F_2 L_2 \cos\theta_2 \tag{11}$$

Therefore, the equations of motion for the system is

$$\begin{cases} m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 \sin\theta_1 + ka^2 \sin(\theta_1 - \theta_2) = F_1 L_1 \cos\theta_1 \\ m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \sin\theta_2 - ka^2 \sin(\theta_1 - \theta_2) = F_2 L_2 \cos\theta_2 \end{cases} \tag{12}$$

For the small motions on the plane, the equations of motion for the system is

$$\begin{cases} m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 \theta_1 + ka^2 (\theta_1 - \theta_2) = F_1 L_1 \cos\theta_1 \\ m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \theta_2 - ka^2 (\theta_1 - \theta_2) = F_2 L_2 \cos\theta_2 \end{cases} \tag{13}$$

Problem 2

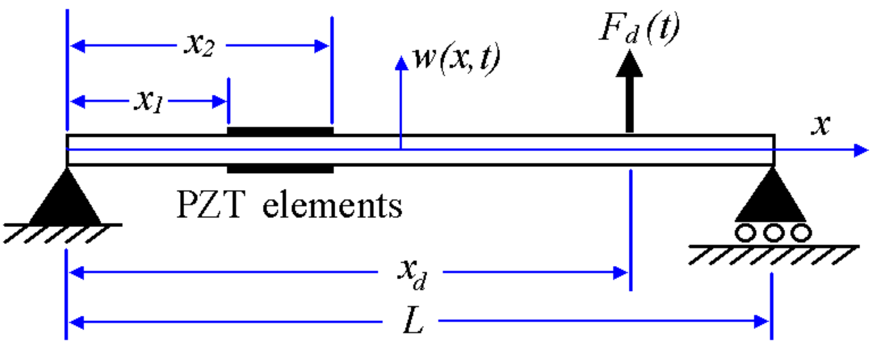
(60 points) Consider a simply-supported uniform beam with PZT actuators mounted on its top and bottom surfaces between $x = x_1$ and $x = x_2$. The patches are activated so as to produce pure bending in the beam. A discrete force $F_d(t)$ has been applied at $x = x_d$.

(a) Derive the partial differential equation for the transverse beam response $w(x,t)$ using extended Hamilton's Principle.

(b) Applying Galerkin's method with comparison functions (Hint: you may use $\phi_r(x) = \sin(r\pi x/L)$), determine the discretized ordinary differential equations (use three expansion terms, i.e., N = 3). Assume the damping matrix $C = \alpha M + \beta K$, where $\alpha = 0.6$ and $\beta = 1.2\times10^{-6}$.

(c) Changing the second order differential equations into the state space form, then use state feedback $u = -K_c x$ for the system, where control gain is given as Kc = [-55400 -22549 15848 -753 -249 174]. Under an impulse excitation for $F_d(t)$with magnitude 1/100 N.sec, plot the time response of transverse displacement at $x = 0.6L$ for the cases without and with control (0 to 0.5 sec). Also plot the corresponding voltage for the controlled case (0 to 0.5 sec).

The rectangular cross sections of the beam and PZT are given as: width b = 2 cm, thickness t_b = 2 mm, and t_p = 0.6 mm, respectively. Other parameters are given as follows: L = 50 cm, x_1 = 15 cm, x_2 = 24 cm, x_d = 30 cm, $d_{31} = -175\times10^{-12}$ m/V, $\rho_b = 2700$ kg/m^3, $\rho_p = 7600$ kg/m^3, $E_b = 7\times10^{10}$ N/m^2, $E_p = 6.5\times10^{10}$ N/m^2.



Solution:

(a) Potential energies:

$$V_b = \frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{14}$$

$$V_p = \int_0^L E_p I_p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \left[H(x-x_1) - H(x-x_2)\right] dx \tag{15}$$

where H is the Heaviside's function.

Kinetic energies:

$$T_b = \frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx \tag{16}$$

$$T_p = \int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 \left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx \tag{17}$$

Virtual work:

$$\delta W_d = \int_0^L f\left(x,t\right)\delta w\left(x,t\right) dx \tag{18}$$

From the constitutive equation of the piezoelectric materials

$$S_1 = s_{11}^E T_1 + d_{31} E_3 \tag{19}$$

$$T_1 = E_p\left(S_1 - d_{31}E_3\right) \tag{20}$$

where $E_p = \frac{1}{s_{11}^E}$, $E_3 = \frac{v(t)}{t_p}$.

The virtual work done by the induced strain (force) is:

$$\delta W_p = 2\int_0^L E_p d_{31} b v\left(t\right)\delta\left(\frac{\partial u_p}{\partial x}\right)\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx \tag{21}$$

where b is the width of beam and piezo layer. and

$$u_p = -\left(\frac{t_b + t_p}{2}\right)\frac{\partial w}{\partial x} \tag{22}$$

Let $a = \frac{t_b+t_p}{2}$,

$$\delta W_p = -2\int_0^L E_p d_{31} a b v\left(t\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx \tag{23}$$

Apply extended Hamilton's principle:

$$\int_{t_1}^{t_2}\left(\delta T - \delta V + \delta W_{NC}\right) dt = 0 \tag{24}$$

$$\begin{aligned}
&\int_{t_1}^{t_2}\left(\delta\left\{\frac{1}{2}\int_0^L \rho_b A_b\left(\frac{\partial w}{\partial t}\right)^2 dx\right\} + \delta\left\{\int_0^L \rho_p A_p\left(\frac{\partial w}{\partial t}\right)^2\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]dx\right\}\right)dt\\
&-\int_{t_1}^{t_2}\left(\delta\left\{\frac{1}{2}\int_0^L E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\} + \delta\left\{\int_0^L E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)^2\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]dx\right\}\right)dt\\
&+\int_{t_1}^{t_2}\left(\int_0^L f\left(x,t\right)\delta w\left(x,t\right)dx - 2\int_0^L E_p d_{31} a b v\left(t\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]dx\right)dt = 0
\end{aligned} \tag{25}$$

- $$\begin{aligned}
&\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \rho_b A_b \left(\frac{\partial w}{\partial t}\right)^2 dx\right\} dt \\
&\quad= \int_0^L \int_{t_1}^{t_2} \rho_b A_b \frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right) dt dx \\
&\quad= \int_0^L \int_{t_1}^{t_2} \rho_b A_b \frac{\partial w}{\partial t}\frac{\partial\left(\delta w\right)}{\partial t} dt dx \\
&\quad= \int_0^L \left(\rho_b A_b \frac{\partial w}{\partial t}\right)\delta w|_{t_1}^{t_2} dx - \int_0^L \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\rho_b A_b \frac{\partial w}{\partial t}\right)\delta w dt dx \\
&\quad= -\int_0^L \int_{t_1}^{t_2} \rho_b A_b \frac{\partial^2 w}{\partial t^2}\delta w dt dx
\end{aligned} \tag{26}$$

- $$\begin{aligned}
&\int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L E_b I_b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right\} dt \\
&\quad= \int_{t_1}^{t_2}\int_0^L \frac{1}{2}E_b I_b \cdot 2\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right) dx dt \\
&\quad= \int_{t_1}^{t_2}\left(E_b I_b \frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left(E_b I_b \frac{\partial^2 w}{\partial x^2}\right)\frac{\partial\left(\delta w\right)}{\partial x} dx dt \\
&\quad= \int_{t_1}^{t_2}\left(E_b I_b \frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt - \int_{t_1}^{t_2} \frac{\partial}{\partial x}\left(E_b I_b \frac{\partial^2 w}{\partial x^2}\right)\delta w|_0^L dt \\
&\qquad\qquad + \int_{t_1}^{t_2}\int_0^L \frac{\partial^2}{\partial x^2}\left(E_b I_b \frac{\partial^2 w}{\partial x^2}\right)\delta w dx dt \\
&\quad= \int_{t_1}^{t_2} E_b I_b \frac{\partial^2 w}{\partial x^2}\,\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt - \int_{t_1}^{t_2} E_b I_b \frac{\partial^3 w}{\partial x^3}\,\delta w|_0^L dt \\
&\qquad\qquad + \int_{t_1}^{t_2}\int_0^L E_b I_b \frac{\partial^4 w}{\partial x^4}\delta w dx dt \\
&\quad= \int_{t_1}^{t_2}\int_0^L E_b I_b \frac{\partial^4 w}{\partial x^4}\delta w dx dt
\end{aligned} \tag{27}$$

- Similar to the derivation of Equation (26), the second term in Equation (25) can be degenerated into

$$\begin{aligned}
&\int_{t_1}^{t_2} \delta\left\{\int_0^L \rho_p A_p \left(\frac{\partial w}{\partial t}\right)^2 \left[H\left(x-x_1\right) - H\left(x-x_2\right)\right] dx\right\} dt \\
&\quad= -2\int_0^L \int_{t_1}^{t_2} \rho_p A_p \frac{\partial^2 w}{\partial t^2}\left[H\left(x-x_1\right) - H\left(x-x_2\right)\right]\delta w dt dx
\end{aligned} \tag{28}$$

-

$$
\begin{aligned}
-\int_{t_1}^{t_2}\delta\left\{\int_0^L E_pI_p\left(\frac{\partial^2 w}{\partial x^2}\right)^2\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]dx\right\}dt\\
&=-\int_{t_1}^{t_2}\int_0^L E_pI_p\cdot 2\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\delta\left(\frac{\partial^2 w}{\partial x^2}\right)dxdt\\
&=-2\int_{t_1}^{t_2}\left(E_pI_p\frac{\partial^2 w}{\partial x^2}\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_0^L dt\\
&\quad+2\int_{t_1}^{t_2}\int_0^L\frac{\partial}{\partial x}\left(E_pI_p\frac{\partial^2 w}{\partial x^2}\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\right)\frac{\partial\left(\delta w\right)}{\partial x}dxdt\\
&=-2\int_{t_1}^{t_2}E_pI_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_0^L dt\\
&\quad+2\int_{t_1}^{t_2}\int_0^L E_pI_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\delta\left(\frac{\partial w}{\partial x}\right)dxdt\\
&\quad+2\int_{t_1}^{t_2}\int_0^L E_pI_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\delta\left(\frac{\partial w}{\partial x}\right)dxdt\\
&=2\int_{t_1}^{t_2}E_pI_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\delta w|_0^L dt\\
&\quad-2\int_{t_1}^{t_2}\int_0^L E_pI_p\left(\frac{\partial^4 w}{\partial x^4}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\delta w dxdt\\
&\quad-2\int_{t_1}^{t_2}\int_0^L E_pI_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\delta w dxdt\\
&\quad+2\int_0^L E_pI_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\delta w|_0^L dt\\
&\quad-2\int_{t_1}^{t_2}\int_0^L E_pI_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\delta w dxdt\\
&\quad-2\int_{t_1}^{t_2}\int_0^L E_pI_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]\delta w dxdt
\end{aligned}
\tag{29}
$$

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$$
\begin{aligned}
-\int_{t_1}^{t_2}2\int_0^L E_pd_{31}abv\left(t\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]dxdt\\
&=-2\int_{t_1}^{t_2}\int_0^L E_pd_{31}abv\left(t\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]\delta w dxdt
\end{aligned}
\tag{30}
$$

Substituting Equation (26)(27)(28)(29)(30) into Equation (25) yields that

$$\begin{aligned}\int_{t_1}^{t_2}\Bigg\{\int_0^L \Bigg(&-\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)-2\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\\&-E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)-2E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\\&-4E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\\&-2E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]+f\left(x,t\right)\\&-2E_p d_{31} a b v\left(t\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]\Bigg)\delta w dx\\&-E_b I_b\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L+E_b I_b\left(\frac{\partial^3 w}{\partial x^3}\right)\delta w\Big|_0^L\Bigg\}dt=0\end{aligned} \tag{31}$$

For arbitrary δw in $0<x<L$, the equation of motion is

$$\begin{aligned}\rho_b A_b\left(\frac{\partial^2 w}{\partial t^2}\right)&+\left[2\rho_p A_p\left(\frac{\partial^2 w}{\partial t^2}\right)+2E_p I_p\left(\frac{\partial^4 w}{\partial x^4}\right)\right]\left[H\left(x-x_1\right)-H\left(x-x_2\right)\right]\\&+E_b I_b\left(\frac{\partial^4 w}{\partial x^4}\right)+4E_p I_p\left(\frac{\partial^3 w}{\partial x^3}\right)\left[H'\left(x-x_1\right)-H'\left(x-x_2\right)\right]\\&+2E_p I_p\left(\frac{\partial^2 w}{\partial x^2}\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]\\&+2E_p d_{31} a b v\left(t\right)\left[H''\left(x-x_1\right)-H''\left(x-x_2\right)\right]=f\left(x,t\right)\end{aligned} \tag{32}$$

with boundary conditions

At $x=0$, $\delta w=0$, $\frac{\partial^2 w}{\partial x^2}=0$

At $x=L$, $\delta w=0$, $\frac{\partial^2 w}{\partial x^2}=0$

(b) Assume

$$w\left(x,t\right)=\sum_{i=1}^{n}\phi_i\left(x\right)q_i\left(t\right) \tag{33}$$

where $\phi_i\left(x\right)$ satisfies all boundary conditions.

Substituting Equation (33) into Equation (32) yields that

$$\begin{aligned}
&\rho_b A_b \sum_{i=1}^{n} \phi_i(x)\ddot{q}_i(t) + E_b I_b \sum_{i=1}^{n} \phi_i^{(4)}(x) q_i(t) \\
&\quad + \left[2\rho_p A_p \sum_{i=1}^{n} \phi_i(x)\ddot{q}_i(t) + 2E_p I_p \sum_{i=1}^{n} \phi_i^{(4)}(x) q_i(t)\right][H(x-x_1) - H(x-x_2)] \\
&\quad + 4E_p I_p \sum_{i=1}^{n} \phi_i^{(3)}(x) q_i(t)[H'(x-x_1) - H'(x-x_2)] \\
&\quad + 2E_p I_p \sum_{i=1}^{n} \phi_i''(x) q_i(t)[H''(x-x_1) - H''(x-x_2)] \\
&\quad + 2E_p d_{31} abv(t)[H''(x-x_1) - H''(x-x_2)] - f(x,t) = \varepsilon
\end{aligned} \tag{34}$$

Min ε by $\langle \varepsilon, \phi_j \rangle = 0$,

$$\implies \langle \varepsilon, \phi_j \rangle = \int_0^L \varepsilon(x,t)\phi_j(x)\,dx = 0 \qquad j = 1, 2, \ldots, n \tag{35}$$

Equation (34) becomes

$$\begin{aligned}
&\Bigg[\rho_b A_b \left(\sum_{i=1}^{n} \int_0^L \phi_i(x)\phi_j(x)\,dx\right) \\
&+ 2\rho_p A_p \left(\sum_{i=1}^{n} \int_0^L \phi_i(x)\phi_j(x)[H(x-x_1) - H(x-x_2)]\,dx\right)\Bigg]\ddot{q}_i(t) \\
&+ \Bigg[E_b I_b \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(4)}(x)\phi_j(x)\,dx\right) \\
&\quad + 2E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(4)}(x)\phi_j(x)[H(x-x_1) - H(x-x_2)]\,dx\right)\Bigg] q_i(t) \\
&+ \left[4E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i^{(3)}(x)\phi_j(x)[H'(x-x_1) - H'(x-x_2)]\,dx\right)\right] q_i(t) \\
&+ \left[2E_p I_p \left(\sum_{i=1}^{n} \int_0^L \phi_i''(x)\phi_j(x)[H''(x-x_1) - H''(x-x_2)]\,dx\right)\right] q_i(t) \\
&+ 2E_p d_{31} abv(t) \int_0^L \phi_j(x)[H''(x-x_1) - H''(x-x_2)]\,dx \\
&- \int_0^L f(x,t)\phi_j(x)\,dx = 0
\end{aligned} \tag{36}$$

- $$\int_0^L \phi_i^{(4)}\phi_j(x)\,dx = \int_0^L \phi_i''(x)\phi_j''(x)\,dx \tag{37}$$
- $$\int_0^L \phi_i^{(4)}(x)\phi_j(x)[H(x-x_1) - H(x-x_2)]\,dx \tag{38}$$

-

$$\begin{aligned}
&\int_0^L \phi_i^{(3)}(x)\,\phi_j(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx \\
&\quad= \phi_i^{(3)}(x)\,\phi_j(x)\,\left[H(x-x_1)-H(x-x_2)\right]\Big|_0^L \\
&\qquad- \int_0^L \phi_i^{(4)}(x)\,\phi_j(x)\,[H(x-x_1)-H(x-x_2)]\,dx \\
&\qquad- \int_0^L \phi_i^{(3)}(x)\,\phi_j'(x)\,[H(x-x_1)-H(x-x_2)]\,dx
\end{aligned} \tag{39}$$

-

$$\begin{aligned}
&\int_0^L \phi_i''(x)\,\phi_j(x)\,[H''(x-x_1)-H''(x-x_2)]\,dx \\
&\quad= \phi_i''(x)\,\phi_j(x)\,\left[H'(x-x_1)-H'(x-x_2)\right]\Big|_0^L \\
&\qquad- \int_0^L \phi_i^{(3)}(x)\,\phi_j(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx \\
&\qquad- \int_0^L \phi_i''(x)\,\phi_j'(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx \\
&\quad= \int_0^L \phi_i^{(4)}(x)\,\phi_j(x)\,[H(x-x_1)-H(x-x_2)]\,dx \\
&\qquad+ \int_0^L \phi_i^{(3)}(x)\,\phi_j'(x)\,[H(x-x_1)-H(x-x_2)]\,dx \\
&\qquad+ \int_0^L \phi_i^{(3)}(x)\,\phi_j'(x)\,[H(x-x_1)-H(x-x_2)]\,dx \\
&\qquad+ \int_0^L \phi_i''(x)\,\phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx
\end{aligned} \tag{40}$$

-

$$\begin{aligned}
&\int_0^L \phi_j(x)\,[H''(x-x_1)-H''(x-x_2)]\,dx \\
&\quad= \phi_j(x)\,\left[H'(x-x_1)-H'(x-x_2)\right]\Big|_0^L \\
&\qquad- \int_0^L \phi_j'(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx \\
&\quad= -\phi_j(x)\,\left[H(x-x_1)-H(x-x_2)\right]\Big|_0^L \\
&\qquad+ \int_0^L \phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx \\
&\quad= \int_0^L \phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx = \phi_j'(x_2)-\phi_j'(x_1)
\end{aligned} \tag{41}$$

From Equation (38), (39), and (40), we can get that

$$
\begin{aligned}
&2\int_0^L \phi_i^{(4)}(x)\,\phi_j(x)\,[H(x-x_1)-H(x-x_2)]\,dx \\
&+4\int_0^L \phi_i^{(3)}(x)\,\phi_j(x)\,[H'(x-x_1)-H'(x-x_2)]\,dx \\
&+2\int_0^L \phi_i''(x)\,\phi_j(x)\,[H''(x-x_1)-H''(x-x_2)]\,dx \\
&=2\int_0^L \phi_i''(x)\,\phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx
\end{aligned}
\tag{42}
$$

Substituting Equation (37), (41), (42) into (36) yields

$$
\begin{aligned}
&\left[\rho_b A_b\left(\sum_{i=1}^n \int_0^L \phi_i(x)\,\phi_j(x)\,dx\right)\right.\\
&\left.+2\rho_p A_p\left(\sum_{i=1}^n \int_0^L \phi_i(x)\,\phi_j(x)\,[H(x-x_1)-H(x-x_2)]\,dx\right)\right]\ddot{q}_i(t)\\
&+\left[E_b I_b\left(\sum_{i=1}^n \int_0^L \phi_i''(x)\,\phi_j''(x)\,dx\right)\right.\\
&\left.+2E_p I_p\left(\sum_{i=1}^n \int_0^L \phi_i''(x)\,\phi_j''(x)\,[H(x-x_1)-H(x-x_2)]\,dx\right)\right]q_i(t)\\
&+2E_p d_{31} a b v(t)\left(\phi_j'(x_2)-\phi_j'(x_1)\right)=\int_0^L f(x,t)\,\phi_j(x)\,dx
\end{aligned}
\tag{43}
$$

Therefore, we can get that

$$
\sum_{r=1}^n m_{sr}\ddot{q}_r(t)+\sum_{s=1}^n k_{sr}q_r(t)=f_{c_s}(t)+f_{d_s}(t),\quad s=1,2,\dots,n \tag{44}
$$

where

$$
\begin{aligned}
m_{sr} &= \rho_b A_b \int_0^L \phi_r(x)\,\phi_s(x)\,dx + 2\rho_p A_p \int_{x_1}^{x_2} \phi_r(x)\,\phi_s(x)\,dx\\
k_{sr} &= E_b I_b \int_0^L \phi_r''(x)\,\phi_s''(x)\,dx + 2E_p I_p \int_{x_1}^{x_2} \phi_r''(x)\,\phi_s''(x)\,dx\\
f_{c_s}(t) &= 2E_p d_{31} a b v(t)\left(\phi_s'(x_1)-\phi_s'(x_2)\right)\\
f_{d_s}(t) &= \int_0^L f(x,t)\,\phi_s(x)\,dx = \int_0^L F_d(t)\,\phi_s(x)\,\delta(x-x_d)\,dx = F_d(t)\,\phi_s(x_d)
\end{aligned}
$$

Choosing $\phi_r=\sin(r\pi x/L), r=1,2,\dots,n$ satisfy all boundary conditions

At $x=0$, $\delta w=0$, $\frac{\partial^2 w}{\partial x^2}=0$

At $x=L$, $\delta w=0$, $\frac{\partial^2 w}{\partial x^2}=0$

Because $C=\alpha M+\beta K$, where $\alpha=0.6$ and $\beta=1.2\times10^{-6}$

$$
M\{\ddot{q}\}+C\{\dot{q}\}+K\{q\}=\{f_c\}+\{f_d\} \tag{45}
$$

for $s=r$,

$$
m_{rr}=\frac{\rho_b A_b L}{2}+\rho_p A_p(x_2-x_1)+\frac{\rho_p A_p L}{2\pi r}\left[\sin\left(\frac{2\pi r x_1}{L}\right)-\sin\left(\frac{2\pi r x_2}{L}\right)\right] \tag{46}
$$

$$
k_{rr}=\left(\frac{\pi r}{L}\right)^4\left\{\frac{E_b I_b L}{2}+E_p I_p(x_2-x_1)+\frac{E_p I_p L}{2\pi r}\left[\sin\left(\frac{2\pi r x_1}{L}\right)-\sin\left(\frac{2\pi r x_2}{L}\right)\right]\right\} \tag{47}
$$

for $s \neq r$,

$$m_{sr} = \frac{2\rho_p A_p L}{\pi} \left[\frac{r \sin\left(\frac{s\pi x}{L}\right) \cos\left(\frac{r\pi x}{L}\right)}{s^2 - r^2} + \frac{s \cos\left(\frac{s\pi x}{L}\right) \sin\left(\frac{r\pi x}{L}\right)}{r^2 - s^2} \right] \Bigg|_{x_1}^{x_2} \tag{48}$$

$$k_{sr} = \frac{2E_p I_p L}{\pi} \left(\frac{sr\pi^2}{L^2}\right)^2 \left[\frac{r \sin\left(\frac{s\pi x}{L}\right) \cos\left(\frac{r\pi x}{L}\right)}{s^2 - r^2} + \frac{s \cos\left(\frac{s\pi x}{L}\right) \sin\left(\frac{r\pi x}{L}\right)}{r^2 - s^2} \right] \Bigg|_{x_1}^{x_2} \tag{49}$$

And

$$f_c(t) = 2E_p d_{31} abv(t) \left(\frac{\pi s}{L}\right) \left(\cos\left(\frac{s\pi x_1}{L}\right) - \cos\left(\frac{s\pi x_2}{L}\right)\right) \tag{50}$$

$$f_d(t) = F_d(t)\,\phi_s(x_d) = F_d(t) \sin\left(\frac{s\pi x_d}{L}\right) \tag{51}$$

(c) Let

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{52}$$

Equation (45) becomes

$$\{\ddot{q}\} = -M^{-1}C\{\dot{q}\} - M^{-1}K\{q\} + M^{-1}\{f_c\} + M^{-1}\{f_d\} \tag{53}$$

In state-space form:

$$\begin{aligned} \dot{x} &= Ax + Bu + \hat{B}u_d \\ y &= C_0 x + Du \end{aligned} \tag{54}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{55}$$

$$B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} 2E_p d_{31} ab \begin{bmatrix} \frac{\pi}{L}\left(\cos\left(\frac{\pi}{L}x_1\right) - \cos\left(\frac{\pi}{L}x_2\right)\right) \\ \frac{2\pi}{L}\left(\cos\left(\frac{2\pi}{L}x_1\right) - \cos\left(\frac{2\pi}{L}x_2\right)\right) \\ \vdots \\ \frac{n\pi}{L}\left(\cos\left(\frac{n\pi}{L}x_1\right) - \cos\left(\frac{n\pi}{L}x_2\right)\right) \end{bmatrix} \tag{56}$$

$$\hat{B} = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \begin{bmatrix} \sin\left(\frac{\pi}{L}x_d\right) \\ \sin\left(\frac{2\pi}{L}x_d\right) \\ \vdots \\ \sin\left(\frac{n\pi}{L}x_d\right) \end{bmatrix} \tag{57}$$

Because $y = w(x,t) = \sum_{i=1}^{n} \phi_r(x_0)\, q_r(t)$ where $x_0 = 0.6L$

$$\begin{aligned} C_0 &= \begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) & \cdots & \phi_n(x_0) & 0 & 0 & \cdots & 0 \end{bmatrix}_{1\times 2n} \\ &= \begin{bmatrix} \sin(0.6\pi) & \sin(1.2\pi) & \cdots & \sin(0.6n\pi) & 0 & 0 & \cdots & 0 \end{bmatrix}_{1\times 2n} \end{aligned} \tag{58}$$

and

$$D = 0 \tag{59}$$

If we use the state feedback $u = -K_c x$, the state-space of the sytem is

$$\begin{aligned} \dot{x} &= (A - BK_c)\,x + \hat{B}u_d \\ y &= (C_0 - DK_c)\,x \end{aligned} \tag{60}$$

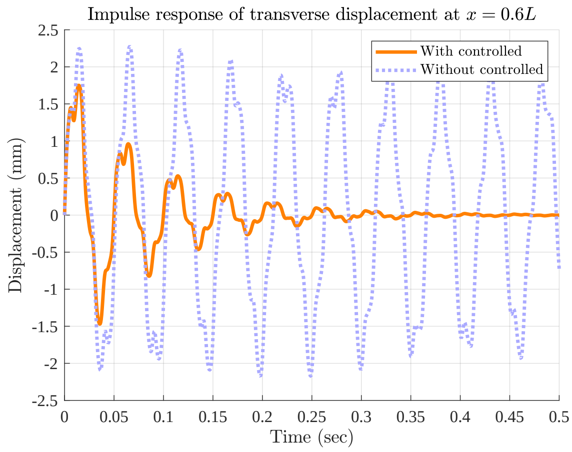
The following codes are used to simulate the response of the system:

```
clc; clf; clear all;

% initialize

b = 2e-2;
L = 0.50;
x1 = 0.15; x2 = 0.24;
xd = 0.30;

Eb = 7e10;
pb = 2700; tb = 2e-3;

Ec = 6.5e10;
pc = 7600; tc = 0.6e-3;
d31 = -175e-12;

Ac = b*tc;  Ab = b*tb;
Ib = b*tb^3/12; Ic = b*tc^3/12+Ac*(tb+tc)^2/4;
a = (tb+tc)/2;

% stiffness and mass matrices

N = 3; % no. of expansion terms

K = zeros(N);
M = zeros(N);
C = zeros(N);
Fc = zeros(N,1); Fd = zeros(N,1);

for r = 1:N
    for s = 1:N
        if r == s
            K(r,s) = (pi*r/L)^4*(Eb*Ib*L/2+Ec*Ic*(x2-x1)+...
                Ec*Ic*L/(2*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L)));
            M(r,s) = pb*Ab*L/2+pc*Ac*(x2-x1)+...
                pc*Ac*L/(2*pi*r)*(sin(2*pi*r*x1/L)-sin(2*pi*r*x2/L));
        else
            K(r,s) = 2*Ec*Ic*L/pi*(pi^2*r*s/L^2)^2*...
                ((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
                (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
```

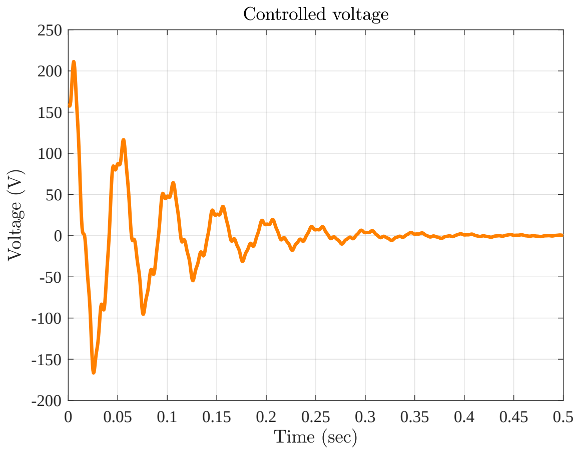
```
                ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
                (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
            M(r,s) = 2*pc*Ac*L/pi*((r*sin(s*pi*x2/L)*cos(r*pi*x2/L))/(s^2-r^2)+...
                (s*sin(r*pi*x2/L)*cos(s*pi*x2/L))/(r^2-s^2)-...
                ((r*sin(s*pi*x1/L)*cos(r*pi*x1/L))/(s^2-r^2)+...
                (s*sin(r*pi*x1/L)*cos(s*pi*x1/L))/(r^2-s^2)));
        end
    end

    % due to voltage input
    Fc(r) = 2*a*Ec*d31*b*(pi*r/L)*(cos(r*pi*x1/L)-cos(r*pi*x2/L));
    % due to discrete force with magnitude 1/100
    Fd(r) = 1/100*sin(r*pi*xd/L);
end

% add internal damping

C = 0.6*M+1.2e-6*K;

% state-space model

AL = -inv(M)*K;
AR = -inv(M)*C;
A = [zeros(N) eye(N);...
   AL AR];
BL1 = inv(M)*Fc; BL2 = inv(M)*Fd;
B1 = [zeros(N,1);BL1];
B2 = [zeros(N,1);BL2];
for r = 1:N
    CCw(1,r) = sin(r*pi*0.6);  % displacement w at midpoint (x=L/2)
end
CC = [CCw zeros(1,N)];
D = [0];

% control gain
Kc = [-55400 -22549 15848 -753 -249 174];
Ac=A-B1*Kc;

% impulse response

t = 0:0.0005:0.5;
IU = 1;
[y,x,t] = impulse(A,B2,CC,D,IU,t); % uncontrolled response
[yc,x,t] = impulse(Ac,B2,CC,D,IU,t); % controlled response
u = -Kc*x'; % controlled voltage

```

```
% plot results

figure(1);
hold on;
plot(t,yc*1000,'color',[1 0.5 0],'LineWidth',2.5)
plot(t,y*1000,':','color',[0.667 0.667 1],'LineWidth',2.5) % unit (mm)
hold off;
grid on;
title('Impulse response of transverse displacement at $x = 0.6L$' ...
    ,'interpreter','latex');
xlabel('Time (sec)','interpreter','latex');
ylabel('Displacement (mm)','interpreter','latex');
legend('With controlled','Without controlled','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q4-2-tyyc"+".pdf";
saveas(gcf,filename);

figure(2);
plot(t,u,'color',[1 0.5 0],'LineWidth',2.5);
grid on;
title('Controlled voltage','interpreter','latex');
xlabel('Time (sec)','interpreter','latex');
ylabel('Voltage (V)','interpreter','latex');
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gca,'position',[0.15 0.20 0.6 0.6]);
set(gcf,'position',[100 100 800 600]);
set(gcf,'renderer','painters');
filename = "Q4-2-tu"+".pdf";
saveas(gcf,filename);
```

The simulation results are shown in Figure 1.



(a)



(b)

Figure 1: Simulation results for the response of the system.